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Application of Fuzzy Numbers in Binomial Tree Model and Time Complexity

Abstract

Discrete binomial models are powerful tools for options valuation. For simple pay-off options they can be viewed as an approximation of famous Black-Scholes option valuation formula. By increasing the quantity of periods in binomial model (i.e. decreasing the length of the period), the results converge to the continuous model. However this approximation is very computationally costly, thus the analytical solution to the valuation is preferable. Nevertheless, the analytical solution does not exist for more complicated pay-off options. In the article we assume the valuation of project with the possibility to change the quantity of products produced. Some input parameters (concretely the volatility and initial cash-flows) are assumed to be uncertain and stated as a fuzzy numbers. Illustrative example is provided in the paper. In this example we examine the time complexity of the algorithm and the influence of the imprecision of input parameters on the appraisal imprecision. From the results it is apparent that the complexity of the model is quadratic. Thus by increasing the quantity of periods in the binomial model it becomes unreasonably time demanding.

Key Words

finance, valuation, investment analysis, fuzzy sets, real options, binomial model

JEL Classification: C63

Introduction

For project valuation the real options methodology could be considered as a generalized approach encompassing both the risk and flexibility aspects simultaneously. The papers and books focused on the real options valuation are for instance [3, 6, 11, 12, 16].

For options valuation one can utilize both the discrete models (binomial or trinomial trees) and analytical continuous version, based on famous Black-Scholes model [1]. For simple European options the value can be stated analytically. However, for American options, exotic and real options with complicated payoff functions the numerical approximation in the form of discrete models has to be applied.

While for the real options the analytical valuation formula usually could not be found, these models are solved mostly by discrete methods. It is also sometimes impossible to state input parameters as the real numbers. Thus, these models are in the paper assumed as a fuzzy-stochastic (in line with [18]). Therefore hybrid (fuzzy-stochastic) binomial option model will be utilized in this paper. We can find a few papers dealing

with fuzzy binomial models methodology approaches. There is supposed a fuzzy volatility (see [8, 9, 13, 14]) or simultaneously fuzzy volatility and fuzzy risk-free rate (see [7]). While the stochasticity of the underlying variables is connected to the risk, the fuzzy approach allows us to work with some vagueness and uncertainty.

In this paper we assume the real investment project under the flexible switch options methodology similar to [17]. This means, that in each moment the project can be switched into another state, in which the underlying asset (resp. free cash flow) changes. These changes are charged by the switch costs (which can be actually negative, i.e. profit).

The goal of the paper is to propose hybrid (fuzzy-stochastic) binomial option model for project valuation and determine the time complexity of the valuation algorithm. The paper proceeds as follows. In the next section the methodology utilized for valuation of a project with switch costs is defined. Then, in the second section the illustrative example is provided and time complexity of the valuation algorithm is examined on this example.

1. Methodology

Valuation of the project is usually based on discounting free cash-flows obtained during its lifetime. The free cash-flow (henceforth FCF) in particular year can be obtained as a net profit plus depreciation of the long term assets minus investments and change of net working capital. Since it is usually difficult to forecast the free cash-flows to distant future, the valuation is usually divided into two phases: (i) in the first phase the cash flow is projected for each year, (ii) in the second phase the cash flow is assumed to be stable or growing by a steady rate. The valuation on the basis of net present value is then as follows,

$$V = \sum_{t=1}^T \frac{FCF_t}{1+r} + \frac{FCF_T}{r(1+r)^T}, \quad (1)$$

where FCF_t is the free cash-flow in year t and r is the required rate of return¹.

1.1 Project valuation with random free cash-flows and switch costs

The formula (1) is the simplest way how to appraise the project based on its predicted free cash-flows. The problem occurs, when we add the flexibility to the project (such as below described possibility to switch to the increased/decreased state of the production). Then discrete models, such as a binomial model, should be utilized.

¹ For the sure FCF the risk-free rate should be utilized. For risky FCF (in this case the mean of the FCF is assumed) the rate taking into account also the risk should be utilized (Risk Adjusted Cost of Capital).

In the previous model (1) the free-cash flows cannot be influenced by the entrepreneur. Assume now that the entrepreneur can change the quantity of goods produced. Thus we assume states which correspond with the utilization of production capacity (e.g. normal state, increased production state and decreased production state). Then in each period the entrepreneur can choose if he stays in actual state or switches to the other state. This change is connected with expenses, so the matrix of switch costs $\mathbf{C} = [c_{i,j}]$ should be defined. Costs $c_{i,j}$ are one-time costs needed to switch from state i to state j .

With some simplicity we can divide the free cash-flow into three parts: (i) variable part of the free cash-flow dependent on some random variable such as a price of the product, inputs etc., (henceforth variable income, x), (ii) stable part of the free cash-flow (e.g. fixed costs, henceforth fc) and (iii) above defined switch costs $c_{i,j}$. We assume that x follows geometric Brownian motion.¹ The project can then be appraised by means of binomial tree model with one risk (random) factor. The model is of discrete version and for the sake of simplicity an intra-interval continuous compounding is applied.

There are several ways how to calibrate the binomial model (see e.g. [2, 5, 10]). In this paper we apply the approach of Cox et al. [5]. In this model the indexes of up (down) movement u (d) are computed from volatility σ and the chosen period length τ as follows,

$$u = e^{\sigma\sqrt{\tau}}, \quad (2)$$

$$d = e^{-\sigma\sqrt{\tau}}. \quad (3)$$

The other approaches with very illustrative algorithms can be found e.g. in [4].

In our model the variable income $x(s,t,n)$ at period t assuming $t-n$ increases and n decreases and in the state s would be as follows,

$$x(s,t,n) = x_{0,s} \cdot e^{(t-2n)\sigma\sqrt{\tau}}, \quad (4)$$

The project value at the end of the first phase T is computed as the perpetuity of free cash-flows in the second phase,

$$V(s,T,n) = \frac{x_{0,s} \cdot e^{(T-2n)\sigma\sqrt{\tau}} - fc}{r}, \quad (5)$$

where n is the quantity of decreases of variable income, T is the chosen number of periods with length τ , r is the required rate of return for one period. During the first phase the project value is computed by means of the backward recurrent procedure

¹ If the price of the product follows geometric Brownian motion then also the sales follow geometric Brownian motion.

thought the binomial tree, so that the value at time t and with n decreases can be expressed as follows,

$$V(s,t,n) = \max_i \left[x_{0,s} e^{(t-2n)\sigma\sqrt{t}} - fc - c_{s,i} + p \frac{V(i,t+1,n)}{1+r} + (1-p) \frac{V(i,t+1,n+1)}{1+r} \right], \quad (6)$$

where $V(s,t,n)$ is the value of the project after t periods and with n decreases and p is calculated risk neutral probability.

1.2 Project valuation with random free cash-flows and switch costs under fuzzy inputs

In this section the volatility σ and also initial variable incomes $x(s,0,0)$ for particular states will be assumed to be a fuzzy numbers (fuzzy sets). Fuzzy sets, firstly introduced by Zadeh [15], are the extension of classical set theory. While in the traditional sets the object either is or is not belonging to the set, in the fuzzy theory there is a membership function $\mu_{\tilde{A}}(x)$ which specifies the degree with which the x belongs to the fuzzy set \tilde{A} . Thus, fuzzy set can be utilized to express and handle vagueness or impreciseness mathematically.

The membership function $\mu_{\tilde{A}}(x)$ can possess various shapes. The most commonly utilized fuzzy numbers are those with triangular shape, i.e. triangular fuzzy numbers. Triangular fuzzy number \tilde{N} can be defined as a triplet (l,m,n) with the membership function $\mu_{\tilde{N}}$ as follows,

$$\mu_{\tilde{N}}(x) = \begin{cases} 0 & \text{for } x < l \\ \frac{x-l}{m-l} & \text{for } l \leq x \leq m \\ \frac{n-x}{n-m} & \text{for } m < x \leq n \\ 0 & \text{for } x > n \end{cases}, \quad (7)$$

where l, m, n are real numbers such that $l < m < n$. Also some properties of the fuzzy sets can be defined such as the support, width, nucleus, height etc. Important tool is α -cut,

$$\tilde{A}^\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}, \quad \forall \alpha \in [0,1]. \quad (8)$$

Below the model from section 1.1 will be updated. The volatility σ and the initial variable incomes $x(s,0,0)$ are assumed to be fuzzy numbers. The numerical computation is made by means of the α -cuts. Assuming $\tilde{\sigma}$ as a fuzzy set of volatility and $\tilde{x}_{0,s}$ as a

fuzzy set of initial variable cash flow, the previous formulas should be changed as follows,

$$\tilde{u}^\alpha = [u^{\alpha-}, u^{\alpha+}], \tilde{d}^\alpha = [d^{\alpha-}, d^{\alpha+}], \tilde{p}^\alpha = [p^{\alpha-}, p^{\alpha+}], \quad (9)$$

$$\tilde{x}^\alpha(s, t, n) = [x^{\alpha-}(s, t, n), x^{\alpha+}(s, t, n)] \quad (10)$$

$$\tilde{V}^\alpha(s, T, n) = [V^{\alpha-}(s, T, n), V^{\alpha+}(s, T, n)] \quad (11)$$

$$\tilde{V}^\alpha(s, t, n) = [V^{\alpha-}(s, t, n), V^{\alpha+}(s, t, n)] \quad (12)$$

where,¹

$$u^{\alpha-} = e^{\sigma^{\alpha-}\sqrt{\tau}}, u^{\alpha+} = e^{\sigma^{\alpha+}\sqrt{\tau}}, d^{\alpha-} = e^{-\sigma^{\alpha-}\sqrt{\tau}}, d^{\alpha+} = e^{-\sigma^{\alpha+}\sqrt{\tau}}, \quad (13)$$

$$p^{\alpha-} = \frac{1+r-d^{\alpha-}}{u^{\alpha-}-d^{\alpha-}}, p^{\alpha+} = \frac{1+r-d^{\alpha+}}{u^{\alpha+}-d^{\alpha+}} \quad (14)$$

$$x^{\alpha-}(s, t, n) = \min \left[x_{0,s}^{\alpha-} \cdot e^{(t-2n)\sigma^{\alpha-}\sqrt{\tau}}, x_{0,s}^{\alpha-} \cdot e^{(t-2n)\sigma^{\alpha+}\sqrt{\tau}} \right] \quad (15)$$

$$x^{\alpha+}(s, t, n) = \max \left[x_{0,s}^{\alpha+} \cdot e^{(t-2n)\sigma^{\alpha-}\sqrt{\tau}}, x_{0,s}^{\alpha+} \cdot e^{(t-2n)\sigma^{\alpha+}\sqrt{\tau}} \right] \quad (16)$$

$$V^{\alpha-}(s, T, n) = \frac{x^{\alpha-}(s, T, n) - fc}{r}, V^{\alpha+}(s, T, n) = \frac{x^{\alpha+}(s, T, n) - fc}{r} \quad (17)$$

$$V^{\alpha-}(s, t, n) = \max_i \left[\begin{aligned} & x^{\alpha-}(s, t, n) - fc - c_{s,i} + p^{\alpha-} \frac{V^{\alpha-}(i, t+1, n)}{1+r} + \\ & + (1-p^{\alpha-}) \frac{V^{\alpha-}(i, t+1, n+1)}{1+r} \end{aligned} \right] \quad (18)$$

$$V^{\alpha+}(s, t, n) = \max_i \left[\begin{aligned} & x^{\alpha+}(s, t, n) - fc - c_{s,i} + p^{\alpha+} \frac{V^{\alpha+}(i, t+1, n)}{1+r} + \\ & + (1-p^{\alpha+}) \frac{V^{\alpha+}(i, t+1, n+1)}{1+r} \end{aligned} \right] \quad (19)$$

2. Application

In the paper the simple application is assumed. We assume the project characterized by four possible states with corresponding annual variable incomes stated as a triangular fuzzy numbers: (i) normal state (95,100,105), (ii) the decrease of production (70,75,80), (iii) the increase of production (120,125,130) and the closure of the project with crisp number 0. The volatility of variable income (σ) is also stated as fuzzy number (0.09,0.12,0.15). Switch costs between the states are as shown in Tab. 1. The fixed costs are assumed to be 75 p.a. for first three states and 0 after the closure. We assume the rate of return r to

¹ These equations should hold: $u^{\alpha-} \cdot d^{\alpha-} = 1$, $u^{\alpha+} \cdot d^{\alpha+} = 1$, $p^{\alpha-} \cdot u^{\alpha-} + (1-p^{\alpha-}) \cdot d^{\alpha-} = 1+r$, $p^{\alpha+} \cdot u^{\alpha+} + (1-p^{\alpha+}) \cdot d^{\alpha+} = 1+r$.

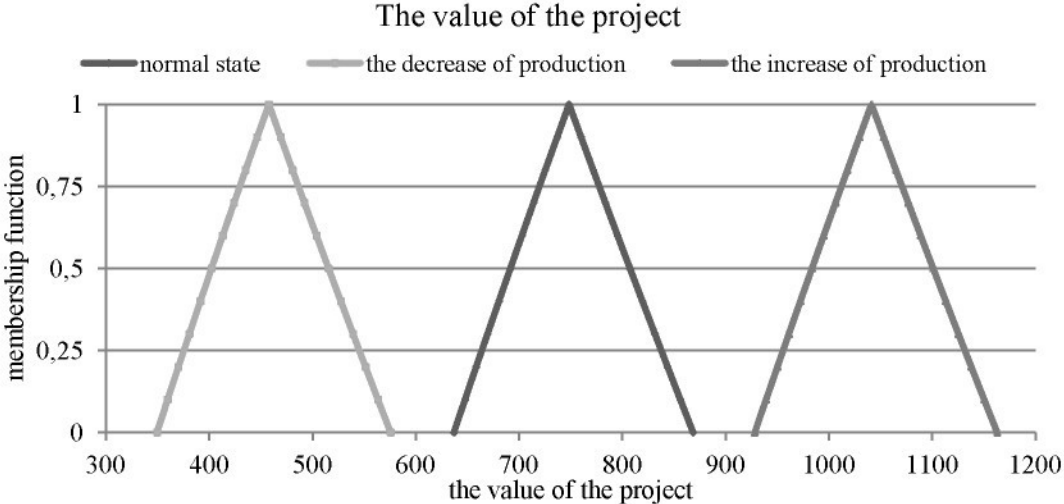
be 7 % p.a. and the length of the first phase T to be 10 years. At the second valuation phase the constant free cash-flows are assumed. The first phase was divided into 512 periods and the project was appraised by means of the methodology described in section 1.2.

Tab. 1 Switch costs

State	Normal	Decrease	Increase	Closure
Normal	0	-225	300	-450
Decrease	300	0	600	-225
Increase	-225	-450	0	-675
Closure	∞	∞	∞	0

Source: own calculation

Fig. 1 The project fuzzy valuation for particular initial states



Source: own calculation

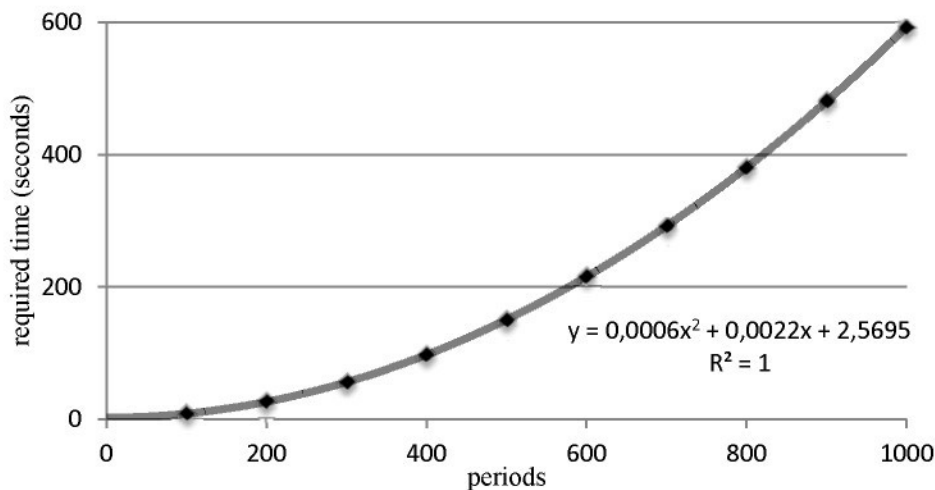
The resulting fuzzy numbers are (due to the multiplication operations) not ideal triangular shapes, but are very close to triangular numbers (see Fig. 1). We can conclude that the resulting valuation in terms of triangular fuzzy numbers are: (i) in normal state (636.9, 748.4, 868.6), (ii) the decrease of the production (349.3, 457.6, 575.5), (iii) the increase of the production (928.3, 1041.3, 1163). It can be seen that differences of valuation between particular states are approximately equal to the switch costs between these states. This is predictable, because the state of the project can be switched in the time of valuation. If we compute with the precise input parameters we will get the appraisal as a single number, (i.e. 748.4, 457.6 and 1041.3 respectively for particular states). But while we are uncertain about the precise value of input parameters also the results are not precise. We know that the project appraisal is in computed intervals. These intervals are stated as the α -cuts (the more uncertain we are, the lower α). For 0-cut the intervals are (636.9, 868.6), (349.3, 575.5) and (928.3, 1163) for normal state, increase and decrease of production respectively. We can see, that the uncertainty in value of the variable cash-flow ± 5 and of the volatility ± 0.03 cause the uncertainty of the appraisal approximately ± 115 . The same interpretation can be done for different α -cuts. See the Tab. 2.

Tab. 2 Imprecision of valuation in dependence on input data imprecision.

α -cut	$\pm x_{0,s}$	$\pm \sigma$	$\pm V(1,0,0)$	$\pm V(2,0,0)$	$\pm V(3,0,0)$
0	5.0	0.030	115.8	113.1	117.4
0.1	4.5	0.027	104.3	101.9	105.6
0.2	4.0	0.024	92.7	90.6	93.9
0.3	3.5	0.021	81.1	79.3	82.1
0.4	3.0	0.018	69.5	68.0	70.4
0.5	2.5	0.015	57.9	56.6	58.6
0.6	2.0	0.012	46.3	45.3	46.9
0.7	1.5	0.009	34.8	34.0	35.2
0.8	1.0	0.006	23.2	22.7	23.5
0.9	0.5	0.003	11.6	11.3	11.7
1	0.0	0.000	0.0	0.0	0.0

Source: own calculation

The valuation algorithm was run on the PC with Intel Core i5 CPU (only one core was utilized) and RAM 8 GB (DDR3, 1333 MHz) for different quantity of periods in the first phase. The times needed to appraise the project are depicted in Fig. 2.

Fig. 2 Time complexity for fuzzy-stochastic project appraisal algorithm

Source: own calculation

As can be seen the time complexity of the algorithm is clearly $O(n^2)$, i.e. quadratic time complexity. The valuation in the case of 1,000 periods (i.e. the period length τ equal to approximately 3.7 days) takes 10 minutes of the processor time. If we decrease period length τ to the one day (i.e. 3,653 periods) the time needed for computations will increase to 2 hours and 14 minutes. In order to come even closer to the continuous valuation, if we decrease the period length τ to the one hour interval (i.e. 87,660 periods), the time needed for computations will increase to 53 days. This is unreasonably huge amount of time needed for calculations. Thus, it can be concluded that this discrete approximation is very time consuming. Some parallelization of calculations should be introduced.

Conclusion

Appraisal of the project as a real option is flexible and useful way encompassing both the risk and the flexibility simultaneously. Both the continuous version and discrete version (i.e. the binomial and trinomial trees) are useful tool for real option appraisal. In this paper we focused on the binomial tree model.

Since it is difficult to state the input parameters precisely, we assumed that the part of the projected free cash-flows and volatility are stated imprecisely as fuzzy numbers. The project appraisal is then also stated as a fuzzy number. Under this approach it can be studied how the initial uncertainty about the input parameters influences (the uncertainty of) the appraisal.

Simple example of the project with three different states and also the closure possibility was provided. The project was appraised as a real option under fuzzy input parameters. For different α -cuts (the different levels of uncertainty of input parameters) the intervals of appraisal was provided. It was shown that the initial impreciseness of input parameters is reflected in the imprecision of the appraisal.

It was found that the time complexity of the appraisal algorithm is quadratic. This means that with the increase of periods quantity (i.e. decrease of period length) the time needed to appraise the project increases with the second power. It was found out that time demands for appraisal with the standard PC is still reasonable for 3,653 periods (2 hours and 14 minutes). By utilizing more periods the computation time demands become unreasonable (for 87,660 periods the required computation time is approximately 54 days).

The proposed appraisal algorithm was not programmed for parallel computations. Thus, further research should be made to consider possibilities of the parallelization. In accordance with the presence of multi-core processors the parallelization should decrease the time needed for computations (i.e. the possibility to increase the quantity of periods for the same computation time).

Acknowledgements

This paper has been elaborated in the framework of the project Opportunity for young researchers, reg. no. CZ.1.07/2.3.00/30.0016, supported by Operational Programme Education for Competitiveness and co-financed by the European Social Fund and the state budget of the Czech Republic (the first author) and the project IT4Innovations Centre of Excellence, reg. no. CZ.1.05/1.1.00/02.0070, supported by Operational Programme Research and Development for Innovations funded by Structural Funds of the European Union and state budget of the Czech Republic (the second author). The support by a SGS research project of VŠB-Technical University of Ostrava under registration number SP2013/173 is also acknowledged.

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