DOMINATION IN BIPARTITE GRAPHS AND IN THEIR COMPLEMENTS

Bohdan Zelinka, Liberec

(Received October 23, 1998)

Abstract. The domatic numbers of a graph G and of its complement $\overline{G}$ were studied by J. E. Dunbar, T. W. Haynes and M. A. Henning. They suggested four open problems. We will solve the following ones:

Characterize bipartite graphs G having $d(G) = d(\overline{G})$.

Further, we will present a partial solution to the problem:

Is it true that if G is a graph satisfying $d(G) = d(\overline{G})$, then $\gamma(G) = \gamma(\overline{G})$?

Finally, we prove an existence theorem concerning the total domatic number of a graph and of its complement.

Keywords: bipartite graph, complement of a graph, domatic number

MSC 2000: 05C69

We consider finite undirected graphs without loops and multiple edges. Mostly we treat bipartite graphs. The bipartition classes of such a graph will be denoted by P and Q and their cardinalities by p and q respectively; the notation will be chosen so that $p \geqslant q$. By $N_G(x)$ we denote the open neighbourhood of a vertex x in a graph G, i.e. the set of all vertices which are adjacent to x in G.

A subset D of the vertex set $V(G)$ of a graph G is called dominating (or total dominating) in G, if for each $x \in V(G) - D$ (or for each $x \in V(G)$, respectively) there exists $y \in D$ adjacent to x. A domatic (or total domatic) partition of G is a partition of $V(G)$, all of whose classes are dominating (or total dominating, respectively) sets in G. The domination number (or total domination number) of G is the minimum number of vertices of a dominating (or total dominating, respectively) set in G. The domatic [1] (or total domatic [2]) number of G is the maximum number of classes of a domatic (or total domatic, respectively) partition of G. The domination number of G is denoted by $\gamma(G)$, its total domination number by $\gamma_t(G)$, its domatic number by $d(G)$, its total domatic number by $d_t(G)$.

Before solving the first mentioned problem we exclude certain cases.

Lemma 1. *Let G be a graph with an isolated vertex. Then $d(G) \neq d(\overline{G})$.*

P r o o f. Let v be an isolated vertex in G. It is contained in all dominating sets in G and thus no two of them may be disjoint and $d(G) = 1$. In $\overline{G}$ there exists the domatic partition $\{\{v\}, V(G) - \{v\}\}$ and thus $d(\overline{G}) = 2$. □

If $q = 1$ for a bipartite graph G, then either G or $\overline{G}$ has an isolated vertex. Therefore the following proposition holds.

Proposition. *Let G be a bipartite graph in which one bipartition class consists of one element. Then $d(G) \neq d(\overline{G})$.*

Lemma 2. *Let G be a bipartite graph with bipartition classes P, Q, let $p = |p|$, $q = |Q|$, $p \geqslant q \geqslant 2$. Then $d(G) \leqslant q \leqslant d(\overline{G})$.*

P r o o f. No proper subset of P or of Q is dominating in G. Therefore if D is a dominating set in G, then either $D = P$, or $D = Q$, or $D \cap P \neq 0$ and $D \cap Q \neq 0$. A domatic partition of G is either $\{P, Q\}$, therefore with two classes, or has the property that each of its classes has a non-empty intersection with Q and thus it has at most q classes; this implies $d(G) \leqslant q$. In the complement $\overline{G}$ the sets P, Q induce complete subgraphs and therefore each union of a non-empty subset of P and a non-empty subset of Q is dominating in $\overline{G}$. We have $p \geqslant q$ and therefore there exists a partition $\{M_1, \ldots, M_q\}$ of P with Q classes. If $Q = \{y_1, \ldots, y_q\}$, we may take the partition $\{M_1 \cup \{y_1\}, \ldots, M_q \cup \{y_q\}\}$ of $V(G)$ and this is a domatic partition of G. Therefore $q \leqslant d(\overline{G})$. □

Now we prove a theorem.

Theorem 1. *Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let $p = |P|$, $q = |Q|$, $p \geqslant q \geqslant 2$. The equality $d(G) = d(\overline{G})$ holds if and only if the following conditions are satisfied:*

(i) *The degree of each vertex of P in G is at least $q - 1$.*

(ii) *The number of vertices of P of degree q is greater than or equal to the number of vertices of Q of degree p.*

(iii) *Either $p \leqslant 2q - 1$, or there exists at least one vertex of Q of degree p.*

P r o o f. Let the conditions (i), (ii), (iii) hold. Let $y_1, \ldots, y_q$ be the vertices of Q. Let $M_0 = \{x \in P \mid N_G(x) = Q\}$ and $M_i = \{x \in P \mid y_i \notin N_G(x)\}$

for $i = 1, \ldots, q$. The condition (i) implies that the sets $M_0, M_1, \ldots, M_q$ are pairwise disjoint; some of them may be empty. Let $J_0 = \{i \in \{1, \ldots, q\} \mid M_i = 0\}$, $J_1 = \{i \in \{1, \ldots, q\} \mid M_i \neq 0\}$. For $i \in J_0$ the vertex x_i is adjacent to all vertices of P and its degree is p. By (ii) we have $|M_0| \geqslant |J_0|$ and thus there exists a partition $\{L_i \mid i \in J_0\}$ of M_0. Now define sets D_i for $i = 1, \ldots, q$. If $i \in J_0$, then $D_i = L_i \cup \{y_i\}$. If $i \in J_1$, then $D_i = M_i \cup \{y_i\}$. The partition $\mathcal{D} = \{D_1, \ldots, D_q\}$ is a domatic partition of G and thus $d(G) \geqslant q$ and, by Lemma 2, $d(G) = q$. The partition $\mathcal{D}$ is also a domatic partition of $\overline{G}$ and thus $d(\overline{G}) \geqslant q$. Suppose that $d(\overline{G}) \geqslant q + 1$ and let $\mathcal{D}'$ be the corresponding domatic partition of $\overline{G}$. At most q classes of $\mathcal{D}'$ may have non-empty intersections with Q and therefore there exists a class D' of $\mathcal{D}'$ which is a subset of P. Each vertex of Q is adjacent in $\overline{G}$ and thus non-adjacent in G to a vertex of D'. If there exists a vertex of Q of degree p (condition (iii)), then this vertex is adjacent in G to all vertices of P and thus also to all of D', which is a contradiction. If such a vertex does not exist, then $p \leqslant 2q - 1$ by (iii). By (i) each vertex of D' is adjacent in G to at most one vertex of Q (to exactly one, if D' is minimal with respect to inclusion), therefore $|D'| \leqslant q$. No proper subset of Q is dominating in G, because for each vertex of Q there exists a vertex of D' adjacent in G only to it. Hence each class of $\mathcal{D}'$ has a non-empty intersection with P. As D' contains at least q vertices of P, the number of all other classes of $\mathcal{D}'$ is at most $p - q$ and $|\mathcal{D}'| \leqslant p - q + 1$. By (iii) then $|\mathcal{D}'| \leqslant q$, which is a contradiction. Therefore $d(\overline{G}) = q$ and $d(G) = d(\overline{G})$.

Now suppose that (i) does not hold. There exists a vertex $x_0 \in P$ whose degree is at most $q-2$ and therefore there exist vertices $y_1 \in Q$, $y_2 \in Q$ which are not adjacent to x_0. Suppose that $d(G) = q$ and let $\mathcal{D} = D_1, \ldots, D_q$ be the corresponding domatic partition. Each class of $\mathcal{D}$ has exactly one element in common with Q; without loss of generality let $D_1 \cap Q = y_1$, $D_2 \cap Q = y_2$. But then both D_1, D_2 must contain x_0, which is a contradiction. Therefore $d(G) < q \leqslant d(\overline{G})$.

Suppose that (ii) does not hold; by our notation this means $|M_0| < |J_0|$. Suppose that $d(G) = q$ and let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be the corresponding partition. We use the notation $Q = \{y_1, \ldots, y_q\}$ and without loss of generality we suppose that $D_i \cap Q = \{y_i\}$ for $i = 1, \ldots, q$. If $i \in J_1$, then $M_i \subseteq D_i - \{y_i\}$. Therefore if $i \in J_0$, then $D_i \cap P \subseteq M_0$. As $|M_0| < |J_0|$ and all these intersections must be non-empty and pairwise disjoint, we have a contradiction. Therefore again $d(G) < q \leqslant d(\overline{G})$.

Now suppose that (iii) does not hold; therefore $p \geqslant 2q$ and $J_0 = \emptyset$, which means $M_i \neq \emptyset$ for each $i \in \{1, \ldots, q\}$. In each M_i we choose a vertex x_i and denote $A = \{x_1, \ldots, x_q\}$. In G the vertices x_i, y_i are adjacent for each $i \in \{1, \ldots, q\}$, therefore A is a dominating set in G. As $p \geqslant 2q$, the set $P - A$ has at least q elements and we may choose a partition $\{S_1, \ldots, S_q\}$ of $P - A$ with q classes. Evidently $S_i \cup \{y_i\}$

is a dominating set in G for each $i \in \{1, \ldots, q\}$ and $\{A, S_1 \cup \{y_1\}, \ldots, S_q \cup \{y_q\}\}$ is a domatic partition of G. We have $d(\overline{G}) \geqslant q + 1 > q \geqslant d(G)$. □

The problem whether $d(G) = d(\overline{G})$ implies $\gamma(G) = \gamma(\overline{G})$ will be solved only for bipartite graphs.

Theorem 2. *Let G be a bipartite graph such that $d(G) = d(\overline{G})$. Then $\gamma(G) = \gamma(\overline{G})$.*

Proof. Again we may restrict our considerations to graphs with $q \geqslant 2$ and without isolated vertices. According to Theorem 1 the equality $d(G) = d(\overline{G})$ implies the validity of the conditions (i), (ii), (iii) and $d(G) = d(\overline{G}) = q$. If there exists at least one vertex $y \in Q$ of degree p, then by (ii) there exists at least one vertex $x \in P$ of degree q. The set $\{x, y\}$ is dominating in G. We have $q \geqslant 2$ and therefore no one-element set may be dominating in G and $\gamma(G) = 2$. If vertex y exists, then $p \leqslant 2q - 1$ must hold by (iii). We use the notation from the proof of Theorem 1. We have $M_i \neq \emptyset$ for all $i \in \{1, \ldots, q\}$. As the sets M_i are pairwise disjoint subsets of P and $p \leqslant 2q - 1$, there exists some $j \in \{1, \ldots, q\}$ such that $|M_j| = 1$. Let $M_j = \{x\}$. The set $\{x, y_j\}$ is dominating in G and $\gamma(G) = 2$. In the graph $\overline{G}$ each two-element set consisting of a vertex of P and a vertex of Q is dominating, because P and Q induce complete subgraphs of $\overline{G}$. No vertex is adjacent in $\overline{G}$ to all others, because such a vertex would be isolated in G. Therefore $\gamma(\overline{G}) = 2 = \gamma(G)$. □

In the case of the total domatic number the situation is more complicated. We will give a full characterization only for the case $q = 2$; for a general case we will prove only an existence theorem. From our considerations we must exclude graphs with isolated vertices, because for them the total domatic number is not well-defined. In particular, for bipartite graphs we exclude the case $q = 1$, because in this case the complement contains an isolated vertex.

For $q = 2$ we can give a full characterization.

Theorem 3. *Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let $p = |P|$, $q = |Q| = 2$, $p \geqslant 2$. The equality $d_t(G) = d_t(\overline{G})$ holds if and only if exactly one vertex of Q has degree p.*

Proof. Let $Q = \{y_1, y_2\}$. Suppose (without loss of generality) that y_1 has degree p, while y_2 has not. Then there exists a vertex $x \in P$ non-adjacent to y_2. Its degree in G is 1. In [2] it is stated that $d_t(G)$ cannot exceed the minimum degree of a vertex in G and therefore $d_t(G) = 1$. In G the vertex y_1 has degree 1 and thus $d_t(G) = 1$ and $d_t(G) = d_t(\overline{G})$.

If none of the vertices of Q has degree p, then there exists a vertex $x_1 \in P$ non-adjacent to y_1 and a vertex $x_2 \in P$ non-adjacent to y_2. We have $x_1 \neq x_2$, otherwise this vertex would be isolated. Both x_1, x_2 have degree 1 and thus $d_t(G) = 1$. If we put $D_1 = \{x_1, y_1\}$, $D_2 = (A - \{x_1\}) \cup \{y_2\}$, then $\{D_1, D_2\}$ is a total domatic partition of $\overline{G}$ and thus $d_t(\overline{G}) \geqslant 2$ and $d_t(\overline{G}) \neq d_t(G)$. If both vertices of Q have degree p, then choose $x \in P$ and put $D_1' = \{x, y_1\}$, $D_2' = (A - \{x\}) \cup \{y_1\}$. The partition $\{D_1', D_2'\}$ is domatic in G and thus $d_t(G) = 2$ (the degrees of vertices of P are equal to 2). In $\overline{G}$ both vertices of Q have degree 1 and thus $d_t(\overline{G}) = 1$ and $d_t(G) \neq d_t(\overline{G})$. □

Now we prove a lemma.

Lemma 3. *Let G be a bipartite graph without isolated vertices and with bipartition classes P, Q, let $p = |P|$, $q = |Q|$, $p \geqslant q \geqslant 2$. Then $d(\overline{G}) \geqslant \lfloor \frac{1}{2}q \rfloor$.*

Proof. The sets P, Q induce complete subgraphs in G. Denote $r = \lfloor \frac{1}{2}q \rfloor$. Choose an arbitrary partition $\{Q_1, \ldots, Q_r\}$ of Q such that at most one class has three elements and all others have two elements each; such a partition has r classes. As $p \geqslant q$, also p can be partitioned into r classes, each of which has at least two elements. Let this partition be $\{P_1, \ldots, P_r\}$. Then $\{P_1 \cup Q_1, \ldots, P_r \cup Q_r\}$ is a domatic partition of G, which implies the assertion. □

Now we prove the existence theorem.

Theorem 4. *Let p, q, s be positive integers, $p \geqslant q \geqslant 3$. There exists a bipartite graph G with the bipartition classes P, Q such that $|P| = p$, $|Q| = q$ and $d_t(G) = d_t(\overline{G}) = s$ if and only if $\frac{1}{2}q \leqslant s \leqslant \frac{3}{4}q$.*

Proof. Let $\frac{1}{2}q \leqslant s \leqslant \frac{3}{4}q$. First we shall investigate the case $s = \frac{1}{2}q$; then obviously q is even. Denote $r = \frac{1}{2}q$. Take two disjoint sets $P = \{x_1, \ldots, x_p\}$, $Q = \{y_1, \ldots, y_p\}$; the vertex set of G will be $V(G) = P \cup Q$. Join each vertex of P with each vertex of Q by an edge, except the pairs $\{x_1, y_i\}$ for $i = 1, \ldots, r$. Thus G is constructed. The vertex x_1 has degree $\frac{1}{2}q$ and thus $d_t(G) \leqslant \frac{1}{2}q$. Put $D_i = \{x_{r+i}, y_{r+i}\}$ for $i = 1, \ldots, r-1$ and $d_r = V(G) - \bigcup_{i=1}^{r-1} D_i$. The partition $\{D_1, \ldots, D_r\}$ is total domatic in G and thus $d_t(G) = r = \frac{1}{2}q$. In $\overline{G}$ no subset of P is total dominating and thus each total dominating set in $\overline{G}$ has a non-empty intersection with Q. If this intersection consists of one element, then this element must be some of the vertices $y_1, \ldots, y_r$ and moreover this total dominating set must contain a vertex of P adjacent to this vertex; such a vertex is only x_1. Therefore a total domatic partition of $\overline{G}$ can contain at most one class having only one vertex in common with Q, all others must have at least two. The number of classes is at most r and $d_t(\overline{G}) \leqslant r$. There exists the same total domatic partition of G as in the proof of Lemma 3 and thus $d_t(G) = r = \frac{1}{2}q$ and $d_t(G) = d_t(\overline{G})$.

Now let $\lfloor \frac{1}{2}q \rfloor + 1 \leqslant \frac{3}{4}q$; we will denote $r = \lfloor \frac{1}{2}q \rfloor$. Take again $V(G) = P \cup Q$, where $P = \{x_1, \ldots, x_p\}$, $Q = \{y_1, \ldots, y_q\}$. Let $m = 2s - q$; we have $2 \leqslant m \leqslant r$. We construct first the complement $\overline{G}$. It contains the edges $x_i y_i$ for $i = 1, \ldots, m$ and in addition the edges $x_i y_{2m+j}$, where $1 \leqslant j \leqslant p - 2m$, $j \equiv i \pmod{m}$, again for $i = 1, \ldots, m$ and for all j satisfying the condition (such j need not exist). Further, $\overline{G}$ obviously contains all edges joining two vertices of P and all edges joining two vertices of Q. In $\overline{G}$ no subset of P is total dominating and thus each total dominating set in $\overline{G}$ must have a non-empty intersection with Q. This intersection may consist of one vertex, only if this vertex is adjacent in $\overline{G}$ to a vertex of P; moreover, the mentioned total dominating set must contain also a vertex of P adjacent to this vertex. Only the vertices $x_1, \ldots, x_m$ are adjacent in $\overline{G}$ to vertices of Q and thus in each total domatic partition of $\overline{G}$ at most m classes have one vertex in common with Q; the others have at least two and the number of classes is at most $m + \frac{1}{2}(q - m) = s$. Therefore $d_t(\overline{G}) \leqslant s$. Let $L_i = \{y_{m+2i-1}, y_{m+2i}\}$ for $i = 1, \ldots, s - m$. Let $\{M_1, \ldots, M_{s-m}\}$ be an arbitrary partition of $P - \{x_1, \ldots, x_m\}$ into $s - m$ classes. Put $\overline{D}_i = \{x_i, y_i\}$ for $i = 1, \ldots, m, \overline{D}_i = L_{i-m} \cup M_{i-m}$ for $i = m + 1, \ldots, m + s$. The partition $\{\overline{D}_1, \ldots, \overline{D}_s\}$ is a total domatic partition of $\overline{G}$ and $d_t(\overline{G}) = s$.

Also each total dominating set in G has a non-empty intersection with Q. It has one vertex in common with Q, only if this vertex has degree p in Q; otherwise it has at least two. There are m vertices of degree p in Q, namely $y_{m+1}, \ldots, y_{2m}$. Analogously as in the case of $\overline{G}$ we have $d_t(G) \leqslant m + \frac{1}{2}(q - m) = s$. Put $D_i = \{x_{m+i}, y_{m+i}\}$ for $i = 1, \ldots, m$. Further, for q even (and thus also m even) put $D_i = \{x_{2(i-m)-1}, x_{2(i-m)}; y_{2(i-m)-1}, y_{2(i-m)}\}$ for $i = m + 1, \ldots, \frac{3}{2}m$, $D_i = \{x_{2i-m-1}, x_{2i-m}, y_{2i-m-1}, y_{2i-m}\}$ for $i = \frac{3}{2}m + 1, \ldots, s$. For q odd we have $D_i = \{x_{2(i-m)-1}, x_{2(i-m)}, y_{2(i-m)-1}, y_{2(i-m)}\}$ for $i = m + 1, \ldots, \frac{1}{2}(3m - 1)$, $D_i = \{x_m, x_{2m+1}, y_m, y_{2m+1}\}$ for $i = \frac{1}{2}(3m+1)$, $D_i = \{x_{2i-m-1}, x_{2i-m}, y_{2i-m-1}, y_2 i - m\}$ for $i = \frac{1}{2}(3m + 1) + 1, \ldots, s$. Then $\{D_1, \ldots, D_s\}$ is a total domatic partition of G and we have $d_t(G) = d_t(\overline{G}) = s$.

Now consider the cases when a does not satisfy the above mentioned inequality. By Lemma 3 for $s < \lfloor \frac{1}{2}q \rfloor$ the required graph does not exist. For q odd consider the case $s = \lfloor \frac{1}{2}q \rfloor = \frac{1}{2}(q - 1) < \frac{1}{2}q$. We have $d_t(\overline{G}) = s$ in the case when G is a complete bipartite graph $K_{p,q}$, but then $d_t(G) = q \neq s$. Suppose that G is a bipartite graph on P, Q with $|P| = p$, $|Q| = q$ which is not $K_{p,q}$. Then there exists $x \in P$ and $y \in Q$ such that x, y are non-adjacent in G and thus adjacent in $\overline{G}$. Let $\{L_1, \ldots, L_s\}$ be a partition of $Q - \{y\}$ into two-element sets, let $\{M_1, \ldots, M_s\}$ be a partition of $P - \{x\}$ into sets with at least two vertices. Put $D_i = L_i \cup M_i$ for $i = 1, \ldots, s$, $D_{s+1} = \{x, y\}$. The partition $\{D_1, \ldots, D_{s+1}\}$ is total domatic in $\overline{G}$ and $d_t(\overline{G}) \geqslant s + 1$. This excludes the case $s = \frac{1}{2}(q - 1)$.

Suppose $s > \frac{3}{4}q$. With the notation introduced above, we have $m = 2s - q > \frac{1}{2}q$. As we have seen in the first part of the proof, for $d_t(G) = s$ we must have at least m vertices of degree p in Q; they are non-adjacent to any vertex in G. For $d_t(\overline{G}) = s$ we must have at least m vertice of Q which are adjacent to some vertex of P in $\overline{G}$. As $m > \frac{1}{2}q$, these two conditions cannot be satisfied simultaneously and thus for $s > \frac{3}{4}q$ the required graph does not exist. □

At the end we prove a theorem which concerns graphs in general, not only bipartite graphs.

Theorem 5. *No disconnected graph G with $d_t(G) = d_t(\overline{G})$ exists.*

Proof. Let G be a disconnected graph. If G contains isolated vertices, then $d_t(G)$ is not defined; therefore suppose that G has no isolated vertex. Let H_1 be a connected component of G with the minimum number of vertices; let $H_2 = G - H_1$. Let h be the number of vertices of H_1. In $\overline{G}$ each vertex of H_1 is adjacent to each vertex of H_2. Let the vertices of H_1 be $v_1, \ldots, v_h$ and choose h pairwise distinct vertices $w_1, \ldots, w_h$ in H_2. Put $\overline{D}_i = \{v_i, w_i\}$ for $i = 1, \ldots, h-1$ and $\overline{D}_h = V(G) - \bigcup_{i=1}^{n-1} \overline{D}_i$. Then $\{\overline{D}_1, \ldots, \overline{D}_h\}$ is a total domatic partition of $\overline{G}$ and $d_t(\overline{G}) \geqslant h$. The total domatic number of G is the minimum of total domatic numbers of the connected components of G and thus $d_t(G) \leqslant d_t(H_1)$. Any total dominating set in a graph has at least two vertices and thus $d_t(G) \leqslant d_t(H_1) \leqslant \frac{1}{2}h < h \leqslant d(\overline{G})$. □

References

[1] *E. J. Cockayne and S. T. Hedetniemi*: Towards the theory of domination in graphs. Networks *7* (1977), 247–261.

[2] *E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi*: Total domination in graphs. Networks *10* (1980), 211–219.

[3] *J. E. Dunbar, T. W. Haynes and M. A. Henning*: The domatic number of a graph and its complement. Congr. Numer. *8126* (1997), 53–63.

Author's address: Technická universita, Kat. aplikované matematiky, Voroněžská 13, 461 17 Liberec, Czech Republic.