SIGNED AND MINUS DOMINATION IN BIPARTITE GRAPHS

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Abstract. The paper studies the signed domination number and the minus domination number of the complete bipartite graph $K_{p,q}$.

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Here we shall study two numerical invariants of graphs concerning domination, namely the signed domination number and the minus domination number [1].

If f is a function which maps the vertex set V of a graph G into some set of numbers and $S \subseteq V$, then $f(S) = \sum_{x \in S} f(x)$.

Let $f\colon V \to \{-1, 1\}$. If for the closed neighbourhood $N[V]$ of any vertex $v \in V$ we have $f(N[v]) \geqslant 1$, then f is called a signed dominating function (SDF) of G. The value $f(V)$ is called the weight $w(f)$ of f. The minimum of $w(f)$ taken over all SDF's is called the signed domination number $\sigma_{\mathrm{sg}}(G)$ of G.

If in this definition we replace the set $\{-1, 1\}$ by $\{-1, 0, 1\}$ we obtain the definition of the minus dominating function (MDF) and of the minus domination number $\sigma^-(G)$ of G.

We shall study $\sigma_{\mathrm{sg}}(K_{p,q})$ and $\sigma^-(K_{p,q})$ for the complete bipartite graph $K_{p,q}$. We suppose always that $q \leqslant p$.

We start with the signed domination number. If a SDF f on $K_{p,q}$ is given, we use the following notation:

The bipartition classes of $K_{p,q}$ are P, Q with $|P| = p$, $|Q| = q$. We define $V^+ = \{v \in V\colon f(v) = 1\}$, $V^- = \{v \in V\colon f(v) = -1\}$. Further $P^+ = V^+ \cap P$,

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$P^- = V^- \cap P$, $Q^+ = V^+ \cap Q$, $Q^- = V^- \cap Q$ and $p^+ = |P^+|$, $p^- = |P^-|$, $q^+ = |Q^+|$, $q^- = |Q^-|$. Therefore $w(f) = p^+ + q^+ - p^- - q^-$.

Now we express a theorem.

Theorem 1. *Let $K_{p,q}$ be a complete bipartite graph with the bipartition classes P, Q such that $|P| = p$, $|Q| = q$, $q \leqslant p$. Let $\sigma_{\mathrm{sg}}(K_{p,q})$ be the signed domination number of $K_{p,q}$. Then*

(i) *for $q = 1$ there is $\sigma_{\mathrm{sg}}(K_{p,q}) = p + 1$;*

(ii) *for $2 \leqslant q \leqslant 3$ there is $\sigma_{\mathrm{sg}}(K_{p,q}) = q$ for p even and $\sigma_{\mathrm{sg}}(K_{p,q}) = q + 1$ for p odd;*

(iii) *for $q \geqslant 4$ there is $\sigma_{\mathrm{sg}}(K_{p,q}) = 4$ for both p and q even, $\sigma_{\mathrm{sg}}(K_{p,q}) = 6$ at both p, q odd and $\sigma_{\mathrm{sg}}(K_{p,q}) = 5$ for one of the numbers p, q even and the other odd.*

P r o o f. First we prove (i). Let $q = 1$. Then $K_{p,q}$ is either K_2, or a star with p edges. For the first case the assertion is evident. Thus let $K_{p,q}$ be a star. Then $Q = \{c\}$, where c is the central vertex and P is the set of vertices of degree 1. Let $x \in P$. Then $N[x] = \{x, c\}$ and $f(N[x]) = f(x) + f(c) \geqslant 2$ for any SDF f. This implies $f(x) = f(c) = 1$. As x was chosen arbitrarily, $K_{p,q}$ has the unique SDF f which has the value 1 in all vertices. Thus $w(f) = p + 1$ and also $\sigma_{\mathrm{sg}}(K_{p,q}) = p + 1$.

The continuation of the proof will consist from a series of claims.

Claim 1. *Let $Q^- = \emptyset$. Then if f is a* SDF, *then $w(f) \geqslant q$ for p even and $w(f) \geqslant q + 1$ for p odd.*

P r o o f. Let f be a SDF and $Q^- = \emptyset$. Then $Q = Q^+$ and $f(Q) = q$. Let $x \in Q$. Then $N[x] = \{x\} \cup P$ and $f(N[x]) = f(x) + f(P) = 1 + f(P)$. The inequality $f(N[x]) \geqslant 1$ holds only if $f(P) \geqslant 0$. We have $f(P) = p^+ - p^-$, $p = p^+ + p^-$ and this implies $f(P) = 2p^+ - p$. If $f(P) \geqslant 0$ and p is even, then $p^+ \geqslant \frac{1}{2}p$, $p^- \leqslant \frac{1}{2}p$, $f(P) \geqslant 0$. If p is odd, then $p^+ \geqslant \frac{1}{2}(p+1)$, $p^- \leqslant \frac{1}{2}(p-1)$ and $f(P) \geqslant 1$. This implies the assertion. □

Claim 2. *Let $P^- = \emptyset$. Then if f is a* SDF, *then $w(f) \geqslant p$ for q even and $w(f) \geqslant p + 1$ for q odd.*

P r o o f. The proof of this claim is analogous to that of Claim 1. Note that $q \leqslant p$ and thus such a lower bound is greater than of equal to the bound from Claim 1. □

Claim 3. *Let $Q \neq \emptyset$. Then $f(P) \geqslant 2$ for p even and $f(P) \geqslant 3$ for p odd.*

Proof. Let $x \in Q^-$. Then $f(N[x]) = f(P) - f(x) = f(P) - 1$. Further considerations are analogous to those from the proof of Claim 1. We obtain here $2p^+ - p \geqslant 2$ and $p^+ \geqslant \frac{1}{2}p + 1$, $p^- \leqslant \frac{1}{2}p - 1$ for p even and $p^+ \geqslant \frac{1}{2}(p+3)$, $p^- \leqslant \frac{1}{2}(p-3)$ for p odd. In the case of p even we have $f(P) = p^+ - p^- \geqslant 2$, in the case of p odd we have $f(P) \geqslant 3$. □

Claim 4. *Let $P \neq \emptyset$. Then $f(Q) \geqslant 2$ for q even and $f(Q) \geqslant 3$ for q odd.*

Proof. The proof of this claim is quite analogous to that of Claim 3. □

Claim 5. *If $P^- \neq \emptyset$ and $Q \neq \emptyset$, then for every SDF f we have $w(f) \geqslant 4$ for both p, q even, $w(f) \geqslant 6$ for both p, q odd and $w(f) \geqslant 5$ for one of the numbers p, q even and the other odd.*

Proof. This follows from Claim 3 and Claim 4, noting that $w(f) = f(P) + f(Q)$. □

Conclusion of the proof of Theorem 1. For $q = 1$ the proof is ready. For $q \geqslant 6$ evidently the lower bound for $w(f)$ from Claim 5 is less than that from Claim 1 and Claim 2. Evidently also for $2 \leqslant q \leqslant 3$ the converse is true. By considering particular cases we see that for $4 \leqslant q \leqslant 5$ both bounds coincide. Therefore it remains to construct a SDF f for which the equality occurs. For $2 \leqslant q \leqslant 3$ we put $f(x) = 1$ for each $x \in Q$ and for $\frac{1}{2}p$ vertices of P for p even or $\frac{1}{2}(p+1)$ vertices x of P for p odd. For $q \geqslant 4$ we assign the value 1 to $\frac{1}{2}p + 1$ vertices of P for p even or $\frac{1}{2}(p+3)$ vertices of P for p odd and analogously to $\frac{1}{2}q + 1$ vertices of Q for q even of $\frac{1}{2}(q+3)$ vertices of Q for q odd. This implies the assertion. □

In the sequel we shall study the minus domination number. We still use the notation F, Q, p, q and a MDF will be denoted by g.

Theorem 2. *Let $K_{p,q}$ be a complete bipartite graph with the bipartition classes P, Q such that $|P| = p$, $|Q| = q$, $q \leqslant p$. Let $\sigma^-(K_{p,q})$ be the minus domination number of $K_{p,q}$. Then*
(i) *for $q = 1$ there is $\sigma^-(K_{p,q}) = 1$;*
(ii) *for $2 \leqslant q \leqslant p$ there is $\sigma^-(K_{p,q}) = 2$.*

Proof. First we prove (i). Let $q = 1$. Then $K_{p,q}$ is either K_2, or a star with p edges. For the first case the assertion is evident. Thus let $K_{p,q}$ be a star. Then $Q = \{c\}$, where c is the central vertex and P is the set of vertices of degree 1. Let $x \in P$, and let g be a MDF of $K_{p,q}$. Then $N[x] = \{x, c\}$ and $g(N[x]) = g(x) + g(c)$.

This is possible only if one of the vertices x, c has the value 1 and the other 0 or 1. Therefore $w(g) \geqslant 1$. We construct MDF g with $w(g) = 1$. It suffices to put $f(c) = 1$ and $f(x) = 0$ for each $x \in P$. This implies the assertion.

Now we prove (ii). Let $2 \leqslant q \leqslant p$. Suppose that there exists a MDF g with $w(g) \leqslant 1$. We have $w(g) = g(P) + g(Q)$; this implies that at least one of these values, say $g(Q) \leqslant 0$. Let $x \in P$. We have $g(N[x]) = g(x) + g(Q) \leqslant 1 + 0 = 1$. This is possible only if $g(x) = 1$ and $g(Q) = 0$. As x was chosen arbitrarily, we have $g(x) = 1$ for each $x \in P$ and $g(P) = p$. Then $w(g) = p \geqslant 2$, which is a contradiction. Therefore $w(g) \geqslant 2$ for each MDF g. A MDF g with $w(g) = 2$ can be obtained by choosing $u \in P$, $v \in Q$ and putting $g(u) = g(v) = 1$, $f(x) = 0$ for any $x \in V - \{u, v\}$. This implies the assertion. $\square$

References

[1] *W. T. Haynes, S. T. Hedetniemi, P. J. Slater*: Fundamentals of Domination in Graphs. Marcel Dekker, New York-Basel-Hong Kong, 1998. Zbl 0890.05002

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