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Topics in Mathematical Fluid Mechanics and Shape Optimization

Habilitation Thesis

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Abstract

This habilitation thesis is based on the author's contributions to the mathematical theory of incompressible fluids and to the shape optimization in fluid mechanics. In Chapter 2, an overview of basic equations, theory of weak solutions and finite-element approximation for incompressible fluids is given. Chapter 3 is devoted to the formulation of shape optimization problems, to the questions of existence of solutions, approximation and differentiability. Finally, in Chapter 4 some results obtained by the author are mentioned, namely on the theory of non-Newtonian piezoviscous fluids, applied shape optimization and sensitivity analysis.

Contents

Chapter 1

Introduction

The important role of fluids in human life has led many scientists and engineers to develop theories, experimental and computational methods that would lead to better understanding of their behaviour. Despite a large effort of even the most distinguished experts, some of fundamental questions in the mathematical description of fluids (e.g. existence of smooth solutions, characterization of turbulence) as well as problems in numerical solution (large Reynolds number and turbulent flows) are still not completely resolved even for the "simple" model of water. The main difficulty is related to the inertial effects due to the fact that fluids undergo large deformations. In addition, the rheology of fluids can be more complicated than that of water, where the stress-strain relation is linear. These so-called non-Newtonian fluids are materials whose viscosity may depend on quantities like temperature, shear rate, pressure, or on the history of deformation. Polymeric melts, oils, asphalt, glaciers, blood or toothpaste are a few examples of materials that can be described as non-Newtonian fluids, depending also on the chosen time scale.

This thesis is dealing with mathematical modeling of fluids in general, having in mind particular applications. Mathematical modeling is one approach to study the properties of fluids can provide both qualitative and quantitative results in details that are often not achievable by experiments, provided that the modelling error is acceptable. An important task is to keep under control or at least estimate the errors due to discretization and finite-precision arithmetics. In this respect, it is an interdisciplinary discipline whose success often relies on a strong cooperation of experts from several fields.

The mathematical challenges in studying non-Newtonian fluids are mostly related to two nonlinearities: one is due to the dependence of stress tensor on other quantities and the other one is the convective term. We try to answer the questions of well-posedness of models as well as of their approximations and the error estimation for so-called power-law and piezoviscous fluids, i.e. fluids whose viscosity depends on the shear rate and/or on the pressure. We also address the question of proper boundary conditions, which are in a sense constitutive relations inherent to the fluid as well as the solid boundary.

The second main topic of this thesis will be the shape optimization, which is related to the question of how the flow of a fluid depends on the geometry of the flow domain. Shape optimization has applications e.g. in designing particular devices, identification and inverse problems as well as in calibration of models. Here the role of modelling and simulation is even more stressed as the method of trial and error for the construction of a new design is often too expensive and inefficient. We shall address the well-posedness, approximate schemes and their convergence for the shape optimization problems with models of nonlinear fluid mechanics as the state problem.

One of the author's goals is to help reducing the gaps between mathematical theory and applied sciences. Despite that scientists in each specific field have to appropriate extensive specialized knowledge and techniques, it is of significant interest to keep as much overview of related areas as possible. The present thesis is an attempt to demonstrate that for a growing set of models in fluid mechanics a rigorous theory is possible which can then put the achievements in numerical computations to a more solid ground.

The structure of the thesis is as follows. In Chapter 2 we recall the basic equations of fluid mechanics with attention to some non-Newtonian and nonlinear models. Then, on the example of the Navier-Stokes equations we explain the main steps in the proof of existence and uniqueness of weak solutions, the finite-element discretization and convergence analysis. Chapter 3 is devoted to the formulation of shape optimization problems, their mathematical and numerical analysis and solution. Again, the theoretical considerations are demonstrated on the Navier-Stokes equations. Finally, in Chapter 4 we present reprints of several author's publications from the field of mathematical fluid mechanics and shape optimization and comment on their contribution to the scientific community.

Chapter 2

Mathematical and Numerical Analysis in Fluid Mechanics

In this chapter we recall basic concepts of continuum mechanics and state the governing equations as well as initial and boundary conditions for several important models of fluids. We focus on the incompressible case and do not consider thermal effects. The notion of a weak solution is introduced for the classical Navier-Stokes equations and the main steps towards the existence and the uniqueness of weak solutions are discussed. We also mention differences that have to be tackled when considering some non-Newtonian models and non-trivial boundary conditions. In the last part of the chapter we describe the finite-element approximation of the Navier-Stokes equations and results on the existence of discrete solutions as well as their convergence to the weak solution.

2.1 Overview of models in fluid mechanics

We shall start by introducing the concept of a deformable body, Lagrangian and Eulerian description and balance laws for continuum. We discuss constitutive relations for Newtonian and non-Newtonian fluids, physically relevant initial and boundary conditions. For more details on derivation and discussion of the models we refer to [33].

2.1.1 Basic equations of continuum mechanics

Let $\mathcal{B}_R \subset \mathbb{R}^3$ denote the reference configuration in three-dimensional Euclidean space of a body under consideration. The motion of the body can be represented by a sufficiently smooth mapping $\chi : \mathbb{R} \times \mathcal{B}_R \to \mathbb{R}^3$ which for given time t maps $\mathcal{B}_R$ isomorphically onto a new configuration $\mathcal{B}_t := \chi(t, \mathcal{B}_R)$. Then, one can introduce the inverse mapping χ^{-1} so that any point $X \in \mathcal{B}_R$ can be uniquely identified with $x := \chi(t, X) \in \mathcal{B}_t$ and conversely, $X = \chi^{-1}(t, x)$.

A scalar function φ associated with the abstract body will be denoted by the same symbol in the Lagrangean as well as the Eulerian description, i.e. $\varphi(t, X) \equiv \varphi(t, x)$. The Lagrangean and Eulerian time derivative of φ is defined as follows:

$$\dot{\varphi} := \frac{\partial \varphi}{\partial t}(t, X) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, \chi(t, X)), \quad \varphi_{,t} := \frac{\partial \varphi}{\partial t}(t, x). \tag{2.1}$$

We also define the deformation gradient $\mathbb{F}$, the velocity $\boldsymbol{v}$, the velocity gradient $\mathbb{L}$ and its symmetric part $\mathbb{D}$:

$$\begin{aligned} \mathbb{F} = \nabla_X \chi := \frac{\partial \chi}{\partial X}, \quad \boldsymbol{v} := \frac{\partial \chi}{\partial t}, \\ \mathbb{L} := \nabla_x \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial x}, \quad \mathbb{D} = \mathbb{D}\boldsymbol{v} := \frac{1}{2}\left(\mathbb{L} + \mathbb{L}^\top\right). \end{aligned} \tag{2.2}$$

By chain rule of differentiation one gets the relation:

$$\dot{\varphi} = \varphi_{,t} + \nabla_x \varphi \cdot \boldsymbol{v}. \tag{2.3}$$

For the derivation of the balance laws we shall use the notion of a control volume, namely an open set $\Omega_R \subset \mathcal{B}_R$, for which we define $\Omega_t := \chi(t, \Omega_R)$.

Balance of mass. Incompressibility, homogeneity. Let ϱ denote the density field. The balance of mass in the Eulerian form reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \varrho(t, x)\, dx = 0 \quad \text{for all control volumes } \Omega_R \subset \mathcal{B}_R. \tag{2.4}$$

By Reynolds transport theorem and localization one obtains from (2.4) the *continuity equation*:

$$\varrho_{,t} + (\nabla_x \varrho) \cdot \boldsymbol{v} + \varrho \operatorname{div} \boldsymbol{v} = \varrho_{,t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0. \tag{2.5}$$

If the abstract body is incompressible then

$$\int_{\Omega_R} dX = \int_{\Omega_t} dx \quad \text{for all control volumes } \Omega_R \subset \mathcal{B}_R, \tag{2.6}$$

which implies

$$\det \mathbb{F}(t, X) = 1 \text{ in } \mathcal{B}_R, \tag{2.7}$$

and since

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\mathbb{F} = \operatorname{div}\boldsymbol{v}\det\mathbb{F}, \tag{2.8}$$

the incompressibility is equivalent to the constraint

$$\operatorname{div}\boldsymbol{v} = \operatorname{tr}\mathbb{D} = 0. \tag{2.9}$$

Hence, the balance of mass for an incompressible material reads:

$$\varrho_{,t} + (\nabla_x\varrho)\cdot\boldsymbol{v} = \dot{\varrho} = 0. \tag{2.10}$$

The above identity implies that the density is a constant function of time for each material point X. However, it permits variations of the density in space, in which case we speak about an *inhomogeneous incompressible material.* A typical example of such materials are granular fluids. For a homogeneous incompressible fluid (i.e. $\varrho(t,X) = const.$), the balance of mass follows directly from (2.9).

Balance of linear and angular momentum. The balance of linear momentum is an analogy of the second Newton's law in classical mechanics. It states that for each control volume $\Omega_R \subset \mathcal{B}_R$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\varrho\boldsymbol{v}\,dx = \int_{\Omega_t}\varrho\boldsymbol{f}\,dx + \int_{\partial\Omega_t}\mathbb{T}^\top\boldsymbol{n}\,dS, \tag{2.11}$$

where $\mathbb{T}$ represents the Cauchy stress tensor, $\boldsymbol{n}$ denotes the unit outward normal vector and $\boldsymbol{f}$ is the density of body forces.

The balance of angular momentum requires that the Cauchy stress is symmetric, i.e.

$$\mathbb{T} = \mathbb{T}^\top. \tag{2.12}$$

The localized form of the balance of linear and angular momentum then reads:

$$\varrho\dot{\boldsymbol{v}} = \varrho\boldsymbol{f} + \operatorname{div}\mathbb{T}. \tag{2.13}$$

Multiplying (2.5) by $\boldsymbol{v}$ and adding to (2.13) we obtain

$$(\varrho\boldsymbol{v})_{,t} + \operatorname{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) = \varrho\boldsymbol{f} + \operatorname{div}\mathbb{T}. \tag{2.14}$$

The balance laws expressed using the above considerations are summarized in the following systems of partial differential equations:

- *compressible fluids:*

$$\varrho_{,t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0, \quad (\varrho\boldsymbol{v})_{,t} + \operatorname{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) = \varrho\boldsymbol{f} + \operatorname{div}\mathbb{T}; \tag{2.15}$$

- *incompressible inhomogeneous fluids:*

$$\operatorname{div} \boldsymbol{v} = 0, \quad \dot{\varrho} = 0, \quad (\varrho \boldsymbol{v})_{,t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \varrho \boldsymbol{f} + \operatorname{div} \mathbb{T}; \tag{2.16}$$

- *incompressible homogeneous fluids:*

$$\operatorname{div} \boldsymbol{v} = 0, \quad \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = \boldsymbol{f} + \frac{1}{\varrho} \operatorname{div} \mathbb{T}, \quad \varrho = const. \tag{2.17}$$

These systems, accomplished by an appropriate relation for $\mathbb{T}$, are usually called the equations of motion.

2.1.2 Constitutive relations for Newtonian fluids and some of their generalizations

The systems governing the motion of fluids have to be closed by a constitutive relation for the Cauchy stress $\mathbb{T}$. If $\mathbb{T}$ depends linearly on the velocity gradient $\mathbb{L}$ then we speak about Newtonian fluids, all other relations are denoted non-Newtonian.

In the standard approach employed in classical textbooks of continuum mechanics (e.g. [50]), one a priori assumes the stress tensor as a function of certain quantities, in particular

$$\mathbb{T} = \mathcal{F}(\varrho, \mathbb{L}). \tag{2.18}$$

The frame indifference then yields the general form of the relation — in the above case one gets $\mathcal{F}(\varrho, \mathbb{L}) = \alpha_1 \mathbb{I} + \alpha_2 \mathbb{D} + \alpha_3 \mathbb{D}^2$, α_i being functions of the density and the invariants of $\mathbb{D}$, $i = 1, 2, 3$. Requiring in addition that $\mathbb{T}$ is linear with respect to $\mathbb{D}$, we obtain the constitutive law of compressible Newtonian fluid:

$$\mathbb{T} = -p(\varrho)\mathbb{I} + \lambda(\varrho)(\operatorname{tr} \mathbb{D})\mathbb{I} + 2\mu(\varrho)\mathbb{D}. \tag{2.19}$$

Here p is the pressure (related to ϱ by a state equation), λ and μ are the bulk and shear moduli of viscosity. To obey the second law of thermodynamics, $\mu(\varrho)$ and $\lambda(\varrho) + \frac{2}{3}\mu(\varrho)$ have to be non-negative. Similarly, for an incompressible homogeneous Newtonian fluid with the assumption $\mathbb{T} = \mathcal{G}(\mathbb{L})$ one obtains:

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}. \tag{2.20}$$

Here the pressure p plays a different role than in (2.19), namely it is the Lagrange multiplier (or the reaction force) to the constraint of incompressibility of the fluid.

It has to be noticed that by the above approach one cannot derive models of incompressible fluids whose viscosity parameter μ depends on the pressure. For this reason we shall mention an alternative way of deriving the constitutive laws. It is especially well-suited for incompressible non-Newtonian fluids (see [33] for more details). Instead of assuming certain form of the tensor $\mathbb{T}$ it may be more convenient to consider a specific form of the rate of dissipation

$$\xi := \mathbb{T} : \mathbb{D}, \tag{2.21}$$

which is a scalar quantity. The second law of thermodynamics requires that $\xi \geq 0$. For a particular choice

$$\xi = 2\nu(p, \varrho, |\mathbb{D}|^2)|\mathbb{D}|^2, \tag{2.22}$$

where $p := -\frac{1}{3}\operatorname{tr}\mathbb{T}$, $|\mathbb{D}|^2 = \mathbb{D} : \mathbb{D}$ and $\nu(p, \varrho, |\mathbb{D}|^2) > 0$, this requirement is automatically satisfied. Maximizing ξ with respect to $\mathbb{D}$ with the constraints (2.21) and (2.9) leads to

$$\mathbb{T} = -p\mathbb{I} + 2\nu(p, \varrho, |\mathbb{D}|^2)\mathbb{D}. \tag{2.23}$$

This represents a class of non-Newtonian models whose viscosity may vary with pressure, density and shear rate. Using this approach one can derive even more general models (both compressible and incompressible) with implicit constitutive relations between $\mathbb{T}$ and $\mathbb{D}$ (see e.g. [35], [34]). Among the mostly used constitutive relations for fluids are the following ones:

- *compressible Newtonian fluids:*

$$\mathbb{T} = -p(\varrho)\mathbb{I} + \lambda(\varrho)(\operatorname{tr}\mathbb{D})\mathbb{I} + 2\mu(\varrho)\mathbb{D}; \tag{2.24}$$

- *incompressible homogeneous Newtonian fluids:*

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}; \tag{2.25}$$

- *incompressible homogeneous fluids with shear-rate-dependent viscosity:*

$$\mathbb{T} = -p\mathbb{I} + 2\nu(|\mathbb{D}|^2)\mathbb{D}; \tag{2.26}$$

- *incompressible homogeneous fluids with pressure- and shear-rate-dependent viscosity:*

$$\mathbb{T} = -p\mathbb{I} + 2\nu(p, |\mathbb{D}|^2)\mathbb{D}. \tag{2.27}$$

The equations of motion (2.9), (2.14) accomplished by (2.25) are usually denoted the Navier-Stokes equations for incompressible fluids.

2.1.3 Initial and boundary conditions

For a complete description of the motion of a fluid one has to know its initial state. For this reason one has to prescribe the initial conditions:

$$\varrho(0,\cdot) = \varrho_0, \quad \boldsymbol{v}(0,\cdot) = \boldsymbol{v}_0, \tag{2.28}$$

where ϱ_0, $\boldsymbol{v}_0$ is the density and the velocity field, respectively, at initial time. Of course $(2.28)_1$ makes sense only for compressible or inhomogeneous fluids as the density of an incompressible homogeneous fluid is constant.

When the domain occupied by the fluid has a (internal or external) boundary, then it is necessary to specify boundary conditions. Their proper choice is not always straightforward, they can be viewed as another constitutive properties of the fluid and the surrounding environment (see [21] for a brief overview of historical developments in this direction).

The most common condition on a solid impermeable wall is

$$\boldsymbol{v} = \boldsymbol{v}_{wall}, \tag{2.29}$$

i.e. the velocity of the fluid equals the velocity of the wall (usually $\boldsymbol{v}_{wall} = \boldsymbol{0}$). This is called the *no-slip* or *Dirichlet* boundary condition. It was however pointed out already by Stokes [46] that no-slip is only an approximation of the real case. In fact the molecules of the fluid do not adhere to the surface and the sliding effects can play an important role especially when the mean free path of the molecules is comparable to the characteristic length of the domain (see [30]).

In situations when the motion of the fluid along the surface is not negligible (e.g. in micro- or nanofluidics, flows along hydrophobic surfaces, to name a few examples), one has to consider some kind of *slip* condition. One of the simplest and mostly used is the condition derived by Navier [36]:

$$(\mathbb{T}\boldsymbol{n})_\tau = -\alpha \boldsymbol{v}_\tau, \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0, \tag{2.30}$$

where $\alpha > 0$ is the slip friction coefficient, $\boldsymbol{n}$ stands for the unit outward normal vector to the boundary and $\boldsymbol{v}_\tau := \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ denotes the tangential part of $\boldsymbol{v}$. A combination of Navier's and no-slip condition is the *threshold* condition:

$$\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = 0, \\ &|(\mathbb{T}\boldsymbol{n})_\tau| \le g, \quad g\boldsymbol{v}_\tau = -|\boldsymbol{v}_\tau|(\mathbb{T}\boldsymbol{n})_\tau. \end{aligned} \tag{2.31}$$

The slip bound $g > 0$ indicates the transition from slip to no-slip regime. It may either be constant or a function of $|\boldsymbol{v}_\tau|$ and $\mathbb{T}\boldsymbol{n}\cdot\boldsymbol{n}$.

For practical purposes the physical domain is often truncated to a region where the fluid motion is of major interest. Then some parts of the boundary are artificial and can represent inflow or outflow zone. The choice of inflow and outflow boundary conditions for general compressible models is a delicate issue which will be omitted in this work. In the remainder of this section we shall address some popular choices for incompressible models. On inflow one typically prescribes the velocity:

$$\boldsymbol{v} = \boldsymbol{v}_{in} \tag{2.32}$$

(a typical form of $\boldsymbol{v}_{in}$ in case of Newtonian fluids has a parabolic profile). The velocity and pressure inside the domain is usually not too sensitive to small variations of the inflow velocity.

A more delicate issue is the selection of an outflow condition (see e.g. [28]). Here it is usually not possible to guess the velocity because an improper choice could result in completely wrong solution inside the domain. The most common is the so-called *do-nothing* condition

$$\mathbb{T}\boldsymbol{n} = \boldsymbol{0}, \tag{2.33}$$

yielding satisfactory results e.g. in channel flows with free outflow. Some other possibilities such as conditions involving the *Bernoulli pressure*:

$$(p + \frac{1}{2}|\boldsymbol{v}|^2)\boldsymbol{n} - \mathbb{S}\boldsymbol{n} = \boldsymbol{h} \tag{2.34}$$

or *non-reflecting* conditions:

$$-\mathbb{T}\boldsymbol{n} = \boldsymbol{h} + \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{n})^{-}\boldsymbol{v} \tag{2.35}$$

can be found in the literature. Their validity is often restricted to particular situations; they are in general not justified for universal usage.

2.2 Well-posedness

One of the fundamental questions in mathematical theory of fluid mechanics is the existence and uniqueness of a regular solution (i.e. solution that satisfies the equations of motion, initial and boundary conditions at every point and time). The question is far from being completely answered. Even for the classical Navier-Stokes equations for incompressible fluids it is still open, despite enormous effort of many researchers.

In the theory of partial differential equations, several notions of solution appear. With regard to their regularity, we speak about classical, strong, renormalized, weak or very weak solutions. Roughly speaking, existence is easier to prove for weaker types of solution while uniqueness can be proved easier for stronger ones.

In what follows, we shall mention some aspects of the existence analysis for weak solutions to the incompressible fluids. For this purpose we consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ and introduce the notation of the following function spaces (see e.g. [1, 37, 31] for their precise definitions and more properties):

- the space $\mathcal{C}(K)$ of continuous functions on a closed set K;
- the *Lebesgue space* $L^q(\Omega)$, $q \in [1,\infty]$, of measurable functions which are integrable with q-th power. The norm in $L^q(\Omega)$ is

$$\|f\|_{q,\Omega} := \begin{cases} \left(\int_\Omega |f|^q\right)^{1/q} & \text{if } q \in [1,\infty), \\ \min\{C > 0;\ |f(x)| \le C\ \forall \text{a.e. } x \in \Omega\} & \text{if } q = \infty; \end{cases}$$

- the subspace $L^q_0(\Omega)$ of functions with zero integral mean, i.e.

$$L^q_0(\Omega) := \left\{ f \in L^q(\Omega);\ \int_\Omega f = 0 \right\};$$

- the *Sobolev space* $W^{1,q}(\Omega)$ with the norm

$$\|f\|_{1,q,\Omega} := \left(\|f\|^q_{q,\Omega} + \|\nabla f\|^q_{q,\Omega}\right)^{1/q};$$

- its subspace of functions with vanishing traces

$$W^{1,q}_0(\Omega) := \{ f \in W^{1,q}(\Omega);\ f_{|\partial\Omega} = 0 \}; \tag{2.36}$$

- the space of divergence-free functions with vanishing traces

$$W^{1,q}_{0,\mathrm{div}}(\Omega) := \left\{ \boldsymbol{w} \in W^{1,q}_0(\Omega;\mathbb{R}^d);\ \operatorname{div} \boldsymbol{w} = 0 \right\}; \tag{2.37}$$

Vector-valued analogues of the above spaces will be denoted $L^q(\Omega;\mathbb{R}^d)$, $W^{1,q}(\Omega;\mathbb{R}^d)$ etc.

Let us also mention several auxiliary tools and inequalities related to the function spaces that will be used in what follows. In all of them it is required that the domain Ω has some minimal regularity, namely it is a domain with Lipschitz boundary. It means, roughly speaking, that the boundary can be locally represented as a graph of a continuous function with bounded first-order derivatives (see [37] for a rigorous definition).

- *Hölder's inequality.* For all $f \in L^q(\Omega)$, $g \in L^s(\Omega)$, where $\frac{1}{q} + \frac{1}{s} = 1$, it holds:
$$\int_\Omega fg \le \|f\|_{q,\Omega}\|g\|_{s,\Omega}. \tag{2.38}$$

- *Imbedding of $W^{1,q}(\Omega)$ into $L^s(\Omega)$.* For every
$$s \in \begin{cases} [1, \frac{dq}{d-q}] & \text{if } q < d, \\ [1,\infty) & \text{if } q = d, \\ [1,\infty] & \text{if } q > d, \end{cases}$$
there exists a constant $C_{Is} = C_{Is}(\Omega, q) > 0$ such that
$$\forall \varphi \in W^{1,q}(\Omega): \ \|\varphi\|_{s,\Omega} \le C_{Is}\|\varphi\|_{1,q,\Omega}. \tag{2.39}$$
If in addition $s < \begin{cases} \frac{dq}{d-q} & \text{if } q < d \\ \infty & \text{if } q \ge d \end{cases}$ then the imbedding $W^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$ is compact.

- *Friedrichs' inequality.* There exists a constant $C_F := C_F(\Omega, q) > 0$ such that
$$\forall \varphi \in W_0^{1,q}(\Omega): \ \|\varphi\|_{q,\Omega} \le C_F\|\nabla\varphi\|_{q,\Omega}. \tag{2.40}$$

- *Korn's inequality.* There exists a constant $C_K := C_K(\Omega, q) > 0$ such that
$$\forall \boldsymbol{\varphi} \in W_0^{1,q}(\Omega;\mathbb{R}^d): \ \|\boldsymbol{\varphi}\|_{q,\Omega} \le C_K\|\mathbb{D}\boldsymbol{\varphi}\|_{q,\Omega}. \tag{2.41}$$

- *Solution operator for divergence equation.* For all $q \in (1,\infty)$ there exists a bounded linear mapping $\mathcal{B}_\Omega : L_0^q(\Omega) \to W_0^{1,q}(\Omega;\mathbb{R}^d)$ (the so-called Bogovskiĭ operator) with the following properties:
$$\operatorname{div}(\mathcal{B}_\Omega f) = f \text{ in } \Omega, \quad \|\mathcal{B}_\Omega f\|_{1,q,\Omega} \le C_B\|f\|_{q,\Omega}, \tag{2.42}$$
where $C_B := C_B(\Omega, q) > 0$. We note that the operator $\mathcal{B}_\Omega$ is the same for all q.

We shall simplify the notation of norms by dropping the symbol Ω where it makes no confusion. The symbol $C(\ldots) > 0$ will denote a generic constant depending only on the indicated quantities, whose meaning can differ from line to line.

2.2.1 Model problem: steady Navier-Stokes equations

Let us demonstrate the main steps in the proof of existence and uniqueness of weak solutions on the steady Navier-Stokes equations for incompressible fluids with the no-slip boundary condition:

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } \Omega, \tag{2.43a}$$

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \mu\Delta\boldsymbol{v} + \frac{1}{\varrho}\nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2.43b}$$

$$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{2.43c}$$

For simplicity of notations we shall denote $p := p/\varrho$ to eliminate the density from the system. The term $\mu\Delta\boldsymbol{v}$ is used in (2.43b) since it is identical to $\operatorname{div}(\mu\mathbb{D}\boldsymbol{v})$ when (2.43a) holds. We also note that since there is only the gradient of the pressure in (2.43), the pressure itself can be determined only up to an additive constant. In what follows we mention the key steps of the analysis, complete results may be found in the classical literature, e.g. [20, 19, 48].

A priori estimate. The first step in the existence analysis is the estimate of the energy. Formally, multiplying (2.43b) by $\boldsymbol{v}$ and integrating over Ω we obtain after integration by parts[1]:

$$\mu\|\nabla\boldsymbol{v}\|_2^2 - \underbrace{\mu\int_{\partial\Omega}(\nabla\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{v}}_{\boldsymbol{v}=0} + \underbrace{\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{v}}_{=0} + \underbrace{\int_{\partial\Omega} p(\boldsymbol{v}\cdot\boldsymbol{n})}_{\boldsymbol{v}=0} - \underbrace{\int_\Omega p\operatorname{div}\boldsymbol{v}}_{\operatorname{div}\boldsymbol{v}=0} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}. \tag{2.44}$$

The boundary integrals and the pressure disappear due to (2.43c) and (2.43a), the convective term vanishes since

$$\begin{aligned}\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{v} &= \int_\Omega (\operatorname{div}\boldsymbol{v})|\boldsymbol{v}|^2 + \int_\Omega \boldsymbol{v}\cdot\nabla\frac{|\boldsymbol{v}|^2}{2}\\ &= \int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\frac{|\boldsymbol{v}|^2}{2} - \int_\Omega (\operatorname{div}\boldsymbol{v})\frac{|\boldsymbol{v}|^2}{2} = 0.\end{aligned} \tag{2.45}$$

[1] More precisely, one has to use the Green theorem:

$$\int_\Omega f\frac{\partial g}{\partial x_i} = \int_{\partial\Omega} fgn_i - \int_\Omega \frac{\partial f}{\partial x_i}g.$$

The right hand side of (2.44) is estimated using Hölder's and Friedrichs' inequality:

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \le \|\boldsymbol{f}\|_2 \|\boldsymbol{v}\|_2 \le C_F \|\boldsymbol{f}\|_2 \|\nabla \boldsymbol{v}\|_2. \tag{2.46}$$

Here $C_F = C_F(\Omega, 2) > 0$ is the constant of the Friedrichs inequality (2.40). Combining (2.44) and (2.46) then leads to the estimate of the velocity:

$$\frac{\|\boldsymbol{v}\|_{1,2}}{C_F} \le \|\nabla \boldsymbol{v}\|_2 \le C_F \frac{\|\boldsymbol{f}\|_2}{\mu}. \tag{2.47}$$

In addition, if the equation (2.43b) is multiplied by $\mathcal{B}p$ and integrated over Ω, one obtains an estimate of the pressure:

$$\|p\|_2 \le C(\Omega, \mu, \|\boldsymbol{f}\|_2). \tag{2.48}$$

Weak formulation. The weak formulation of the boundary-value problem (2.43) is derived by formal multiplication of (2.43a) and (2.43b) by smooth test functions ψ and $\boldsymbol{\varphi}$, respectively, integrating over Ω and using the Green theorem. The a priori estimates (2.47) and (2.48) give the information about suitable function spaces for the velocity and the pressure. The weak formulation of (2.43) then reads:

Velocity-pressure formulation

Find a pair of functions $(\boldsymbol{v}, p) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ *such that*

$$\left.\begin{aligned} &\int_\Omega \mu \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} - \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{\varphi} - p \operatorname{div} \boldsymbol{\varphi} + \psi \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \end{aligned}\right\} \tag{2.49}$$

for every $(\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$.

Here and in what follows, $\mathbb{A} : \mathbb{B} := \sum_{i,j=1}^d [\mathbb{A}]_{ij} [\mathbb{B}]_{ij}$ denotes the scalar product of second order tensors and $[\boldsymbol{w} \otimes \boldsymbol{z}]_{ij} := \boldsymbol{w}_i \boldsymbol{z}_j$ is the dyadic product of vectors.

Alternatively, one can restrict to divergence-free test functions $\boldsymbol{\varphi}$, which leads to an alternative definition without pressure:

Velocity formulation

Find $\boldsymbol{v} \in W_{0,\mathrm{div}}^{1,2}(\Omega)$ *such that*

$$\left.\begin{aligned} &\int_\Omega \mu \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} - \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \end{aligned}\right\} \tag{2.50}$$

for all $\boldsymbol{\varphi} \in W_{0,\mathrm{div}}^{1,2}(\Omega)$.

Problems (2.49) and (2.50) are in fact equivalent, since the pressure can always be reconstructed from the velocity field, as a consequence of de Rham's theorem.

Existence by Galerkin's method. In what follows we prove the following proposition:

> *For any $\mu > 0$ and $\boldsymbol{f} \in L^2(\Omega; \mathbb{R}^d)$ there exists a solution to* (2.50).

The Galerkin method can be characterized by the following steps:

- The function space from the weak formulation is approximated by a sequence of nested finite dimensional subspaces, which leads in the case of a steady problem to a system of nonlinear algebraic equations;
- The existence of a solution of the approximate problem is proved using a variant of the Brouwer fixed-point theorem with help of the a priori estimates;
- For increasing dimension of the finite dimensional spaces we obtain a sequence of approximate solutions. Using the a priori estimates and the reflexivity of the Lebesgue and Sobolev spaces, one can pass to a weakly convergent subsequence. To show that the limit of this sequence is a solution to (2.50), one has to pass to the limit in the integral identities. Linear terms are treated with help of the weak convergence, for the nonlinear convective term one has to use strong convergence of solutions (which follows from the Rellich-Kondrachov theorem on the compact imbedding).

In the case of classical Navier-Stokes system, the Galerkin method is usually applied to the problem (2.50) formulated only in the velocity. The reason for avoiding the pressure is the saddle-point structure of the system (2.43) where the pressure plays the role of a Lagrange multiplier to the incompressibility constraint (2.43a). By eliminating the pressure one recovers the elliptic (positive definite) or parabolic structure of the problem. However, in some problems, such as the models with pressure-dependent viscosity, it is natural to introduce the pressure from the very beginning. In that case, positive definiteness is achieved by regularizing the incompressibility in (2.49).

Let $\{\boldsymbol{\varphi}^i\}_{i=1}^{\infty}$ be a basis of $W^{1,2}_{0,\mathrm{div}}(\Omega)$. We denote $V^n := \mathrm{span}\{\boldsymbol{\varphi}^1, ..., \boldsymbol{\varphi}^n\}$ the finite dimensional space spanned by the first n basis functions. For every $n \in \mathbb{N}$ we define the following problem:

Galerkin approximation

$$\left.\begin{array}{l}\textit{Find a function } \boldsymbol{v}^n \in V^n \textit{ in the form } \boldsymbol{v}^n := \sum_{i=1}^n \alpha^i \boldsymbol{\varphi}^i, \textit{ which satisfies}\\[1em] \qquad\displaystyle\int_\Omega \mu \nabla \boldsymbol{v}^n : \nabla \boldsymbol{\varphi}^i - \boldsymbol{v}^n \otimes \boldsymbol{v}^n : \nabla \boldsymbol{\varphi}^i = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}^i,\\[1em] \textit{for every } i \in \{1, ..., n\}.\end{array}\right\} \tag{2.51}$$

The identity (2.51) represents a set of n equations for the coefficients $\boldsymbol{\alpha} := (\alpha^1, ..., \alpha^n) \in \mathbb{R}^n$, which can be equivalently written as:

$$\boldsymbol{P}(\boldsymbol{\alpha}) = \boldsymbol{0}, \tag{2.52}$$

where $\boldsymbol{P} : \mathbb{R}^n \to \mathbb{R}^n$ is a (nonlinear) function defined by

$$P_i(\boldsymbol{\alpha}) := \int_\Omega \mu \nabla \boldsymbol{v}^n : \nabla \boldsymbol{\varphi}^i - \boldsymbol{v}^n \otimes \boldsymbol{v}^n : \nabla \boldsymbol{\varphi}^i - \boldsymbol{f} \cdot \boldsymbol{\varphi}^i, \; i = 1, ..., n. \tag{2.53}$$

If $\boldsymbol{P}$ is continuous and if there exists $R > 0$ such that

$$\forall \boldsymbol{\alpha} \in \mathbb{R}^n, \; |\boldsymbol{\alpha}| = R : \; \boldsymbol{P}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} \geq 0, \tag{2.54}$$

then a variant of the Brouwer fixed point theorem states that (2.52) has at least one solution. It is not difficult to see that $\boldsymbol{P}$ is continuous. Moreover, from the a priori estimates presented above one can show that

$$\boldsymbol{P}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} = \int_\Omega \mu |\nabla \boldsymbol{v}^n|^2 - \boldsymbol{f} \cdot \boldsymbol{v}^n \geq C_1 |\boldsymbol{\alpha}|^2 - C_2, \tag{2.55}$$

where $C_1, C_2 > 0$ depend only on μ and $\|\boldsymbol{f}\|_2$ (we note that $|\boldsymbol{\alpha}|$ is equivalent to $\|\boldsymbol{v}^n\|_{1,2}$). Hence (2.54) holds for $R = \sqrt{C_2/C_1}$. Consequently there exists at least one $\boldsymbol{v}^n$ satisfying (2.51).

Next we can take the sequence $\{\boldsymbol{v}^n\}_{n=1}^\infty$, which is, due to the a priori estimates, bounded in $W^{1,2}_{0,\text{div}}(\Omega)$. The reflexivity of this space implies that there is a weakly convergent subsequence $\{\boldsymbol{v}^{n_k}\}_{k=1}^\infty$ and a function $\boldsymbol{v} \in W^{1,2}_{0,\text{div}}(\Omega)$ such that

$$\boldsymbol{v}^{n_k} \rightharpoonup \boldsymbol{v} \text{ weakly in } W^{1,2}_{0,\text{div}}(\Omega) \text{ as } k \to \infty.$$

In addition,

$$\boldsymbol{v}^{n_k} \to \boldsymbol{v} \text{ strongly in } L^4(\Omega; \mathbb{R}^d), \; k \to \infty,$$

passing eventually to a subsequence, as follows from the Rellich-Kondrachov theorem[2]. Consequently,

$$\boldsymbol{v}^{n_k} \otimes \boldsymbol{v}^{n_k} \to \boldsymbol{v} \otimes \boldsymbol{v} \text{ strongly in } L^2(\Omega).$$

[2]The Rellich-Kondrachov theorem states that $W^{1,2}(\Omega)$ is compactly embedded into $L^q(\Omega)$, $q \in [1, 2d/(d-2))$.

The above convergence properties are sufficient to pass to the limit in (2.51) and thus prove that $\boldsymbol{v}$ satisfies (2.50). We also have the estimate (2.47) for any weak solution.

Uniqueness. We show first that the pressure is uniquely determined by the velocity:

> *If* $(\boldsymbol{v}, p^1)$ *and* $(\boldsymbol{v}, p^2)$ *are two solutions to* (2.49) *then* $p^1 = p^2$.

Indeed, from (2.49) we obtain:

$$\int_\Omega (p^1 - p^2) \operatorname{div} \boldsymbol{\varphi} = 0 \ \forall \boldsymbol{\varphi} \in W_0^{1,2}(\Omega; \mathbb{R}^d).$$

Using $\boldsymbol{\varphi} := \mathcal{B}(p^1 - p^2)$ gives

$$\|p^1 - p^2\|_2^2 = 0.$$

Next we show that the velocity is unique under the assumption of 'small data'.

> *There exists a positive constant* $C = C(\Omega)$ *such that if* $\|\boldsymbol{f}\|_2 \le C\mu^2$, *then the solution to* (2.50) *is unique.*

Let us assume that $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ are two solutions of (2.50). We take the test function $\boldsymbol{\varphi} = \boldsymbol{w} := \boldsymbol{v}^1 - \boldsymbol{v}^2$, subtract the resulting integral identities and obtain:

$$\mu\|\nabla \boldsymbol{w}\|_2^2 - \int_\Omega (\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) : \nabla \boldsymbol{w} = 0. \tag{2.56}$$

The second term is estimated in absolute value using the following rearrangement:

$$\int_\Omega (\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) : \nabla \boldsymbol{w} = \int_\Omega \boldsymbol{v}^1 \otimes \boldsymbol{w} : \nabla \boldsymbol{w} + \int_\Omega \boldsymbol{w} \otimes \boldsymbol{v}^2 : \nabla \boldsymbol{w} =: I_1 + I_2. \tag{2.57}$$

Indeed, with help of the Hölder inequality, the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and the estimate (2.47) we obtain:

$$|I_1| \le \|\boldsymbol{v}^1\|_4 \|\boldsymbol{w}\|_4 \|\nabla \boldsymbol{w}\|_2 \le C_{I4} \|\boldsymbol{v}^1\|_{1,2} \|\nabla \boldsymbol{w}\|_2^2 \le C_{I4} C_F^2 \frac{\|\boldsymbol{f}\|_2}{\mu} \|\nabla \boldsymbol{w}\|_2^2. \tag{2.58}$$

An argument similar to (2.45) yields $I_2 = 0$. Hence (2.56)–(2.58) yields:

$$\left(\mu - \frac{C_{I4}C_F^2\|\boldsymbol{f}\|_2}{\mu}\right)\|\nabla \boldsymbol{w}\|_2^2 \leq 0,$$

which leads to the conclusion that $\boldsymbol{w} = \boldsymbol{0}$, i.e. $\boldsymbol{v}^1 = \boldsymbol{v}^2$ provided that

$$\|\boldsymbol{f}\|_2 \leq \frac{\mu^2}{C_{I4}C_F^2}, \tag{2.59}$$

i.e. if the forcing is sufficiently small with regard to the viscosity.

2.2.2 Generalizations

For generalizations and modifications of the problem (2.43), further refined or specialized methods have to be used. We briefly comment on some of them.

Non-Newtonian power-law models. The mathematical analysis can be done for nonlinear models with shear dependent viscosity, i.e.

$$\mathbb{T} = -p\mathbb{I} + \mathbb{S}, \quad \mathbb{S} = 2\nu(|\mathbb{D}\boldsymbol{v}|^2)\mathbb{D}\boldsymbol{v}.$$

If $\mathbb{S}$ has the following polynomial growth in $\mathbb{D}\boldsymbol{v}$:

$$\mathbb{S} \approx (\kappa + |\mathbb{D}\boldsymbol{v}|^2)^{\frac{r-2}{2}}\mathbb{D}\boldsymbol{v}, \quad \kappa \in \{0,1\},\ r > 1 \tag{2.60}$$

(we speak about power-law fluids), then one can apply the theory of monotone operators and a variant of Lebesgue dominated convergence theorem to pass to the limit in the term $\int_\Omega \mathbb{S} : \mathbb{D}\boldsymbol{\varphi}$. The key property is the inequality

$$\mathbb{S} : \mathbb{D}\boldsymbol{v} \geq \begin{cases} c|\mathbb{D}\boldsymbol{v}|^r & \text{if } \kappa = 0 \text{ or } r > 2, \\ c|\mathbb{D}\boldsymbol{v}|^2 & \text{if } \kappa = 1 \text{ and } r < 2, \end{cases}$$

which together with the Korn inequality (2.41) permits to obtain the a priori bounds of the velocity and the pointwise or strong convergence of $\mathbb{D}\boldsymbol{v}$.

For $r > 2$ the fluids are called shear thickening (viscosity increases with increasing shear rate), for $r < 2$ shear thinning (viscosity decreases with increasing shear rate), the case $r = 2$ reduces to Newtonian fluids. There is a critical value $r^* = 3d/(d+2)$ such that $\boldsymbol{v} \in W^{1,r^*}(\Omega;\mathbb{R}^d)$ implies $\boldsymbol{v}\otimes\boldsymbol{v} : \nabla\boldsymbol{v} \in L^1(\Omega)$. If $r \geq r^*$ then one can use the solution $\boldsymbol{v}$ as a test function and thus prove the a priori estimates. In the so-called supercritical case $r < r^*$ one has to apply more refined techniques (namely L^∞ or Lipschitz truncation) to overcome this difficulty, see [17, 18].

Non-Newtonian piezoviscous models. For fluids with shear-rate- and pressure-dependent viscosity, also called piezoviscous fluids, the analysis can be done under more restrictive assumptions:

$$\sum_{i,j,k,l=1}^{d} \frac{\partial \mathbb{S}_{ij}(p,\mathbb{D})}{\partial \mathbb{D}_{kl}} \mathbb{A}_{ij}\mathbb{A}_{kl} \approx (1+|\mathbb{D}|^2)^{\frac{r-2}{2}}|\mathbb{A}|^2 \quad \forall \mathbb{A} \in \mathbb{R}^{d\times d}, \tag{2.61a}$$

$$\left|\frac{\partial \mathbb{S}(p,\mathbb{D})}{\partial p}\right| \leq \gamma_0 (1+|\mathbb{D}|^2)^{\frac{r-2}{4}} \tag{2.61b}$$

with $r \in (1,2)$ and sufficiently small $\gamma_0 > 0$, i.e. the fluid must be shear-thinning and slightly pressure-thickening, see Figure 2.1.

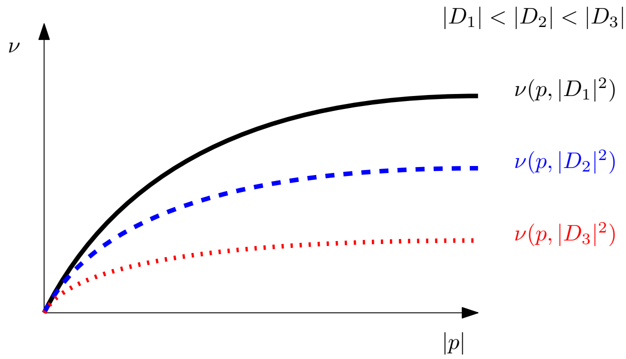

Figure 2.1: Example of viscosity-pressure relation for different fixed values of shear-rate.

In this model the meanvalue of the pressure $\int_\Omega p$ is one of input parameters, since it influences the viscosity and consequently also the velocity field. Hence instead of $L^q_0(\Omega)$ we use the space $L^q(\Omega)$ for the pressure. Also the pressure cannot be eliminated from the system by restricting onto divergence-free spaces and thus one has to incorporate it to the approximate systems from the beginning, taking e.g. the regularized continuity equation

$$-\varepsilon|p|^\alpha p + \operatorname{div} \boldsymbol{v} = 0$$

for certain power α and regularization parameter $\varepsilon > 0$ and making an extra limit passage for $\varepsilon \to 0+$. Since the viscosity is a nonlinear function of the pressure, the limit passages in approximate schemes also require strong convergence of the pressure. This is possible due to the special growth conditions (2.61). We refer to [33, 16, 11] for overview of recent results in this field.

Boundary conditions. In practice, the homogeneous Dirichlet boundary condition is not satisfactory. Indeed, considering a domain with walls, inflow and outflow, a combination of several boundary conditions has to be used. This requires also some modifications in the existence analysis. We comment on a selection of frequently used conditions:

- At inflow, usualy non-homogeneous Dirichlet condition

  $$\boldsymbol{v} = \boldsymbol{v}_{in}$$

  is used. Since the velocity then does not vanish on the boundary, the weak formulation has to be modified. The estimate of the convective term also relies on a suitable extension of $\boldsymbol{v}_{in}$ to Ω satisfying

  $$\forall \boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega): \quad \int_\Omega \boldsymbol{\varphi} \otimes \boldsymbol{\varphi} : \nabla \boldsymbol{v}_{in} \leq \frac{\mu}{2} \|\nabla \boldsymbol{\varphi}\|_2^2.$$

- Solid walls permitting slippage are modeled mostly by the Navier condition. In the estimates of the velocity one has to use a modified Friedrichs' or Korn's inequality which takes into account only vanishing normal component of functions.

- Friction-type boundary conditions such as (2.31) are used for surfaces to which the fluid adheres provided the shear stress is below some threshold. This leads to a non-smooth problem that can be formulated as a variational inequality with the incompressibility constraint.

- For outflow boundary conditions of the type (2.33) one needs a modified Bogovskiĭ operator $\tilde{\mathcal{B}}_\Omega : L^q(\Omega) \to W^{1,q}_{\Gamma_O}(\Omega; \mathbb{R}^d)$, where $W^{1,q}_{\Gamma_O}(\Omega; \mathbb{R}^d) := \{\boldsymbol{\varphi} \in W^{1,q}(\Omega);\ \boldsymbol{\varphi} = \boldsymbol{0} \text{ on } \partial\Omega \setminus \Gamma_O\}$ and Γ_O is the outflow part of the boundary. As a consequence of this condition, the pressure is completely determined by the velocity. An open question related to the do-nothing outflow condition (2.33) is the estimate of the convective term. When using the solution as a test function, one obtains after integrating by parts:

  $$\int_\Omega \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \boldsymbol{v} = \frac{1}{2} \int_{\Gamma_O} |\boldsymbol{v}|^2 (\boldsymbol{v} \cdot \boldsymbol{n}).$$

  This term does not vanish and is impossible to estimate unless the velocity is known a priori to be sufficiently small. Hence, for Navier-Stokes equations one has to consider a modified condition such as (2.34) or (2.35) that compensates the influence of the convection at outflow.

2.3 Finite element approximation

We shall describe the spatial discretization of steady Navier-Stokes equations by means of the finite element method. In contrast to the analysis of the continuous problem, one is usually interested in the mixed (velocity-pressure) formulation (2.49). The reason is twofold: First, in the majority of situations people are interested in obtaining the pressure, not only the velocity. Second, the velocity-based formulation brings difficulty in discretizing the incompressibility constraint.

Let $\mathcal{T}_h$ be a triangulation of Ω, i.e. non-overlapping partition into d-dimensional simplices, and h be the norm (diameter of largest element) of $\mathcal{T}_h$. We shall consider finite-dimensional spaces $W_h \subset W_0^{1,2}(\Omega;\mathbb{R}^d)$ and $L_h \subset L_0^2(\Omega)$ built on top of $\mathcal{T}_h$. In the discretization of Navier-Stokes equations one has to take into account for the following issues:

- In order to get stable approximation of the pressure, the finite element spaces W_h and L_h must satisfy the discrete inf-sup condition:

$$\inf_{\substack{\psi_h \in L_h \\ \psi_h \neq 0}} \sup_{\substack{\boldsymbol{\varphi}_h \in W_h \\ \boldsymbol{\varphi}_h \neq \mathbf{0}}} \frac{\int_\Omega \psi_h \operatorname{div} \boldsymbol{\varphi}_h}{\|\psi_h\|_2 \|\boldsymbol{\varphi}_h\|_{1,2}} \geq C_{BB} \tag{2.62}$$

  where $C_{BB} := C_{BB}(\Omega) > 0$ is a constant independent of h. Such property holds e.g. for the Taylor-Hood finite elements:

$$W_h := (P_{k+1}(\mathcal{T}_h))^d \cap W_0^{1,2}(\Omega;\mathbb{R}^d), \quad L_h := P_k(\mathcal{T}_h) \cap L_0^2(\Omega), \quad k \geq 1,$$

  where $P_k(\mathcal{T}_h)$ denotes the set of piecewise polynomials on the elements of $\mathcal{T}_h$ with degree up to k and continuous in Ω.

- To retain the uniform estimates of discrete solutions, the discretization of the convective term should obey the skew-symmetry. In particular, in (2.49) the form

$$c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}) := \int_\Omega \boldsymbol{w} \otimes \boldsymbol{u} : \nabla \boldsymbol{z}$$

  is used which is skew-symmetric in the following sense:

$$c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}) = -c(\boldsymbol{u},\boldsymbol{z},\boldsymbol{w}) \; \forall \boldsymbol{u} \in W^{1,2}_{0,\mathrm{div}}(\Omega), \boldsymbol{w},\boldsymbol{z} \in W^{1,2}(\Omega;\mathbb{R}^d).$$

  In the approximate schemes the functions will not be divergence-free and thus c will be replaced by

$$c_h(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}) := \frac{1}{2}\left(c(\boldsymbol{u},\boldsymbol{w},\boldsymbol{z}) - c(\boldsymbol{u},\boldsymbol{z},\boldsymbol{w})\right),$$

which satisfies:

$$c_h(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}) = -c_h(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}) \ \forall \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z} \in W^{1,2}(\Omega; \mathbb{R}^d).$$

In addition, for $\boldsymbol{u} \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, $\boldsymbol{w}, \boldsymbol{z} \in W^{1,2}(\Omega; \mathbb{R}^d)$ it holds:

$$c_h(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}) = c(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{z}).$$

Let us introduce the following forms:

$$a(\boldsymbol{u}, \boldsymbol{w}) := \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{w}, \quad b(q, \boldsymbol{w}) := \int_\Omega q \operatorname{div} \boldsymbol{w}, \quad l(\boldsymbol{w}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}.$$

Replacing $W^{1,2}_0(\Omega; \mathbb{R}^d)$ by W_h, $L^2_0(\Omega)$ by L_h and c by c_h in (2.49) we formally arrive at the discrete problem:

Finite-element approximation

$$\left.\begin{array}{l} \textit{Find } (\boldsymbol{v}_h, p_h) \in W_h \times L_h \textit{ such that} \\[1ex] \quad a(\boldsymbol{v}_h, \boldsymbol{\varphi}_h) - b(p_h, \boldsymbol{\varphi}_h) + b(\psi_h, \boldsymbol{v}_h) - c_h(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{\varphi}_h) = l(\boldsymbol{\varphi}_h) \\[1ex] \textit{for every } (\boldsymbol{\varphi}_h, \psi_h) \in W_h \times L_h. \end{array}\right\} \tag{2.63}$$

In the following subsections we comment on the well-posedness, convergence properties and numerical solution of (2.63).

2.3.1 Numerical analysis

The existence and convergence analysis for the discrete problem (2.63) mimicks in many aspects the Galerkin method for the original problem (2.50). Here we however use spaces that are not necessarily nested, hence an additional condition on the density of the finite-element spaces will be imposed. In addition to the existence and convergence result, it is also important to estimate the error between the discrete and exact solutions to (2.50).

Existence of discrete solutions. Restricting the problem (2.63) to test functions $\boldsymbol{\varphi}_h$ with zero discrete divergence, i.e.

$$\forall \psi_h \in L_h : \ b(\psi_h, \boldsymbol{\varphi}_h) = 0,$$

we eliminate p_h and obtain a system of nonlinear algebraic equations for $\boldsymbol{v}_h$ similar to (2.53). With help of the Brouwer theorem and uniform estimates it can be shown that this system has a solution. The existence of the discrete pressure p_h such that the pair $(\boldsymbol{v}_h, p_h)$ is a solution to (2.63) then follows from the closed range theorem.

Uniform estimates and convergence. Let us consider a sequence of triangulations $\{\mathcal{T}_h\}_{h\to 0+}$ and spaces $\{(W_h, L_h)\}_{h\to 0+}$ satisfying the discrete inf-sup condition (2.62) and the approximation property:

$$\left.\begin{array}{l}\text{For every pair } (\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega) \text{ there exists an ap-}\\ \text{proximating sequence } \{(\boldsymbol{\varphi}_h, \psi_h)\}_{h\to 0+},\ \boldsymbol{\varphi}_h \in W_h,\ \psi_h \in L_h, \text{ such}\\ \text{that}\\ \qquad \boldsymbol{\varphi}_h \to \boldsymbol{\varphi} \text{ in } W^{1,2}(\Omega; \mathbb{R}^d) \text{ and } \psi_h \to \psi \text{ in } L_0^2(\Omega),\ h \to 0+\,.\end{array}\right\} \tag{2.64}$$

We note that the approximation property (2.64) is satisfied e.g. when the sequence $\{\mathcal{T}_h\}_{h\to 0+}$ is *uniformly regular*, i.e. if the minimal interior angles of all triangles in $\mathcal{T}_h$ are bounded from below uniformly with respect to $h \to 0+$.

Now we are going to estimate the discrete solutions $(\boldsymbol{v}_h, p_h)$ uniformly with respect to $h \to 0+$. Using the test functions $\boldsymbol{\varphi}_h := \boldsymbol{v}_h$ and $\psi_h := p_h$ in (2.63) we obtain after the same manipulations, similarly as in Section 2.2.1, the uniform bound

$$\frac{\|\boldsymbol{v}_h\|_{1,2}}{C_F} \le \|\nabla \boldsymbol{v}_h\|_2 \le C_F \frac{\|\boldsymbol{f}\|_2}{\mu}. \tag{2.65}$$

The estimate of the pressure follows from the discrete inf-sup condition (2.62):

$$\begin{aligned}\|p_h\|_2 &\le \frac{1}{C_{BB}} \sup_{\substack{\boldsymbol{\varphi}_h \in W_h\\ \boldsymbol{\varphi}_h \neq \mathbf{0}}} \frac{b(p_h, \boldsymbol{\varphi}_h)}{\|\boldsymbol{\varphi}_h\|_{1,2}}\\ &= \frac{1}{C_{BB}} \sup_{\substack{\boldsymbol{\varphi}_h \in W_h\\ \boldsymbol{\varphi}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{v}_h, \boldsymbol{\varphi}_h) - c_h(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{\varphi}_h) - l(\boldsymbol{\varphi}_h)}{\|\boldsymbol{\varphi}_h\|_{1,2}}\\ &\le C(\Omega, \mu, \|\boldsymbol{f}\|_2),\end{aligned} \tag{2.66}$$

with $C(\Omega, \mu, \|\boldsymbol{f}\|_2) > 0$ independent of h.

Having the uniform estimates (2.65), (2.66) at our disposal, we can deduce that there is a pair $(\hat{\boldsymbol{v}}, \hat{p}) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ and a subsequence $\{h_k\}_{k=1}^{\infty}$, $h_k \to 0+$ as $k \to \infty$, such that

$$\boldsymbol{v}_{h_k} \rightharpoonup \hat{\boldsymbol{v}} \qquad \text{(weakly) in } W^{1,2}(\Omega; \mathbb{R}^d), \tag{2.67}$$

$$\boldsymbol{v}_{h_k} \to \hat{\boldsymbol{v}} \qquad \text{(strongly) in } L^4(\Omega; \mathbb{R}^d), \tag{2.68}$$

$$p_{h_k} \rightharpoonup \hat{p} \qquad \text{(weakly) in } L_0^2(\Omega),\ k \to \infty. \tag{2.69}$$

This is enough to pass to the limit in (2.63) and from the approximation property (2.64) it follows that $\hat{\boldsymbol{v}} = \boldsymbol{v}$ and $\hat{p} = p$, where $(\boldsymbol{v}, p)$ is a solution to (2.49).

Uniqueness and rate of convergence. The uniqueness of the discrete solutions $\{(\boldsymbol{v}_h, p_h)\}$ holds under the same assumptions as in the case of the continuous problem, i.e. pressure is uniquely determined by velocity and velocity is unique under the assumption (2.59) of "small data". Moreover, a slightly more strict assumption guarantees the best approximation property (known as the Céa lemma):

Let

$$\|\boldsymbol{f}\|_2 < \frac{\mu^2}{2C_{I4}^2 C_F}. \tag{2.70}$$

Then there is a constant $C = C(\Omega, \mu, \|\boldsymbol{f}\|_2) > 0$ *independent of* $h \to 0+$ *such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,2} + \|p - p_h\|_2 \le C \inf_{\substack{\boldsymbol{\varphi}_h \in W_h \\ \psi_h \in L_h}} \left(\|\boldsymbol{v} - \boldsymbol{\varphi}_h\|_{1,2} + \|p - \psi_h\|_2\right). \tag{2.71}$$

The proof of this statement relies on the estimation of the difference

$$\begin{aligned}
&|c_h(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\varphi}_h - \boldsymbol{v}_h) - c_h(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{\varphi}_h - \boldsymbol{v}_h)| \\
&\qquad \le C(\Omega, \mu, \|\boldsymbol{f}\|_2)\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_2 \|\nabla(\boldsymbol{v} - \boldsymbol{\varphi}_h)\|_2 + \varepsilon\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_2^2
\end{aligned}$$

with certain sufficiently small $\varepsilon > 0$. This is true under the smallness condition (2.70)

The inequality (2.71) leads together with interpolation estimates in Sobolev spaces to an explicit rate of the error norms:

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,2} \le Ch^r, \quad \|p - p_h\|_2 \le Ch^s,$$

where $C = C(\Omega, \mu, \|\boldsymbol{f}\|_2) > 0$, and $r, s > 0$ depend on the degree of polynomials contained in the finite element spaces W_h, L_h and on the regularity of $\boldsymbol{v}$ and p.

2.3.2 Computation

Let $\{\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^N\}$ and $\{\psi^1, \dots, \psi^M\}$ be the basis of W_h, L_h, respectively. We represent the discrete solution in terms of vectors of coefficients $\boldsymbol{V}$, $\boldsymbol{P}$:

$$\boldsymbol{v}_h := \sum_{j=1}^{N} V_j \boldsymbol{\varphi}^j, \quad p_h := \sum_{j=1}^{M} P_j \psi^j.$$

Then the discrete problem (2.63) has an equivalent algebraic form:

$$\begin{bmatrix} \mathbb{A} + \mathbb{C}(\boldsymbol{V}) & \mathbb{B}^\top \\ \mathbb{B} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{V} \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{bmatrix},$$

where $\mathbb{A} \in \mathbb{R}^{N\times N}$, $\mathbb{B} \in \mathbb{R}^{M\times N}$, $\mathbb{C} : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ and $\boldsymbol{F} \in \mathbb{R}^N$ are defined as follows:

$$[\mathbb{A}]_{ij} := a(\boldsymbol{\varphi}^j, \boldsymbol{\varphi}^i), \quad [\mathbb{B}]_{ij} := b(\psi^j, \boldsymbol{\varphi}^i),$$
$$[\mathbb{C}(\boldsymbol{V})]_{ij} := c_h(\boldsymbol{v}_h, \boldsymbol{\varphi}^j, \boldsymbol{\varphi}^i), [\boldsymbol{F}]_i := l(\boldsymbol{\varphi}^i).$$

This system of nonlinear equations is usually linearized either by Picard or Newton iterations. Their convergence is in case of Navier-Stokes equations guaranteed for small data, i.e. large viscosity. In both cases we arrive at the linearized system with saddle-point structure:

$$\begin{bmatrix} \tilde{\mathbb{A}} & \mathbb{B}^\top \\ \mathbb{B} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{V}} \\ \tilde{\boldsymbol{P}} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{F}} \\ \boldsymbol{0} \end{bmatrix}. \tag{2.72}$$

The matrix $\tilde{\mathbb{A}}$ and the vector $\tilde{\boldsymbol{F}}$ are in general different for each iteration. The matrix of the system (2.72) is indefinite and nonsymmetric. For the numerical solution one can use e.g. a general method such as sparse LU decomposition [13, 2] or GMRES [40] with a suitable preconditioner.

We note that the finite element solution of Navier-Stokes and related equations is to some extent also possible for large data (=small viscosity). However in that case it is necessary to consider a stabilization or a turbulence model that produces an artificial turbulent viscosity. We refer to [15, 39, 8] for more details on stabilization.

Chapter 3

Shape Optimization in Fluid Mechanics

In many real-world applications one is facing the problem of designing the shape of a device which interacts with a fluid (car body, airplane wing, turbine, to name a few examples). In order to meet certain requirements (e.g. reduce drag or energy losses) it is important to know how the shape of the device affects the flow properties. The process of designing a suitable shape can be formulated as a mathematical optimization problem. Successfull solving this problem can significantly simplify the process by suggesting or excluding certain designs.

In this chapter we aim to present the main ideas of shape optimization in a model setting considering the Navier-Stokes equations as the flow problem.

3.1 Formulation of optimization problems

Let us consider a set $\mathcal{O}$ of *admissible domains* in which a fluid can flow. We shall study the problem of minimizing the value of a function, which depends on the domain through the solution of the so-called *state problem*, which in our case will be the Navier-Stokes equations: For every admissible domain $\Omega \in \mathcal{O}$ we solve:

$$\left.\begin{aligned} \operatorname{div} \boldsymbol{v}_\Omega = 0, \quad \operatorname{div}(\boldsymbol{v}_\Omega \otimes \boldsymbol{v}_\Omega) - \mu\Delta\boldsymbol{v}_\Omega + \nabla p_\Omega = \boldsymbol{f} &\quad \text{in } \Omega,\\ \boldsymbol{v}_\Omega = \mathbf{0} &\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{$\mathcal{P}(\Omega)$}$$

Here $\mu > 0$ and $\boldsymbol{f} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ are assumed to be independent of Ω. The *cost function* to be minimized will be denoted J and we assume that it depends on Ω as well as on $\boldsymbol{v}_\Omega$ and p_Ω. For example, one can take one of the following cost functions:

$$J(\Omega, \boldsymbol{v}_\Omega, p_\Omega) := \begin{cases} \displaystyle\int_{\partial\Omega} (-p_\Omega + 2\mu \mathbb{D}\boldsymbol{v}_\Omega)\boldsymbol{n} \cdot \boldsymbol{t}, \ \boldsymbol{t} \in \mathbb{R}^d & \text{(drag functional)}, \\ \displaystyle\int_{\Omega} |\nabla \boldsymbol{v}_\Omega|^2 & \text{(energy functional)}, \\ \displaystyle\int_{\Omega_0} |\boldsymbol{v}_\Omega - \boldsymbol{v}_{opt}|^2, \ \boldsymbol{v}_{opt} \in L^2(\Omega_0) & \text{(least-squares type f.)}. \end{cases} \tag{3.1}$$

In fluid mechanics, the solutions to the state problem are often not unique, as is the case of $(\mathcal{P}(\Omega))$. For this reason we shall define the admissible set

$$\mathcal{A} := \{(\Omega, \boldsymbol{v}_\Omega, p_\Omega);\ \Omega \in \mathcal{O},\ (\boldsymbol{v}_\Omega, p_\Omega) \text{ is a solution of } (\mathcal{P}_w(\Omega))\},$$

where $(\mathcal{P}_w(\Omega))$ is the weak formulation of $(\mathcal{P}(\Omega))$:

$$\left.\begin{aligned} &\textit{Find } (\boldsymbol{v}_\Omega, p_\Omega) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega) \textit{ such that} \\ &\int_\Omega (\mu \nabla \boldsymbol{v}_\Omega - \boldsymbol{v}_\Omega \otimes \boldsymbol{v}_\Omega) : \nabla \boldsymbol{\varphi} - p_\Omega \operatorname{div} \boldsymbol{\varphi} + \psi \operatorname{div} \boldsymbol{v}_\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \\ &\textit{for all } (\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega). \end{aligned}\right\} \quad (\mathcal{P}_w(\Omega))$$

The shape optimization problem then reads:

$$\left.\begin{aligned} &\textit{Find } (\Omega^*, \boldsymbol{v}^*, p^*) \in \mathcal{A} \textit{ such that} \\ &\qquad J(\Omega^*, \boldsymbol{v}^*, p^*) = \min_{(\Omega, \boldsymbol{v}_\Omega, p_\Omega) \in \mathcal{A}} J(\Omega, \boldsymbol{v}_\Omega, p_\Omega). \end{aligned}\right\} \quad (\mathbb{P})$$

In what follows we shall address the following questions:

- Under what assumptions does the problem $(\mathbb{P})$ have a solution?
- Can $(\mathbb{P})$ be approximated by a sequence of finite dimensional optimization problems whose solutions converge to a solution of $(\mathbb{P})$?
- Is the cost function J differentiable? How can one compute its gradient?

3.2 Existence of optimal design

Assuming that for every $\Omega \in \mathcal{O}$ the state problem $(\mathcal{P}_w(\Omega))$ has at least one solution, we are interested in establishing the sufficient conditions under which the function J has a minimizer, i.e. an optimal shape Ω^* with a corresponding solution $(\boldsymbol{v}_{\Omega^*}, p_{\Omega^*})$ of $(\mathcal{P}_w(\Omega^*))$.

The classical way of proving the existence of a minimizers is based on the Bolzano-Weierstrass theorem, i.e. a continuous function on a compact set attains its minimum. In this respect, one has to introduce:

- convergence of domains $\Omega_n \overset{\mathcal{O}}{\to} \Omega$, where $\Omega_n, \Omega \in \mathcal{O}$, $n \in \mathbb{N}$;
- convergence of functions $(\boldsymbol{\varphi}_n, \psi_n) \rightsquigarrow (\boldsymbol{\varphi}, \psi)$, where $(\boldsymbol{\varphi}_n, \psi_n) \in W_0^{1,2}(\Omega_n; \mathbb{R}^d) \times L_0^2(\Omega)$, $n \in \mathbb{N}$, $(\boldsymbol{\varphi}, \psi) \in W^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ and $\Omega_n \overset{\mathcal{O}}{\to} \Omega$

in such a way that the following assumptions are satisfied:

(A1) $\mathcal{O}$ is compact with respect to the convergence "$\overset{\mathcal{O}}{\to}$";

(A2) Solutions to $(\mathcal{P}_w(\Omega))$ are bounded independently of $\Omega \in \mathcal{O}$;

(A3) $\mathcal{A}$ is closed, i.e. if $\Omega_n \overset{\mathcal{O}}{\to} \Omega$ and $(\boldsymbol{v}_{\Omega_n}, p_{\Omega_n})$ are solutions to $(\mathcal{P}_w(\Omega_n))$, $n \in \mathbb{N}$, such that $(\boldsymbol{v}_{\Omega_n}, p_{\Omega_n}) \rightsquigarrow (\overline{\boldsymbol{v}}, \overline{p})$, then $(\overline{\boldsymbol{v}}, \overline{p})$ is a solution to $(\mathcal{P}_w(\Omega))$;

(A4) J is lower semicontinuous in the following sense:

$$\left.\begin{array}{r} \Omega_n \overset{\mathcal{O}}{\to} \Omega \\ (\boldsymbol{\varphi}_n, \psi_n) \rightsquigarrow (\boldsymbol{\varphi}, \psi) \end{array}\right\} \Rightarrow \liminf_{n\to\infty} J(\Omega_n, \boldsymbol{\varphi}_n, \psi_n) \geq J(\Omega, \boldsymbol{\varphi}, \psi).$$

The existence result is then an easy consequence:

> *Let (A1)-(A4) be satisfied. Then* $(\mathbb{P})$ *has a solution.*

Indeed, let us take a sequence $\{(\Omega_n, \boldsymbol{v}_n, p_n)\} \subset \mathcal{A}$ minimizing J. (A1) implies that there is a subsequence (denoted by the same symbol) and a domain $\Omega \in \mathcal{O}$ such that $\Omega_n \overset{\mathcal{O}}{\to} \Omega$. Next, since by (A2) the sequence $\{(\boldsymbol{v}_n, p_n)\}_{n=1}^{\infty}$ is bounded, there is another subsequence (denoted by the same symbol) and a pair $(\boldsymbol{v}, p)$ such that $(\boldsymbol{v}_n, p_n) \rightsquigarrow (\boldsymbol{v}, p)$. From (A3) we get that $(\boldsymbol{v}, p)$ is a solution to $(\mathcal{P}_w(\Omega))$. Finally, (A4) implies that

$$J(\Omega, \boldsymbol{v}, p) \leq \liminf_{n\to\infty} J(\Omega_n, \boldsymbol{v}_n, p_n),$$

which means that $(\Omega, \boldsymbol{v}, p)$ is an optimal triplet for $(\mathbb{P})$.

In the following subsections we address the issue of proper definitions of convergence of domains and functions such that (A1)-(A4) hold true.

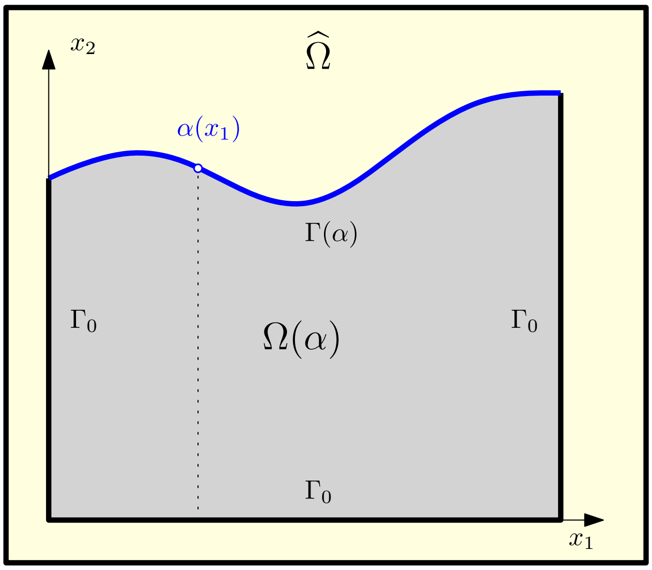

Figure 3.1: Admissible domain $\Omega(\alpha)$ and hold-all domain $\widehat{\Omega}$.

3.2.1 Convergence of domains

In many practical problems, shape optimization involves improving just a part of the boundary of Ω. Then it is usually reasonable to describe the part to be optimized as a graph of a function. In our model setting, we shall restrict for simplicity of presentation to the 2D case where every admissible domain will be of the form

$$\Omega(\alpha) := \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ x_1 \in (0,1),\ x_2 \in (0, \alpha(x_1)) \right\}.$$

Its boundary is decomposed as follows:

$$\partial\Omega(\alpha) = \Gamma_0 \cup \Gamma(\alpha), \quad \Gamma(\alpha) := \{(x_1, \alpha(x_1));\ x_1 \in (0,1)\},$$

see Figure 3.1. The set $\mathcal{O}$ is then represented by a set of functions $\mathcal{U}_{ad} := \{\alpha : [0,1] \to \mathbb{R};\ \Omega(\alpha) \in \mathcal{O}\}$. Hence we can define the convergence of domains via convergence of the corresponding functions in $\mathcal{U}_{ad}$. Let $\mathcal{U}_{ad}$ consist of functions which are bounded together with their derivatives up to order $k+1$, $k \in \{0, 1, \ldots\}$:

$$\mathcal{U}_{ad} := \Big\{ \alpha : [0,1] \to \mathbb{R};\ \forall x_1 \in [0,1], l \in \{1, \ldots, k+1\} : \\ \alpha_{min} \leq \alpha(x_1) \leq \alpha_{max},\ |\alpha^{(l)}(x_1)| \leq C_l \Big\}.$$

The constants $\alpha_{min}, \alpha_{max}, C_1, \ldots, C_{k+1} > 0$ are assumed to be such that $\mathcal{U}_{ad} \neq \emptyset$. Then the convergence of domains in $\mathcal{O}$ can be introduced as follows:

$$\Omega(\alpha_n) \xrightarrow{\mathcal{O}} \Omega(\alpha) \quad \text{if and only if} \quad \alpha_n \to \alpha \text{ in } \mathcal{C}^k([0,1]),\ n \to \infty. \tag{3.2}$$

The Arzelà-Ascoli theorem implies that $\mathcal{U}_{ad}$ is compact with respect to convergence in $\mathcal{C}^k([0,1])$. Consequently,

> $\mathcal{O}$ *is compact with respect to the convergence* (3.2).

3.2.2 Extension of functions and uniform bounds

For the convergence of functions which are defined in different domains of definition we shall need:

- a *hold-all domain* $\widehat{\Omega}$ such that $\Omega \subset \widehat{\Omega}$ for every $\Omega \in \mathcal{O}$;
- linear extension operators $E_\Omega : X(\Omega) \to W^{1,2}(\widehat{\Omega}; \mathbb{R}^d)$ whose norms are independent of $\Omega \in \mathcal{O}$, i.e. such that

$$(E_\Omega \boldsymbol{\varphi})_{|\Omega} = \boldsymbol{\varphi},$$

$$\|E_\Omega \boldsymbol{\varphi}\|_{1,2,\widehat{\Omega}} \leq C_E \|\boldsymbol{\varphi}\|_{1,2,\Omega}$$

for all $\Omega \in \mathcal{O}$ and $\boldsymbol{\varphi} \in X(\Omega)$.

Here $X(\Omega)$ is the space where the velocity lives and $C_E > 0$ is independent of $\Omega \in \mathcal{O}$. In the case of $(\mathcal{P}_w(\Omega))$ we have $X(\Omega) := W_0^{1,2}(\Omega; \mathbb{R}^d)$. The extension operators E_Ω must in addition satisfy the *Mosco conditions*:

(M1) If $\Omega_n \xrightarrow{\mathcal{O}} \Omega$ and $E_{\Omega_n} \boldsymbol{\varphi}_n \rightharpoonup \widehat{\boldsymbol{\varphi}}$ (weakly) in $W^{1,2}(\widehat{\Omega}; \mathbb{R}^d)$, where $\boldsymbol{\varphi}_n \in X(\Omega_n)$, then $\widehat{\boldsymbol{\varphi}}_{|\Omega} \in X(\Omega)$;

(M2) If $\Omega_n \xrightarrow{\mathcal{O}} \Omega$ and $\boldsymbol{\varphi} \in X(\Omega)$ then there exists a sequence $\{\boldsymbol{\varphi}_n\}_{n=1}^\infty$, $\boldsymbol{\varphi}_n \in X(\Omega_n)$ such that $E_{\Omega_n} \boldsymbol{\varphi}_n \to E_\Omega \boldsymbol{\varphi}$ (strongly) in $W^{1,2}(\widehat{\Omega})$.

The choice of E_Ω depends on the boundary condition which is prescribed on $\Gamma(\alpha)$. In the case of homogeneous Dirichlet condition we use the zero extension to $\widehat{\Omega}$:

$$(E_\Omega \boldsymbol{\varphi})(x) = \tilde{\boldsymbol{\varphi}} := \begin{cases} \boldsymbol{\varphi}(x) & \text{if } x \in \Omega, \\ \mathbf{0} & \text{if } x \in \widehat{\Omega} \setminus \Omega. \end{cases}$$

Clearly, for $\boldsymbol{\varphi} \in W_0^{1,2}(\Omega;\mathbb{R}^d)$ we have $\tilde{\boldsymbol{\varphi}} \in W_0^{1,2}(\widehat{\Omega};\mathbb{R}^d)$ and $\|\tilde{\boldsymbol{\varphi}}\|_{1,2,\widehat{\Omega}} = \|\boldsymbol{\varphi}\|_{1,2,\Omega}$. The conditions (M1)-(M2) are satisfied due to density of compactly supported functions in $W_0^{1,2}(\Omega;\mathbb{R}^d)$. For other types of boundary conditions on $\Gamma(\alpha)$ such as Navier's condition, the choice of suitable convergence in $\mathcal{O}$ and extension operators is more delicate. For the pressure it is reasonable to take the zero extension so that properties analogous to (M1)-(M2) are satisfied.

Using the extension operator and the characteristic function

$$\chi_\Omega := \begin{cases} 1 & \text{in } \Omega, \\ 0 & \text{elsewhere,} \end{cases}$$

we rewrite $(\mathcal{P}_w(\Omega))$ equivalently using the fixed domain $\widehat{\Omega}$:

$$\left.\begin{aligned} &\textit{Find } (\boldsymbol{v}_\Omega, p_\Omega) \in W_0^{1,2}(\Omega;\mathbb{R}^d) \times L_0^2(\Omega) \textit{ such that} \\ &\int_{\widehat{\Omega}} \chi_\Omega \left(\mu \nabla E_\Omega \boldsymbol{v}_\Omega - E_\Omega \boldsymbol{v}_\Omega \otimes E_\Omega \boldsymbol{v}_\Omega\right) : \nabla E_\Omega \boldsymbol{\varphi} \\ &\qquad\qquad -\tilde{p}_\Omega \operatorname{div}(E_\Omega \boldsymbol{\varphi}) + \tilde{\psi} \operatorname{div}(E_\Omega \boldsymbol{v}_\Omega) = \int_{\widehat{\Omega}} \chi_\Omega \boldsymbol{f} \cdot E_\Omega \boldsymbol{\varphi} \\ &\textit{for all } (\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega) \times L_0^2(\Omega). \end{aligned}\right\} \qquad (\widehat{\mathcal{P}}_w(\Omega))$$

Following the steps from Section 2.2.1, we take $(\boldsymbol{\varphi}, \psi) := (\boldsymbol{v}_\Omega, p_\Omega)$ in $(\widehat{\mathcal{P}}_w(\Omega))$ in order to obtain the uniform estimate of $\boldsymbol{v}$. However, to ensure that the resulting upper bound is independent of Ω, one has to use Friedrichs' inequality in $\widehat{\Omega}$. It is feasible since $E_\Omega \boldsymbol{v}_\Omega \in W_0^{1,2}(\widehat{\Omega})$, so that

$$\begin{aligned} \mu \|\nabla (E_\Omega \boldsymbol{v}_\Omega)\|_{2,\widehat{\Omega}}^2 = \int_{\widehat{\Omega}} \boldsymbol{f} \cdot E_\Omega \boldsymbol{v}_\Omega \le \|\boldsymbol{f}\|_{2,\widehat{\Omega}} \|E_\Omega \boldsymbol{v}_\Omega\|_{2,\widehat{\Omega}} \\ \le C_F \|\boldsymbol{f}\|_{2,\widehat{\Omega}} \|\nabla (E_\Omega \boldsymbol{v}_\Omega)\|_{2,\widehat{\Omega}}, \end{aligned}$$

where $C_F := C_F(\widehat{\Omega}, 2) > 0$ is independent of $\Omega \in \mathcal{O}$.

The uniform estimate of the pressure holds provided that the norm of the Bogovskiĭ operator $\mathcal{B}_\Omega$ is bounded independently of $\Omega \in \mathcal{O}$. This can be proved for example if the admissible domains are uniformly star-shaped or uniformly Lipschitz (see [19, 9]), which is our case. In summary, we have proved:

> *There exists a constant $C > 0$ such that*
>
> $$\forall (\Omega, \boldsymbol{v}_\Omega, p_\Omega) \in \mathcal{A}: \ \|E_\Omega \nabla \boldsymbol{v}_\Omega\|_{2,\widehat{\Omega}} + \|\tilde{p}_\Omega\|_{2,\widehat{\Omega}} \le C. \tag{3.3}$$

3.2.3 Convergence of functions and closedness of admissible set

Let $\Omega_n \xrightarrow{\mathcal{O}} \Omega$. We define the convergence of a sequence $\{(\boldsymbol{\varphi}_n, \psi_n)\}_{n=1}^{\infty}$, where $(\boldsymbol{\varphi}_n, \psi_n) \in W_0^{1,2}(\Omega_n; \mathbb{R}^d) \times L_0^2(\Omega_n)$, $n \in \mathbb{N}$, to the pair $(\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ as follows:

$$(\boldsymbol{\varphi}_n, \psi_n) \rightsquigarrow (\boldsymbol{\varphi}, \psi) \text{ if and only if } \begin{cases} E_{\Omega_n}\boldsymbol{\varphi}_n \rightharpoonup E_\Omega \boldsymbol{\varphi} & \text{in } W^{1,2}(\widehat{\Omega}; \mathbb{R}^d), \\ \tilde{\psi}_n \rightharpoonup \tilde{\psi} & \text{in } L_0^2(\widehat{\Omega}). \end{cases} \tag{3.4}$$

The estimate (3.3) implies that from any sequence $\{(\boldsymbol{v}_n, p_n)\}_{n=1}^{\infty}$, where $(\boldsymbol{v}_n, p_n)$ is a solution to $(\mathcal{P}_w(\Omega_n))$, $n \in \mathbb{N}$, one can extract a subsequence (denoted by the same symbol) such that

$$(E_{\Omega_n}\boldsymbol{v}_n, \tilde{p}_n) \rightharpoonup (\widehat{\boldsymbol{v}}, \widehat{p}) \text{ in } W^{1,2}(\widehat{\Omega}; \mathbb{R}^d) \times L_0^2(\widehat{\Omega}).$$

Then by (M1), we obtain that $(\boldsymbol{v}_n, p_n) \rightsquigarrow (\overline{\boldsymbol{v}}, \overline{p}) := (\widehat{\boldsymbol{v}}_{|\Omega}, \widehat{p}_{|\Omega})$.

Now let us consider an arbitrary sequence $\{(\Omega_n, \boldsymbol{v}_n, p_n)\} \subset \boldsymbol{\mathcal{A}}$ such that $\Omega_n \xrightarrow{\mathcal{O}} \Omega$, $(\boldsymbol{v}_n, p_n) \rightsquigarrow (\overline{\boldsymbol{v}}, \overline{p})$ and a pair of test functions $(\boldsymbol{\varphi}, \psi) \in W_0^{1,2}(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ in $(\widehat{\mathcal{P}}_w(\Omega))$. Thanks to (M2) and the density of compactly supported functions in $L_0^2(\widehat{\Omega})$, we can take approximating sequences $\{\boldsymbol{\varphi}_n\}_{n=1}^{\infty}$, $\{\psi_n\}_{n=1}^{\infty}$ so that $(\widehat{\mathcal{P}}_w(\Omega_n))$ becomes:

$$\int_{\widehat{\Omega}} \chi_{\Omega_n} \left(\mu \nabla(E_{\Omega_n}\boldsymbol{v}_n) - (E_{\Omega_n}\boldsymbol{v}_n) \otimes (E_{\Omega_n}\boldsymbol{v}_n)\right) : \nabla(E_{\Omega_n}\boldsymbol{\varphi}_n)$$
$$- \tilde{p}_n \operatorname{div}(E_{\Omega_n}\boldsymbol{\varphi}_n) + \tilde{\psi}_n \operatorname{div}(E_{\Omega_n}\boldsymbol{v}_n) = \int_{\widehat{\Omega}} \chi_{\Omega_n} \boldsymbol{f} \cdot E_{\Omega_n}\boldsymbol{\varphi}_n.$$

Since $\Omega_n \xrightarrow{\mathcal{O}} \Omega$, the characteristic functions satisfy $\chi_{\Omega_n} \to \chi_\Omega$ in $L^q(\Omega)$ for any $q \in [1, \infty)$. Passing to the limit $n \to \infty$, using the weak and strong convergence of $\{(\boldsymbol{v}_n, p_n)\}$ and $\{(\boldsymbol{\varphi}_n, \psi_n)\}$, respectively, we obtain:

$$\int_{\widehat{\Omega}} \chi_\Omega \left(\mu \nabla(E_\Omega \overline{\boldsymbol{v}}) - (E_\Omega \overline{\boldsymbol{v}}) \otimes (E_\Omega \overline{\boldsymbol{v}})\right) : \nabla(E_\Omega \boldsymbol{\varphi}) - \tilde{\overline{p}} \operatorname{div}(E_\Omega \boldsymbol{\varphi})$$
$$+ \tilde{\psi} \operatorname{div}(E_\Omega \overline{\boldsymbol{v}}) = \int_{\widehat{\Omega}} \chi_\Omega \boldsymbol{f} \cdot E_\Omega \boldsymbol{\varphi},$$

which implies that $(\overline{\boldsymbol{v}}, \overline{p})$ is a solution to $(\widehat{\mathcal{P}}_w(\Omega))$. This completes the proof of (A3).

Finally, since all 3 examples of cost functions from (3.1) are lower semicontinuous with respect to the weak convergence in $W^{1,2}(\widehat{\Omega}; \mathbb{R}^d) \times L^2(\widehat{\Omega})$, (A4) is satisfied.

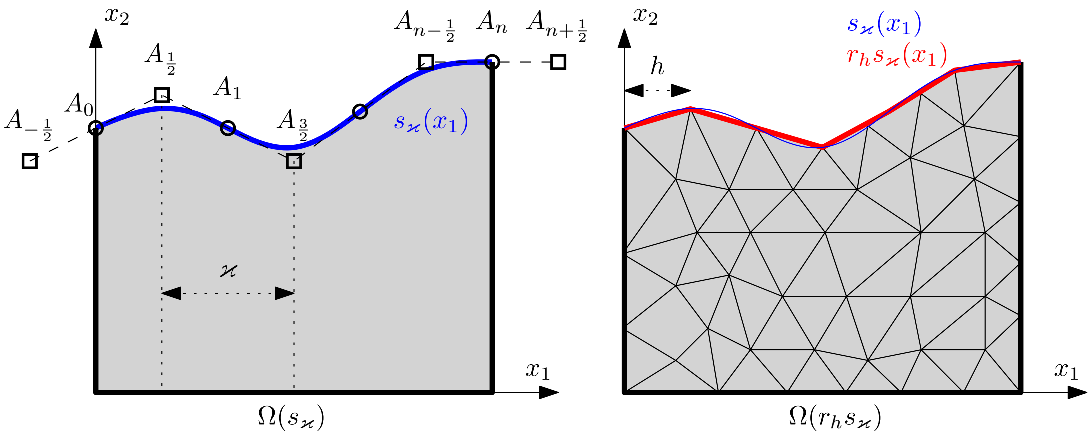

Figure 3.2: Approximation of the boundary of $\Omega(\alpha)$: Discrete design domain $\Omega(s_{\varkappa})$ (left) and discrete computational domain $\Omega(r_h s_{\varkappa})$ with its triangulation $\mathcal{T}_h(s_{\varkappa})$ (right).

3.3 Numerical analysis and computation

We are going to describe the approximation of the shape optimization problem $(\mathbb{P})$. This involves discretization of domains as well as of the state problem. It is usually not difficult to prove that the discrete shape optimization problem has a solution without any additional assumptions. The result for convergence of discrete optimal solutions is however quite weak due to the fact that $(\mathbb{P})$ is in general a non-convex optimization problem.

3.3.1 Discrete shape optimization problem

Every admissible domain $\Omega(\alpha)$ will be approximated by a *discrete design domain* $\Omega(s_{\varkappa})$, where $s_{\varkappa}$ is a function parameterized by $n := n(\varkappa)$ degrees of freedom. We require that $n(\varkappa) \to \infty$ as $\varkappa \to 0+$ and that every $\alpha \in \mathcal{U}_{ad}$ can be approximated by a sequence $\{s_{\varkappa}\}_{\varkappa \to 0+}$ in $\mathcal{C}^k([0,1])$. One can consider for instance piecewise Bézier functions such as in Figure 3.2. There, the degrees of freedom are the vertical positions of points $A_{i-\frac{1}{2}} = \left(\frac{i-\frac{1}{2}}{n}, \alpha_i\right)$, $i = 0, \dots, n+1$. For example, if $\mathcal{U}_{ad}$ consists of functions with bounded derivatives ($k = 0$), then the set of discrete design domains is represented by

$$\boldsymbol{U}_n := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+2};\ \alpha_{min} \le \alpha_i \le \alpha_{max},\ i = 0, ..., n+1; \right. \\ \left. \frac{|\alpha_{i+1} - \alpha_i|}{\varkappa} \le C_1,\ i = 0, ..., n \right\},$$

i.e. the constraints on the derivatives of α are replaced by constraints on differences of the piecewise linear function given by the control points $\{A_{i-\frac{1}{2}}\}_{i=0}^{n+1}$. With every $\boldsymbol{\alpha} \in \boldsymbol{U}_n$ we associate the function $s_\varkappa := s_\varkappa(\boldsymbol{\alpha})$, so that the discrete admissible set is

$$\mathcal{U}_{ad}^{\varkappa} := \{s_\varkappa(\boldsymbol{\alpha});\ \boldsymbol{\alpha} \in \boldsymbol{U}_{n(\varkappa)}\}.$$

Next we turn to the discretization of the state problem. We consider the finite element approximation of $(\mathcal{P}(\Omega))$. For this reason we need to replace the discrete design domain $\Omega(s_\varkappa)$ by its piecewise polygonal approximation $\Omega(r_h s_\varkappa)$, where h is a mesh discretization parameter such that $h \to 0+$ whenever $\varkappa \to 0+$ and r_h is a piecewise linear interpolation operator, see Figure 3.2. For every *discrete computational domain* $\Omega(r_h s_\varkappa)$ we construct a triangulation $\mathcal{T}_h(s_\varkappa)$ with the norm h. The finite element approximation of $(\mathcal{P}(r_h s_\varkappa))$, as described in Section 2.3, will be denoted by $(\mathcal{P}_h(r_h s_\varkappa))$. Let us define the set

$$\mathcal{A}_{\varkappa h} := \{(s_\varkappa, \boldsymbol{v}_h, p_h);\ s_\varkappa \in \mathcal{U}_{ad}^{\varkappa},\ (\boldsymbol{v}_h, p_h) \text{ is a solution to } (\mathcal{P}_h(r_h s_\varkappa))\}.$$

Then the discrete shape optimization problem reads:

$$\left.\begin{aligned} &\textit{Find } (s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) \in \mathcal{A}_{\varkappa h} \textit{ such that} \\ &\qquad J(s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) = \min_{(s_\varkappa, \boldsymbol{v}_h, p_h) \in \mathcal{A}_{\varkappa h}} J(s_\varkappa, \boldsymbol{v}_h, p_h). \end{aligned}\right\} \qquad (\mathbb{P}_{\varkappa h})$$

3.3.2 Existence and convergence of discrete optimal shapes

In order to establish the existence and convergence results, we have to impose additional assumptions on the family of triangulations $\{\mathcal{T}_h(s_\varkappa)\}$, $h, \varkappa \to 0+$, which are listed below.

We will suppose that, for any $h, \varkappa > 0$ fixed, the system $\{\mathcal{T}_h(s_\varkappa)\}$, $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ consists of topologically equivalent triangulations, meaning that

(T1) the triangulation $\mathcal{T}_h(s_\varkappa)$ has the same number of nodes and the nodes still have the same neighbours for any $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$;

(T2) the positions of the nodes of $\mathcal{T}_h(s_\varkappa)$ depend solely and continuously on variations of the design nodes $\{A_{i-\frac{1}{2}}\}_{i=0}^{n+1}$.

For $h, \varkappa \to 0+$ we suppose that

(T3) the family $\{\mathcal{T}_h(s_\varkappa)\}$ is uniformly regular with respect to $h, \varkappa$ and $s_\varkappa \in \mathcal{U}_{ad}^\varkappa$: there is $\theta_0 > 0$ such that $\theta(h, s_\varkappa) \geq \theta_0$, $\forall h, \varkappa > 0$, $\forall s_\varkappa \in \mathcal{U}_{ad}^\varkappa$, where $\theta(h, s_\varkappa)$ is the minimal interior angle of all triangles from $\mathcal{T}_h(s_\varkappa)$.

Due to $(T1)$, one can easily show that $(\mathbb{P}_{\varkappa h})$ leads to the following nonlinear programming problem:

$$\min_{(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) \in \boldsymbol{U}_n \times \mathbb{R}^m} \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) \text{ subject to } \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}, \qquad (\mathbb{P}_n)$$

where $\mathcal{J}$, $\boldsymbol{R}$, $\boldsymbol{q}(\boldsymbol{\alpha})$ is the algebraic representation of J, $(\mathcal{P}_h(r_h s_\varkappa))$ and $(\boldsymbol{v}_h, p_h)$, respectively. It also follows that $m = N + M$, where N, M is the number of degrees of freedom for the velocity and the pressure, respectively.

Using the a priori estimates and limit passage similar as in Section 2.3, one can prove the following continuity of the control-to-state mapping $\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha})$:

Let $\boldsymbol{\alpha}_l \to \boldsymbol{\alpha}$, $l \to \infty$, where $\boldsymbol{\alpha}_l, \boldsymbol{\alpha} \in \boldsymbol{U}_n$, and let $\boldsymbol{q}(\boldsymbol{\alpha}_l) \in \mathbb{R}^m$ satisfy $\boldsymbol{R}(\boldsymbol{\alpha}_l, \boldsymbol{q}(\boldsymbol{\alpha}_l)) = \boldsymbol{0}$. Then there exists a $\boldsymbol{q}(\boldsymbol{\alpha}) \in \mathbb{R}^m$ and a subsequence (denoted by the same symbol) such that

$$\boldsymbol{q}(\boldsymbol{\alpha}_l) \to \boldsymbol{q}(\boldsymbol{\alpha}), \ l \to \infty$$

and $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}$.

Since $\boldsymbol{U}_n$ is compact, we immediately obtain the existence of a discrete optimal shape.

Problem $(\mathbb{P}_n)$ (and equivalently $(\mathbb{P}_{\varkappa h})$) has a solution.

As far as convergence is considered, it is possible to show two kinds of result. Firstly, for $\varkappa, h \to 0+$ solutions to the discrete state problems $(\mathcal{P}_h(r_h s_\varkappa))$ converge (passing eventually to a subsequence) to a solution of $(\mathcal{P}_w(\Omega))$, provided that $\Omega(s_\varkappa) \overset{\mathcal{O}}{\to} \Omega(\alpha)$:

For every sequence $\{(s_\varkappa, \boldsymbol{v}_h, p_h)\}_{\varkappa,h\to 0+}$, $(s_\varkappa, \boldsymbol{v}_h, p_h) \in \mathcal{A}_{\varkappa h}$ *such that* $\Omega(r_h s_\varkappa) \xrightarrow{\mathcal{O}} \Omega(\alpha)$ *there is a triplet* $(\alpha, \hat{\boldsymbol{v}}, \hat{p})$ *and a subsequence (denoted by the same symbol) such that*

$$\begin{aligned} E_{\Omega(r_h s_\varkappa)}\boldsymbol{v}_h &\rightharpoonup \hat{\boldsymbol{v}} && \text{in } W^{1,2}(\widehat{\Omega};\mathbb{R}^d),\\ \tilde{p}_h &\rightharpoonup \hat{p} && \text{in } L^2_0(\widehat{\Omega}),\ \ \varkappa, h \to 0+ \end{aligned}$$

and, in addition, $(\alpha, \hat{\boldsymbol{v}}_{|\Omega(\alpha)}, \hat{p}_{|\Omega(\alpha)}) \in \mathcal{A}$. *If the solution of* $(\mathcal{P}_w(\Omega))$ *is unique then the whole sequence converges in the sense mentioned above.*

The second convergence result ensures that optimal solutions to $(\mathbb{P}_{\varkappa h})$ converge (modulo subsequence) to the optimal solution of $(\mathbb{P})$, however under quite strong assumptions of uniqueness of states and continuity of J:

Let the solutions to $(\mathcal{P}_w(\Omega))$ *be unique for every* $\alpha \in \mathcal{U}_{ad}$ *and* J *be continuous. Then for every sequence* $\{(s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}_{\varkappa,h\to 0+}$ *of optimal triplets of* $(\mathbb{P}_{\varkappa h})$, $\varkappa, h \to 0+$ *there is a subsequence (denoted by the same symbol) such that*

$$\left.\begin{aligned} \Omega(r_h s^*_\varkappa) &\xrightarrow{\mathcal{O}} \Omega(\alpha^*), && \\ E_{\Omega(r_h s^*_\varkappa)}\boldsymbol{v}^*_h &\rightharpoonup \hat{\boldsymbol{v}}^* && \text{in } W^{1,2}(\widehat{\Omega}),\\ \tilde{p}^*_h &\rightharpoonup \hat{p}^* && \text{in } L^2_0(\widehat{\Omega}),\ \ \varkappa, h \to 0+, \end{aligned}\right\} \tag{3.5}$$

where $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha*)})$ *is an optimal triplet for* $(\mathbb{P})$. *In addition, any accumulation point of the sequence in the sense* (3.5) *possesses this property.*

3.3.3 Computation: gradient-based optimization and differentiation of cost function

The discrete shape optimization problem $(\mathbb{P}_n)$ is a mathematical programming problem that can be solved using standard optimization algorithms. Its characteristic property is that the evaluation of the cost function is usually far more time consuming (it requires to solve the state problem) than one step of the optimization algorithm. Depending on application, one may consider in principle two kinds of algorithms:

- Global optimization: The algorithms try to find the true minimizer, however at a high computational cost (usually, thousands of evaluations of the cost function are necessary). On the other hand, some evaluations can be done in parallel. This is reasonable often when there is no natural preference or starting point for the optimization process.

- Local (gradient-based) optimization: Here one tries to find a local minimum using true or approximate gradient (or eventually hessian). This approach requires significantly less evaluations, however the solution is only a local improvement of an initial guess that has to be provided. The gradient has to be either computed using quite sophisticated methods (solution of adjoint equation, differentiation of the algebraic system with respect to coordinates of mesh nodes) or approximated by difference quotients (inaccurate, time consuming). This is reasonable when the optimal solution is presumed to be a minor improvement of an initial design.

In what follows we shall comment on the gradient based approach. The evaluation of the cost function is done by the following chain:

$$\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha}) \mapsto \mathfrak{J}(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})).$$

For simplicity we assume that the first mapping is single valued, i.e. the solver of the state problem

$$\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0} \tag{3.6}$$

gives a unique solution. Differentiating (3.6) with respect to $\boldsymbol{\alpha}$ we get:

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}} + \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}} \nabla_{\boldsymbol{\alpha}} \boldsymbol{q} = \boldsymbol{0}, \text{ i.e. } \nabla_{\boldsymbol{\alpha}} \boldsymbol{q} = -\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}\right)^{-1} \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}. \tag{3.7}$$

Then the gradient of $\mathfrak{J}$ can be expressed as follows:

$$\begin{aligned} \nabla \mathfrak{J} &= \frac{\partial J}{\partial \boldsymbol{\alpha}} + (\nabla_{\boldsymbol{\alpha}} \boldsymbol{q})^{\top} \frac{\partial J}{\partial \boldsymbol{q}} \\ &\overset{(3.7)}{=} \frac{\partial J}{\partial \boldsymbol{\alpha}} - \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}\right)^{\top} \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}\right)^{-\top} \frac{\partial J}{\partial \boldsymbol{q}} \\ &= \frac{\partial J}{\partial \boldsymbol{\alpha}} - \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}\right)^{\top} \boldsymbol{p}(\boldsymbol{\alpha}), \end{aligned} \tag{3.8}$$

where the *adjoint state* $\boldsymbol{p}(\boldsymbol{\alpha})$ is the solution of the linearized problem

$$\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}\right)^{\top} \boldsymbol{p}(\boldsymbol{\alpha}) = \frac{\partial J}{\partial \boldsymbol{q}}. \tag{3.9}$$

Hence, for evaluation of $\nabla\mathfrak{J}$ one has to solve the adjoint equation (3.9), express the derivatives $\partial J/\partial\boldsymbol{\alpha}$, $\partial\boldsymbol{R}/\partial\boldsymbol{\alpha}$ and use (3.8). Computation of the partial derivatives with respect to $\boldsymbol{\alpha}$ is an elaborate and error-prone task. It is done either using algebraic sensitivity analysis or can be simplified with the aid of the automatic differentiation, where the computer code is implemented in such a way that every algebraic operation involves also the computation of the respective derivatives. For more details on automatic differentiation we refer to [22, 23].

3.4 Sensitivity analysis

Computation of gradient of cost function in numerical solution can be done in two ways:

- Discretize-then-differentiate. The state problem and the cost function is discretized and the true gradient of the discretized cost function is computed, as described in Section 3.3.3;

- Differentiate-then-discretize. First, the state problem and the cost function is differentiated with respect to shape. The resulting system is then discretized and its solution formally used in the formula for the gradient of the cost function.

The second approach is easier to implement since one does not need any algebraic sensitivity analysis, however the computed gradient of the cost function is only formal. In any case, precise characterization of the gradient of the cost function is useful on its own. We shall describe the approach based on the material derivative, which is similar to the concept of Lagrangean and Eulerian description in continuum mechanics.

Let $\boldsymbol{T}:\mathbb{R}^d\to\mathbb{R}^d$ be a smooth vector field, $\varepsilon>0$ and $x_\varepsilon := x+\varepsilon\boldsymbol{T}(x)$. Then the mapping $x\mapsto x_\varepsilon$ transforms a domain Ω onto $\Omega_\varepsilon := \{x_\varepsilon;\ x\in\Omega\}$, see Figure 3.3. The field $\boldsymbol{T}$ describes a direction of deformation of Ω which serves for the definition of derivatives of quantities depending on the domain. For a function u_ε defined in Ω_ε, $\varepsilon\ge 0$, we define the *material derivative*:

$$\dot{u}(x) := \lim_{\varepsilon\to 0}\frac{u_\varepsilon(x_\varepsilon)-u(x)}{\varepsilon} = \left.\frac{\mathrm{d}\,(u_\varepsilon(x_\varepsilon))}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0},\ x\in\Omega,$$

denoting $u := u_0$. By the chain rule of differentiation we get

$$\dot{u}(x) = \left.\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}(x) + \nabla u(x)\cdot\frac{\mathrm{d}x_\varepsilon}{\mathrm{d}\varepsilon} = u'(x)+\nabla u(x)\cdot\boldsymbol{T}(x),$$

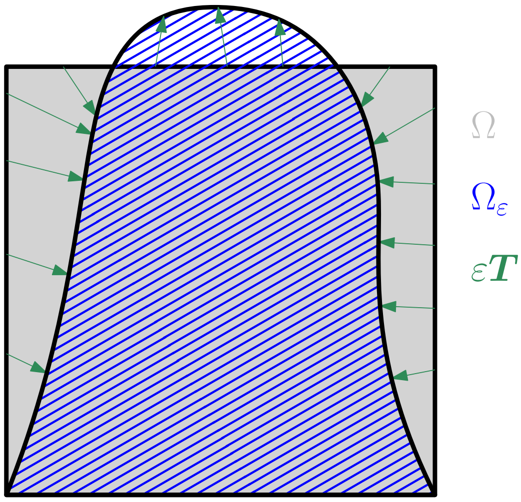

Figure 3.3: An example of a domain Ω and its deformation to Ω_ε using a field $\boldsymbol{T}$.

where u' is the *shape derivative*. The material derivative $\dot{u}$ is thus related to the shape derivative u' by the identity

$$\dot{u} = u' + \nabla u \cdot \boldsymbol{T}.$$

We note that the definition of the shape derivative requires u to be more regular. The directional shape derivative of a functional

$$F_\varepsilon := \int_{\Omega_\varepsilon} f_\varepsilon(x_\varepsilon)$$

is then expressed as

$$\mathrm{d}F(\Omega; \boldsymbol{T}) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F_\varepsilon \bigg|_{\varepsilon=0} = \int_\Omega \dot{f} + f \operatorname{div} \boldsymbol{T} = \int_\Omega f' + \int_{\partial\Omega} f \boldsymbol{T} \cdot \boldsymbol{n}. \tag{3.10}$$

In what follows we shall illustrate how to use this approach for expressing the shape gradient of a cost function depending on the solution of a state problem. Consider now the problem $(\mathcal{P}_w(\Omega))$. Differentiating the integrals in $(\mathcal{P}_w(\Omega_\varepsilon))$ according to (3.10) one obtains, after some effort, that the shape derivatives $(\boldsymbol{v}', p')$ satisfy the linearized problem:

$$\begin{aligned}
\operatorname{div} \boldsymbol{v}' &= 0 && \text{in } \Omega, && (3.11\text{a})\\
\operatorname{div}(\boldsymbol{v}' \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{v}') - \mu \Delta \boldsymbol{v}' + \nabla p' &= \boldsymbol{0} && \text{in } \Omega, && (3.11\text{b})\\
\boldsymbol{v}' &= -((\nabla \boldsymbol{v})\boldsymbol{n})(\boldsymbol{T} \cdot \boldsymbol{n}) && \text{on } \partial\Omega. && (3.11\text{c})
\end{aligned}$$

One could also work with material derivatives, but then the obtained expressions are much more complicated. Let us take the cost function

$$J_\varepsilon := \int_{\Omega_\varepsilon} |\nabla \boldsymbol{v}_\varepsilon|^2.$$

Then its shape gradient can be expressed as follows:

$$\mathrm{d}J(\Omega; \boldsymbol{T}) = \int_\Omega 2\nabla \boldsymbol{v} : \nabla \boldsymbol{v}' + \int_{\partial\Omega} |\nabla \boldsymbol{v}|^2 \boldsymbol{T} \cdot \boldsymbol{n}. \tag{3.12}$$

In order to avoid the shape derivative in the expression, we use the adjoint problem with the solution $(\boldsymbol{w}, q)$:

$$\operatorname{div} \boldsymbol{w} = 0 \qquad \text{in } \Omega, \tag{3.13a}$$

$$2(\mathbb{D}\boldsymbol{w})\boldsymbol{v} + \mu\Delta\boldsymbol{w} + \nabla q = 2\Delta\boldsymbol{v} \qquad \text{in } \Omega, \tag{3.13b}$$

$$\boldsymbol{w} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{3.13c}$$

Then, multiplying (3.13a), (3.13b) by p' and $\boldsymbol{v}'$, respectively, integrating by parts and using (3.11) we obtain:

$$\int_\Omega 2\nabla \boldsymbol{v} : \nabla \boldsymbol{v}' = \int_{\partial\Omega} \nabla \boldsymbol{v} : (\mu\nabla\boldsymbol{w} + q\mathbb{I} - 2\nabla\boldsymbol{v})\,(\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{T}\cdot\boldsymbol{n}.$$

Inserting this into (3.12) we arrive at

$$\mathrm{d}J(\Omega; \boldsymbol{T}) = \int_{\partial\Omega} \nabla \boldsymbol{v} : ((\mu\nabla\boldsymbol{w} + q\mathbb{I} - 2\nabla\boldsymbol{v})\,\boldsymbol{n}\otimes\boldsymbol{n} + \nabla\boldsymbol{v})\,\boldsymbol{T}\cdot\boldsymbol{n}. \tag{3.14}$$

From (3.14) we see that the shape gradient $\mathrm{d}J$ is supported on the boundary $\partial\Omega$ and depends on the normal component of the field $\boldsymbol{T}$. This quite natural property of the shape gradient holds for many functions and is the statement of the so-called structure theorem for shape functions.

The above computations using the shape derivative u' are done only formally. To make them rigorous, one has to identify the material derivative $\dot{u}$ first by reformulating $(\mathcal{P}_w(\Omega_\varepsilon))$ to the fixed domain Ω, showing Lipschitz estimates for the differences $(u_\varepsilon - u)/\varepsilon$ and passing to the limit. Then, expressing the shape gradient $\dot{J}$ as a volume integral which depends continuously and linearly on $\boldsymbol{T}$, one can pass to the shape derivative, provided that it is regular enough.

Chapter 4

Presented Works and Their Novelties

In this last chapter we present selected works documenting the author's contribution to mathematical fluid mechanics and shape optimization. The reprints of publications are divided into groups which are commented separately. Namely, we summarize results on:

- mathematical and numerical analysis of non-Newtonian fluids;
- applied shape optimization for nonlinear fluid models;
- shape optimization involving fluid models with slip boundary conditions;
- shape sensitivity analysis for non-Newtonian fluids.

4.1 Mathematical theory of piezoviscous fluids

It has been known for decades that the viscosity of a fluid can depend on the pressure and the shear-rate. While for water and many common fluids the dependence is negligible, in some areas such as tribology, glaciology or geology in general it may play a significant role. Lubricants in journal bearings are one example of fluids, where the viscosity can grow with the pressure even in an exponential way [6]. The mathematical theory for this class of models is so far limited to quite restrictive cases, where the growth with pressure has to be compensated by the decrease with the shear-rate.

When considering inner flows of incompressible fluids, a mathematical artifact is that the pressure is determined by the velocity up to an additive constant. For Navier-Stokes equations, the value of this constant is not important since it does not influence the velocity field. In the case of pressure-dependent viscosity it is however not true, namely choosing a wrong mean value of pressure yields wrong velocity. In addition, without fixing the pressure one cannot achieve uniqueness of solutions.

We present reprints of the works [32, 29] which bring the following new results:

- In the first paper we resolve the specific issue of fixing the pressure by physically relevant boundary conditions. In particular, certain outflow and filtration conditions that prescribe the pressure on a part of the boundary are presented. We prove the existence and uniqueness of weak solutions with these boundary conditions. The mathematical theory for this type of problems involves modifications due to the fact that test functions as well as the solution do not completely vanish on the boundary.

- In the second paper we study the finite-element approximation for the model without the convective term. We prove the convergence and the error estimates, and verify them by a numerical example. The error estimates are complicated by the nonlinear term containing the viscosity.

Reprints

- M. Lanzendörfer, J. Stebel. On Pressure Boundary Conditions for Steady Flows of Incompressible Fluids with Pressure and Shear Rate Dependent Viscosities. *Applications of Mathematics*, **56**(3):265-285, 2011.

- A. Hirn, M. Lanzendörfer, J. Stebel. Finite element approximation of flow of fluids with shear rate and pressure dependent viscosity. *IMA Journal of Numerical Analysis*, **32**(4):1604-1634, 2012.

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ON PRESSURE BOUNDARY CONDITIONS FOR STEADY FLOWS OF INCOMPRESSIBLE FLUIDS WITH PRESSURE AND SHEAR RATE DEPENDENT VISCOSITIES*

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Abstract. We consider a class of incompressible fluids whose viscosities depend on the pressure and the shear rate. Suitable boundary conditions on the traction at the inflow/outflow part of boundary are given. As an advantage of this, the mean value of the pressure over the domain is no more a free parameter which would have to be prescribed otherwise. We prove the existence and uniqueness of weak solutions (the latter for small data) and discuss particular applications of the results.

Keywords: existence, weak solutions, incompressible fluids, non-Newtonian fluids, pressure dependent viscosity, shear dependent viscosity, inflow/outflow boundary conditions, pressure boundary conditions, filtration boundary conditions

MSC 2010: 35Q35, 35J65, 76D03

1. Introduction

A well-known property of the Navier-Stokes equations describing the motion of an incompressible Newtonian fluid is that the fluid pressure is determined to within a constant. This degree of freedom does not play important role as far as only the pressure gradient is present in the equations of motion. Some generalizations of the Navier-Stokes equations, such as the equations for fluids with shear rate dependent viscosity share this property as well.

It has been observed that under some circumstances the fluid viscosity may depend significantly both on the shear rate and on the pressure. In such case the value of the

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pressure affects the whole solution of the equations. In previous theoretical studies, such as [10], [16], [26], the mean value of the pressure either over the whole domain or over its nontrivial subdomain was prescribed as one of the input parameters. A difficulty of this approach lies in the fact that the pressure mean value is not a proper quantity from the practical point of view, i.e. there is no hint on the value which should be prescribed for a particular application. The objective of this paper is to propose an alternative way of fixing the pressure, namely to use a suitable inflow/outflow boundary condition.

Let us demonstrate the idea on a simple example: Consider the Navier-Stokes equations and the Poiseuille flow in a 2D channel $(0, L) \times (0, 1)$ of length L and height 1, for which the velocity and the pressure are given by

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) &= (v_0 x_2(1 - x_2), 0), \quad v_0 \in \mathbb{R}, \\ p(\boldsymbol{x}) &= p_0 - 2\mu v_0 x_1, \quad p_0 \in \mathbb{R}. \end{aligned}$$

Here μ is the (constant) viscosity and $\frac{1}{4}v_0$ is the peak velocity in the channel centre. The parameter p_0 can be chosen arbitrarily and has no influence on the velocity. If we additionally prescribe a constant normal force h on the channel outlet $\{L\} \times (0, 1)$ by

$$-p + 2\mu \mathbf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n} = h, \tag{1.1}$$

where $\mathbf{D}(\boldsymbol{v})$ is the symmetric velocity gradient and $\boldsymbol{n}$ the unit outer normal to the boundary, then we automatically obtain $p_0 = 2\mu v_0 L - h$ and the pressure is fixed. We will show (see Section 4) that boundary conditions similar to (1.1) have the same effect on weak solutions to fluids with shear rate and pressure dependent viscosity.

In many applications, induced force is prescribed on a part of the boundary:

$$\mathbf{T}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}), \tag{1.2}$$

where $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$ denotes the Cauchy stress, $\boldsymbol{n}$ the outer normal to the boundary and $\boldsymbol{h}$ a given force. As a particular example, often a kind of natural outflow can be achieved in flow simulations by simply prescribing

$$\mathbf{T}\boldsymbol{n} = \mathbf{0};$$

this type of condition (usually referred to as the *do nothing* condition) is easy to use in numerical simulations and yields quite reliable results (see e.g. [20]).

Some existence analysis of the Navier-Stokes equations with the condition (1.2) is available: Local results (i.e. for small data or short time) were obtained e.g. in [24] and in [25] for stationary and for time dependent case, respectively. Global existence analysis is, however, an open problem because (1.2) does not prevent backward flow through the boundary and thus an uncontrolled amount of kinetic energy can be brought into the domain. In [23] the authors showed the existence of weak solutions to the variational inequality involving an explicit constraint imposed on the backward flows.

In this paper we will study boundary conditions involving a surface force depending on the velocity:

$$-\mathbf{T}\boldsymbol{n} = \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{v}), \tag{1.3}$$

where the assumptions on $\boldsymbol{b}$ are specified in Subsection 2.2. Important examples and their motivation are given in Section 5. We follow the approach used e.g. in [13], where

$$\boldsymbol{b} = \boldsymbol{h}(\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{n})^{-}\boldsymbol{v}$$

with $z^{-} := \max\{0, -z\}$ being the negative part of z. Namely, we restrict ourselves to such forms of $\boldsymbol{b}$ in (1.3) that expend all the kinetic energy brought in by the inflow, allowing us to establish standard energy estimates.

The paper is organized as follows. In Section 2 we specify the problem to be analyzed and state the main theorem. The existence and uniqueness of weak solutions is then proved in Section 3 and Section 4, respectively. Finally, Section 5 contains particular applications covered by the theory.

2. Definition of the problem and the main result

We investigate the system of PDEs

$$\left.\begin{aligned} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} + \nabla p &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega,$$

where

$$\mathbf{S} \equiv \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) = \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}). \tag{2.1}$$

Here $\boldsymbol{v}$, p, $\boldsymbol{f}$, $\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)$ is the velocity, the kinematic pressure, the body force and the kinematic viscosity, respectively. The equations describe the motion of an incompressible homogeneous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. The domain boundary consists of three measurable and disjoint parts: $\partial\Omega := \Gamma_D \cup \Gamma_1 \cup \Gamma_2$, on which we prescribe the boundary conditions

$$\boldsymbol{v} = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{2.2}$$

$$p\boldsymbol{n} - \mathbf{S}\boldsymbol{n} = \boldsymbol{b}_1(\boldsymbol{v}) \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\left.\begin{aligned} \boldsymbol{v} &= (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} \\ p - \mathbf{S}\boldsymbol{n}\cdot\boldsymbol{n} &= b_2(\boldsymbol{v}) \end{aligned}\right\} \qquad \text{on } \Gamma_2. \tag{2.4}$$

Throughout the paper we will assume that $\partial\Omega$, Γ_D, Γ_1, and Γ_2 are Lipschitz continuous. Further we will denote $\Gamma := \Gamma_1 \cup \Gamma_2$ and suppose that $|\Gamma_D| > 0$ and $|\Gamma| > 0$, i.e., the Dirichlet condition (2.2) and at least one of the conditions (2.3), (2.4) are present. Note that $|\Gamma_D| > 0$ is needed in order to guarantee the validity of Korn's inequality.

The equations governing the flow of an incompressible fluid with the viscosity depending on the pressure and the shear rate were subject to a number of recent studies. For more details on models of the type (2.1), we refer the reader to [16], [27], [29], [30]. Simple flows and numerical simulations are discussed in [21], [22]. In [9], [10], [26], issues concerning various boundary conditions were studied. In [8], [11], some further generalizations are provided. The proof of existence presented here derives from the one developed in [16], where the existence theory was established for steady flows subject to homogeneous Dirichlet boundary condition only.

2.1. Structural assumptions

The following assumptions on $\mathbf{S}$ are considered.

(A1) For a given $r \in (1,2)$, there exist positive constants C_1 and C_2 such that for all symmetric linear transformations $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{d\times d}$ and all $p \in \mathbb{R}$:

$$\begin{aligned} C_1(1+|\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2 &\leqslant \frac{\partial \mathbf{S}(p,\mathbf{D})}{\partial \mathbf{D}} \cdot (\mathbf{B}\otimes\mathbf{B}) \\ &\leqslant C_2(1+|\mathbf{D}|^2)^{(r-2)/2}|\mathbf{B}|^2, \end{aligned}$$

where $(\mathbf{B}\otimes\mathbf{B})_{ijkl} = \mathbf{B}_{ij}\mathbf{B}_{kl}$.

(A2) For all symmetric linear transformations $\mathbf{D} \in \mathbb{R}^{d\times d}$ and for all $p \in \mathbb{R}$:

$$\left|\frac{\partial \mathbf{S}(p,\mathbf{D})}{\partial p}\right| \leqslant \gamma_0(1+|\mathbf{D}|^2)^{(r-2)/4} \leqslant \gamma_0,$$

with $\gamma_0 > 0$ to be specified later.

For particular examples see the references given above.

We state some useful inequalities following from (A1) and (A2). First, it was proved in [28], Lemma 1.19 of Chapter 5, that for every $p \in \mathbb{R}$ and $\mathbf{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$|\mathbf{S}(p,\mathbf{D}) : \mathbf{D}| \leqslant \frac{C_2}{r-1}(1+|\mathbf{D}|)^{r-1}, \tag{2.5}$$

$$\mathbf{S}(p,\mathbf{D}) : \mathbf{D} \geqslant C_3 \min\{|\mathbf{D}|^2, |\mathbf{D}|^r\}, \tag{2.6}$$

with $C_3 = C_3(r, C_1)$. Next, defining

$$I^{1,2} := |\mathbf{D}^1 - \mathbf{D}^2|^2 \int_0^1 (1 + |\mathbf{D}^1 + s(\mathbf{D}^2 - \mathbf{D}^1)|^2)^{(r-2)/2}\,\mathrm{d}s, \tag{2.7}$$

one can show that (see e.g. Lemma 1.4 in [10])

$$\frac{C_1}{2} I^{1,2} \leqslant (\mathbf{S}(p^1,\mathbf{D}^1) - \mathbf{S}(p^2,\mathbf{D}^2)) : (\mathbf{D}^1 - \mathbf{D}^2) + \frac{\gamma_0^2}{2C_1}|p^1 - p^2|^2, \tag{2.8}$$

$$|\mathbf{S}(p^1,\mathbf{D}^2) - \mathbf{S}(p^2,\mathbf{D}^2)| \leqslant C_2\sqrt{I^{1,2}} + \gamma_0|p^1 - p^2|, \tag{2.9}$$

$$\|1 + |\mathbf{D}^1| + |\mathbf{D}^2|\|_r^{r-2}\|\mathbf{D}^1 - \mathbf{D}^2\|_r^2 \leqslant \int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x}. \tag{2.10}$$

We use the inequality (2.6) in the form

Lemma 2.1. *Assume that* (A1), (A2) *are fulfilled. Let*[1] $\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)$ *and* $F \geqslant 0$. *Then*

$$\begin{aligned}\int_\Omega \mathbf{S}(p,\mathbf{D}(\boldsymbol{u})) : \mathbf{D}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - F\|\mathbf{D}(\boldsymbol{u})\|_r \\ \geqslant C_4\min\{\|\mathbf{D}(\boldsymbol{u})\|_r^2, \|\mathbf{D}(\boldsymbol{u})\|_r^r\} - C_5(F^2 + F^{r'}),\end{aligned} \tag{2.11}$$

where $r' := r/(r-1)$, *and the constants* $C_4, C_5 > 0$ *depend solely on* Ω, r *and* C_3.

Proof. Define $\hat{\Omega} := \{\boldsymbol{x} \in \Omega\colon\ |\mathbf{D}(\boldsymbol{u})| > 1\}$ and $\underline{\Omega} := \Omega \setminus \hat{\Omega}$. Then (2.6) gives

$$\begin{aligned}\int_\Omega \mathbf{S}(p,\mathbf{D}(\boldsymbol{u})) : \mathbf{D}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - F\|\mathbf{D}(\boldsymbol{u})\|_r \\ \geqslant C_3\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2 + C_3\|\mathbf{D}(\boldsymbol{u})|_{\hat{\Omega}}\|_r^r - F(\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r + \|\mathbf{D}(\boldsymbol{u})|_{\hat{\Omega}}\|_r).\end{aligned}$$

Hölder's inequality $\|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r^2 \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2|\underline{\Omega}|^{\frac{1}{2}(2-r)\cdot 2/r} \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_2^2|\Omega|^{\frac{1}{2}(2-r)\cdot 2/r}$, Young's inequality and the fact that $\frac{1}{2}\min\{\|\mathbf{D}(\boldsymbol{u})\|_r^2, \|\mathbf{D}(\boldsymbol{u})\|_r^r\} \leqslant \|\mathbf{D}(\boldsymbol{u})|_{\underline{\Omega}}\|_r^2 + \|\mathbf{D}(\boldsymbol{u})|_{\hat{\Omega}}\|_r^2$ then lead to (2.11). □

[1] In this paper, $\mathrm{W}^{1,r}(\Omega)$, $\mathrm{W}_0^{1,r}(\Omega)$, $\mathrm{L}^q(\Omega)$, $\mathrm{L}_0^q(\Omega)$ stand for the Sobolev space, its subspace of functions with zero trace, the Lebesgue space, and its subspace of functions with zero mean value, respectively. Bold symbols denote the vector counterparts of these spaces. The norms of $\mathrm{W}^{1,r}(\Omega)$, $\mathrm{L}^q(\Omega)$ will be denoted by $\|\cdot\|_{1,r}$, $\|\cdot\|_q$ respectively.

2.2. Boundary assumptions

Concerning the boundary conditions (2.3)–(2.4), we define

$$\langle \boldsymbol{b}(\boldsymbol{v}), \boldsymbol{\varphi}\rangle := \langle \boldsymbol{b}_1(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \langle b_2(\boldsymbol{v}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2}$$

and assume the following conditions:

(B1) With some $\gamma_1 \in \langle 3, r^*)$, the mapping

$$\boldsymbol{b}_1(\cdot)\colon\ \mathbf{L}^{\gamma_1}(\Gamma_1) \to \mathbf{L}^{\gamma_1}(\Gamma_1)^* \tag{2.12}$$

is continuous and bounded. Here $r^* := (d-1)r/(d-r)$ denotes the exponent for which $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{r^*}(\partial\Omega)$.

(B2) With some $\beta_1 \geqslant 0$,

$$\langle \boldsymbol{b}_1(\boldsymbol{u}), \boldsymbol{u}\rangle_{\Gamma_1} \geqslant -\frac{1}{2}\int_{\Gamma_1} (\boldsymbol{u}\cdot\boldsymbol{n})|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} - \beta_1\|\boldsymbol{u}\|_{\gamma_1,\Gamma_1} \tag{2.13}$$

for all $\boldsymbol{u} \in \mathbf{L}^{\gamma_1}(\Gamma_1)$.

(B3) With some $\gamma_2 \geqslant 3$, the mapping

$$b_2(\cdot)\colon\ \mathrm{L}^{\gamma_2}(\Gamma_2) \to \mathrm{L}^{\gamma_2}(\Gamma_2)^* \tag{2.14}$$

is continuous and bounded.

(B4) With some $\beta_2 \geqslant 0$ and $\underline{\beta_2} > 0$,

$$\langle b_2(\boldsymbol{u}\cdot\boldsymbol{n}), \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\Gamma_2} \geqslant -\frac{1}{2}\int_{\Gamma_2} (\boldsymbol{u}\cdot\boldsymbol{n})|\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + \underline{\beta_2}\|\boldsymbol{u}\|^{\gamma_2}_{\gamma_2,\Gamma_2} - \beta_2 \tag{2.15}$$

for all $\boldsymbol{u} \in \mathbf{L}^{\gamma_2}(\Gamma_2)$.

(B5) With some continuous function $m\colon \mathbb{R}^+ \to \mathbb{R}^+$, where $\lim_{x\searrow 0} m(x) = 0$, b_2 is uniformly[2] monotone:

$$\langle b_2(w) - b_2(z), w - z\rangle_{\Gamma_2} \geqslant m(\|w-z\|_{\gamma_2,\Gamma_2}) \tag{2.16}$$

for all $w \neq z \in \mathrm{L}^{\gamma_2}(\Gamma_2)$.

Additionally, in order to prove the uniqueness of solutions we will require that the following stronger conditions hold:

[2] For the sake of simplicity, the uniform monotonicity is assumed here. The readers can verify themselves that the monotonicity of b_2 would also allow to show the existence of a weak solution, with help of the Minty trick.

(B6) With some $\lambda_1 > 0$ and $K_1 > 0$ (to be specified later),

$$\|\boldsymbol{b}_1(\boldsymbol{u}^1) - \boldsymbol{b}_1(\boldsymbol{u}^2)\|_{\gamma_1',\Gamma_1} \leqslant \lambda_1 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{\gamma_1,\Gamma_1} \tag{2.17}$$

for all $\boldsymbol{u}^1, \boldsymbol{u}^2 \in \mathbf{L}^{\gamma_1}(\Gamma_1)$, $\|\boldsymbol{u}^i\|_{\gamma_1,\Gamma_1} \leqslant K_1$, $i = 1, 2$.

(B7) With some $\lambda_2 > 0$ and $K_2 > 0$ (to be specified later),

$$\|b_2(\boldsymbol{u}^1 \cdot \boldsymbol{n}) - b_2(\boldsymbol{u}^2 \cdot \boldsymbol{n})\|_{1,\Gamma_2} \leqslant \lambda_2 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{r^*,\Gamma_2} \tag{2.18}$$

for all $\boldsymbol{u}^1, \boldsymbol{u}^2 \in \mathrm{L}^{\gamma_2}(\Gamma_2)$, $\|\boldsymbol{u}^i\|_{\gamma_2,\Gamma_2} \leqslant K_2$, $i = 1, 2$.

2.3. Weak formulation

We define the following function spaces:

$$\begin{aligned}
\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) &:= \{\boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega);\ \operatorname{tr}\boldsymbol{v}|_{\Gamma_D} = \mathbf{0},\ \operatorname{tr}\boldsymbol{v}|_{\Gamma_2} = (\operatorname{tr}\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} \in \mathrm{L}^{\gamma_2}(\Gamma_2)\},\\
\mathbf{W}^{1,r}_{\mathrm{b.c.,div}}(\Omega) &:= \{\boldsymbol{v} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega\}.
\end{aligned}$$

Note that, due to embedding, $\boldsymbol{v} \in \mathbf{W}^{1,r}(\Omega)$ implies $\boldsymbol{v} \in \mathbf{L}^{\gamma_1}(\Gamma_1)$. Given $\boldsymbol{f} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)^*$, we consider the following weak formulation:

Definition 2.2 (Problem (P)). A pair $(\boldsymbol{v}, p) \in \mathbf{W}^{1,r}_{\mathrm{b.c.,div}}(\Omega) \times \mathrm{L}^{r'}(\Omega)$ is called a weak solution of Problem (P) if and only if

$$\begin{aligned}
\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}&\\
- \int_\Omega p \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}), \boldsymbol{\varphi}\rangle &= \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle
\end{aligned} \tag{2.19}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$.

We close this subsection by recalling the properties of the Bogovskiĭ operator (see [32] or [1], [3] for the reference) and by stating its corollary.

Lemma 2.3 (Bogovskiĭ's operator; [32], Lemma 3.17). *Let $1 < q < \infty$. Then there exists a continuous linear operator $\mathcal{B}\colon \mathrm{L}^q_0(\Omega) \to \mathbf{W}^{1,q}_0(\Omega)$ such that for all $f \in \mathrm{L}^q_0(\Omega)$*

$$\begin{cases} \operatorname{div}(\mathcal{B}f) = f \quad \text{a.e. in } \Omega,\\ \|\mathcal{B}f\|_{1,q} \leqslant C_{\mathrm{div}}(\Omega, q)\|f\|_q. \end{cases} \tag{2.20}$$

Lemma 2.4. *Let* $q \in (1,\infty)$, $s \in \langle 1,\infty\rangle$. *Then there exists a continuous bounded linear operator* $\tilde{\mathcal{B}}\colon \mathrm{L}^q(\Omega) \to \mathbf{W}^{1,q}_{\mathrm{b.c.}}(\Omega)$ *such that for all* $f \in \mathrm{L}^q(\Omega)$

$$\begin{cases} \operatorname{div}(\tilde{\mathcal{B}}f) = f \ \text{ a.e. in } \Omega, \\ \|\tilde{\mathcal{B}}f\|_{1,q} \leqslant \tilde{C}_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,q)\|f\|_q, \\ \|\tilde{\mathcal{B}}f\|_{s,\Gamma} \leqslant C'_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,s)|\int_\Omega f|. \end{cases} \tag{2.21}$$

P r o o f. Let us take an arbitrary function $\boldsymbol{\xi} \in \mathcal{C}^\infty(\overline{\Omega})^d$ such that $\boldsymbol{\xi}|_{\Gamma_D} = \mathbf{0}$, $\boldsymbol{\xi}|_{\Gamma_2} = (\boldsymbol{\xi}\cdot\boldsymbol{n})\boldsymbol{n}$ and $\int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 1$. Then for any $f \in \mathrm{L}^q(\Omega)$ we define $\tilde{\mathcal{B}}(f) := \mathcal{B}\big(f - \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\operatorname{div}\boldsymbol{\xi}\big) + \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\boldsymbol{\xi}$. Since $\tilde{\mathcal{B}}(f)|_{\partial\Omega} = \big(\int_\Omega f\,\mathrm{d}\boldsymbol{x}\big)\boldsymbol{\xi}$, we have that $\tilde{\mathcal{B}}(f) \in \mathbf{W}^{1,q}_{\mathrm{b.c.}}(\Omega)$. It is then easy to verify with help of Lemma 2.3 that such choice meets the statement (2.21). □

2.4. Main result

Theorem 2.5 (Well-posedness of (P))**.**
Let $\boldsymbol{f} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)^*$ *and assume that* (A1)–(A2) *hold for the viscosity,* (B1)–(B5) *hold for the boundary data, with*

$$\frac{3d}{d+2} < r < 2 \quad \text{and} \quad \gamma_0 < \frac{1}{\tilde{C}_{\mathrm{div}}(\Omega,\Gamma_1,\Gamma_2,2)}\,\frac{C_1}{C_1+C_2}. \tag{2.22}$$

Then

(i) *there exists a weak solution to* (P)*;*

(ii) *for any weak solution* $(\boldsymbol{v},p)$ *of* (P)*, the velocity* $\boldsymbol{v}$ *satisfies the estimate*

$$\|\boldsymbol{v}\|_{1,r} + \|\boldsymbol{v}\|_{\gamma_2,\Gamma_2} \leqslant K, \tag{2.23}$$

where $K \searrow 0$ *whenever* $(\|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*}, \beta_1, \beta_2) \searrow \mathbf{0}$*, the other problem data being fixed;*

(iii) *if additionally* (B6), (B7) *are satisfied and if* K *and* λ_1, λ_2 *are small enough, then the weak solution to* (P) *is unique.*

3. The existence of a weak solution

The proof of (i) has the same structure as the proof given in [16] for the problem with the homogeneous Dirichlet boundary condition on $\partial\Omega$: In 3.1, we define an approximate problem (P^ε), derive energy estimates and show the existence of a weak solution to (P^ε) via Galerkin approximations. Also, (ii) follows from the estimates derived in here. In 3.2, we show estimates for the pressure p^ε uniform with respect to ε. This allows us to find sequences $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$, $\varepsilon_n \searrow 0$, weakly converging to a limit $(\boldsymbol{v}, p)$. In 3.3, the strong convergence of p^{ε_n} and $\mathbf{D}(\boldsymbol{v}^{\varepsilon_n})$ is shown and $(\boldsymbol{v}, p)$ is identified as the weak solution to problem (P).

3.1. Approximate problem (P^ε)

We relax the incompressibility constraint and look for a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon) \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) \times W^{1,2}(\Omega)$ satisfying

$$\text{(3.1)}\quad \varepsilon \int_\Omega \nabla p^\varepsilon \cdot \nabla \xi \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^\varepsilon \xi \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon) \xi \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \xi \in W^{1,2}(\Omega),$$

together with

$$\text{(3.2)}\quad \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega).$$

Note that, contrary to the case studied in [16], equation (3.1) does not determine the mean value of the pressure $\frac{1}{|\Omega|} \int_\Omega p^\varepsilon \, \mathrm{d}\boldsymbol{x}$. This is a consequence of the fact that $\boldsymbol{v}^\varepsilon \cdot \boldsymbol{n}|_\Gamma$ is not prescribed.

We show that $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ can be found as a limit of the Galerkin approximations $(\boldsymbol{v}^N, p^N)$ defined as

$$p^N := \sum_{k=1}^N c_k^N \alpha_k \quad \text{and} \quad \boldsymbol{v}^N := \sum_{k=1}^N d_k^N \boldsymbol{a}_k \qquad \text{for } N = 1, 2, \ldots,$$

where $\{\alpha_k\}_{k=1}^\infty$ and $\{\boldsymbol{a}_k\}_{k=1}^\infty$ are bases of $W^{1,2}(\Omega)$ and $\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$, respectively, and where $\boldsymbol{c}^N := (c_1^N, \ldots, c_N^N)$ and $\boldsymbol{d}^N := (d_1^N, \ldots, d_N^N)$ solve the algebraic system

$$\text{(3.3a)}\quad \varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha_k \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^N \alpha_k \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^N) \alpha_k \, \mathrm{d}\boldsymbol{x} = 0, \quad k = 1, \ldots, N,$$

$$\text{(3.3b)}\quad \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{a}_l \, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^N)(\boldsymbol{v}^N \cdot \boldsymbol{a}_l) \, \mathrm{d}\boldsymbol{x} - \int_\Omega p^N \operatorname{div}(\boldsymbol{a}_l) \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{a}_l) \, \mathrm{d}\boldsymbol{x} + \big\langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{a}_l \big\rangle = \langle \boldsymbol{f}, \boldsymbol{a}_l \rangle, \quad l = 1, \ldots, N.$$

Multiplying the kth equation in (3.3a) by c_k^N and the lth equation in (3.3b) by d_l^N and summing for $k, l = 1, \ldots, N$, we obtain

$$\text{(3.4)} \qquad \varepsilon\|p^N\|_{1,2}^2 + \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{v}^N \,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^N)|\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N) \,\mathrm{d}\boldsymbol{x} + \left\langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{v}^N \right\rangle = \left\langle \boldsymbol{f}, \boldsymbol{v}^N \right\rangle.$$

Using Green's theorem, we observe that

$$\text{(3.5)} \qquad \int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N) \cdot \boldsymbol{v}^N \,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^N)|\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Gamma (\boldsymbol{v}^N \cdot \boldsymbol{n})|\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x}.$$

Moreover, from (2.13) and (2.15) it follows that

$$\frac{1}{2}\int_\Gamma (\boldsymbol{v}^N \cdot \boldsymbol{n})|\boldsymbol{v}^N|^2 \,\mathrm{d}\boldsymbol{x} + \left\langle \boldsymbol{b}(\boldsymbol{v}^N), \boldsymbol{v}^N \right\rangle \geqslant \underline{\beta_2}\|\boldsymbol{v}^N\|_{\gamma_2,\Gamma_2}^{\gamma_2} - \beta_1\|\boldsymbol{v}^N\|_{\gamma_1,\Gamma_1} - \beta_2,$$

and thus

$$\varepsilon\|p^N\|_{1,2}^2 + \underline{\beta_2}\|\boldsymbol{v}^N\|_{\gamma_2,\Gamma_2}^{\gamma_2} + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N) \,\mathrm{d}\boldsymbol{x} \leqslant \|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*}\|\boldsymbol{v}^N\|_{1,r} + \beta_1\|\boldsymbol{v}^N\|_{\gamma_1,\Gamma_1} + \beta_2.$$

Using (2.11), Korn's inequality, and the embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$ we finally arrive at

$$\text{(3.6)} \qquad \varepsilon\|p^N\|_{1,2}^2 + \underline{\beta_2}\|\boldsymbol{v}^N\|_{\gamma_2,\Gamma_2}^{\gamma_2} + C_4 \min\{\|\mathbf{D}(\boldsymbol{v}^N)\|_r^2, \|\mathbf{D}(\boldsymbol{v}^N)\|_r^r\} \leqslant K.$$

Here and in what follows, $C > 0$ and $K > 0$ stand for generic constants, independent of N and ε. In addition, $K \searrow 0$ whenever the problem data $\|\boldsymbol{f}\|_{W^{1,r}_{\mathrm{b.c.}}(\Omega)^*}$, β_1, and β_2 tend to zero (while the other data are fixed). From (3.6) it directly follows that

$$\text{(3.7)} \qquad \|\boldsymbol{v}^N\|_{1,r} \leqslant K.$$

Estimates (3.6) and (3.7) imply, with help of the Brouwer fixed point theorem, the solvability of (3.3). Using (2.5) we obtain the estimate

$$\|\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N))\|_{r'} \leqslant C.$$

Due to this and the boundedness of b_2, there is a subsequence of $\{(\boldsymbol{v}^N, p^N)\}$ (denoted by the same symbol) and a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ such that

$$\text{(3.8)} \qquad \begin{cases} \boldsymbol{v}^N \rightharpoonup \boldsymbol{v}^\varepsilon & \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^{\gamma_2}(\Gamma_2), \\ p^N \rightharpoonup p^\varepsilon & \text{weakly in } \mathrm{W}^{1,2}(\Omega), \\ \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \rightharpoonup \overline{\mathbf{S}^\varepsilon} & \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d\times d}, \\ b_2(\boldsymbol{v}^N) \rightharpoonup \overline{b_2^\varepsilon} & \text{weakly in } \mathrm{L}^{\gamma_2'}(\Gamma_2). \end{cases}$$

Moreover, the compact embeddings yield

$$(3.9)\qquad \begin{cases} p^N \to p^\varepsilon & \text{strongly in } \mathrm{L}^2(\Omega), \\ \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon & \text{strongly in } \mathbf{L}^s(\Omega) \text{ for all } s\colon 1 \leqslant s < \dfrac{rd}{d-r}, \\ \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon & \text{strongly in } \mathbf{L}^{\gamma_1}(\Gamma_1). \end{cases}$$

The fact that $r > 3d/(d+2)$, $(3.8)_1$, and (3.9) are sufficient to show that

$$\int_\Omega \operatorname{div}(\boldsymbol{v}^N \otimes \boldsymbol{v}^N)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^N)(\boldsymbol{v}^N\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$. Thus, we can pass to the limit in (3.3) and obtain (3.1) together with

$$(3.10)\qquad \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}_1(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \langle \overline{b_2^\varepsilon}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega).$$

Next, from inequality (2.8) with $p^1 := p^N$ and $p^2 := p^\varepsilon$ (and analogously for $\boldsymbol{v}^1$, $\boldsymbol{v}^2$), (2.10) and $\|\mathbf{D}(\boldsymbol{v}^\varepsilon)\|_r \leqslant \liminf_{N\to\infty} \|\mathbf{D}(\boldsymbol{v}^N)\|_r \leqslant C$ it follows that

$$(3.11)\qquad C\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(v^\varepsilon)\|_r^2 \leqslant \int_\Omega [\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) - \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))] : (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}^\varepsilon))\,\mathrm{d}\boldsymbol{x} + \frac{\gamma_0^2}{2C_1}\|p^N - p^\varepsilon\|_2^2.$$

Similarly to [16], we prove the strong convergence of $\mathbf{D}(\boldsymbol{v}^N)$. Using (3.11), (2.16), and letting $N \to \infty$ we observe (due to (3.8)) that

$$\limsup_{N\to\infty}(\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}^\varepsilon)\|_r^2 + m(\|\boldsymbol{v}^N - \boldsymbol{v}^\varepsilon\|_{\gamma_2,\Gamma_2})) \leqslant \limsup_{N\to\infty}\left(\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N)\,\mathrm{d}\boldsymbol{x} + \langle b_2(\boldsymbol{v}^N\cdot\boldsymbol{n}), \boldsymbol{v}^N\cdot\boldsymbol{n}\rangle_{\Gamma_2}\right) - \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{v}^\varepsilon)\,\mathrm{d}\boldsymbol{x} - \langle \overline{b_2^\varepsilon}, \boldsymbol{v}^\varepsilon\cdot\boldsymbol{n}\rangle_{\Gamma_2}.$$

This can be further estimated from above, with help of (3.4), (3.9), $\liminf_{N\to\infty} \|p^N\|_{1,2} \geqslant \|p^\varepsilon\|_{1,2}$, (3.1), and (3.10), by

$$\begin{aligned}\langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon\rangle &- \langle \boldsymbol{b}_1(\boldsymbol{v}^\varepsilon), \boldsymbol{v}^\varepsilon\rangle_{\Gamma_1} - \varepsilon\|p^\varepsilon\|_{1,2}^2 - \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)\cdot \boldsymbol{v}^\varepsilon \,\mathrm{d}\boldsymbol{x}\\ &+ \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)|\boldsymbol{v}^\varepsilon|^2 \,\mathrm{d}\boldsymbol{x} - \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{v}^\varepsilon)\,\mathrm{d}\boldsymbol{x} - \left\langle \overline{b_2^\varepsilon}, \boldsymbol{v}^\varepsilon\cdot \boldsymbol{n}\right\rangle_{\Gamma_2} = 0.\end{aligned}$$

Therefore, and due to $(3.9)_1$, we have the almost everywhere convergence

$$\mathbf{D}(\boldsymbol{v}^N) \to \mathbf{D}(\boldsymbol{v}^\varepsilon) \text{ a.e. in } \Omega, \quad \boldsymbol{v}^N \to \boldsymbol{v}^\varepsilon \text{ a.e. on } \Gamma_2 \quad \text{and} \quad p^N \to p^\varepsilon \text{ a.e. in } \Omega.$$

Vitali's theorem and the continuity (2.14) of $b_2(\cdot)$ allow us to identify the limits as

$$\begin{gathered}\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x},\\ \left\langle b_2(\boldsymbol{v}^N\cdot \boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2} \to \langle b_2(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \left\langle \overline{b_2^\varepsilon}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2}\end{gathered}$$

for every $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$.

3.2. Uniform estimates for the pressure p^ε and the weak convergence

For any pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ which solves (3.1) and (3.2) we can obtain the same energy estimates as in 3.1:

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + \|\boldsymbol{v}^\varepsilon\|_{\gamma_2,\Gamma_2}^{\gamma_2} + \|\boldsymbol{v}^\varepsilon\|_{1,r} \leqslant K \quad \text{and} \quad \|\mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))\|_{r'} \leqslant C. \tag{3.12}$$

Let us recall Lemma 2.4 and test (3.2) with $\boldsymbol{\varphi}^\varepsilon := \tilde{\mathcal{B}}(|p^\varepsilon|^{r'-2}p^\varepsilon)$. Note that $\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} \leqslant \tilde{C}_{\text{div}}(\Omega, \Gamma_1, \Gamma_2, r)\|p^\varepsilon\|_{r'}^{r'/r}$ and $\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma_2,\Gamma_2} \leqslant C'_{\text{div}}(\Omega, \Gamma_1, \Gamma_2\gamma_2)\|p^\varepsilon\|_{r'/r}^{r'/r}$. Then, using (2.5), Hölder's inequality, (2.12), (2.14), the embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$, and at last the estimate (3.12), we get

$$\begin{aligned}\|p^\varepsilon\|_{r'}^{r'} &= \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon\otimes\boldsymbol{v}^\varepsilon)\cdot\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^\varepsilon), \boldsymbol{\varphi}^\varepsilon\rangle - \langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon\rangle\\ &\leqslant C\|\boldsymbol{v}^\varepsilon\|_{1,r}^2\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \frac{C_2}{r-1}\|1+|\mathbf{D}(\boldsymbol{v}^\varepsilon)|\|_r^{r-1}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|\boldsymbol{f}\|_{W^{1,r}_{\text{b.c.}}(\Omega)^*}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r}\\ &\quad + C\|\boldsymbol{b}_1(\boldsymbol{v}^\varepsilon)\|_{\gamma_1',\Gamma_1}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|b_2(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{n})\|_{\gamma_2',\Gamma_2}\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma_2,\Gamma_2}\\ &\leqslant C\|p^\varepsilon\|_{r'}^{r'/r}.\end{aligned}$$

Since $r > 1$, this implies

$$\|p^\varepsilon\|_{r'} \leqslant C. \tag{3.13}$$

Again, we find a sequence $\varepsilon_n \searrow 0$ and a pair $(\boldsymbol{v}, p)$ such that

$$
(3.14) \qquad \begin{cases}
\boldsymbol{v}^{\varepsilon_n} \rightharpoonup \boldsymbol{v} & \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^{\gamma_2}(\Gamma_2), \\
p^{\varepsilon_n} \rightharpoonup p & \text{weakly in } \mathrm{L}^{r'}(\Omega), \\
\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) \rightharpoonup \overline{\mathbf{S}} & \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d\times d}, \\
b_2(\boldsymbol{v}^{\varepsilon}) \rightharpoonup \overline{b_2} & \text{weakly in } \mathrm{L}^{\gamma_2'}(\Gamma_2), \\
\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} & \text{strongly in } \mathbf{L}^{\gamma_1}(\Gamma_1), \\
\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} & \text{strongly in } \mathbf{L}^{s}(\Omega) \text{ for all } s\colon\ 1 \leqslant s < \dfrac{dr}{d-r}.
\end{cases}
$$

Clearly, due to (3.12), $\boldsymbol{v}$ satisfies (ii) of Theorem 2.5. Note that $(3.14)_1$ and (3.12) together with (3.1) yield

$$
(3.15) \qquad \operatorname{div} \boldsymbol{v} = 0 \quad \text{a.e. in } \Omega.
$$

We can then pass to the limit in (3.2), obtaining

$$
(3.16) \quad \int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}
\\ + \langle \boldsymbol{b}_1(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_{\Gamma_1} + \left\langle \overline{b_2}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega).
$$

Finally, we use Vitali's theorem and the continuity of $b_2(\cdot)$ again, to show that

$$
\int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = \int \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x},
\\ \langle b_2(\boldsymbol{v}^{\varepsilon}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} \to \langle b_2(\boldsymbol{v}\cdot\boldsymbol{n}), \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_{\Gamma_2} = \left\langle \overline{b_2}, \boldsymbol{\varphi}\cdot\boldsymbol{n}\right\rangle_{\Gamma_2}
$$

for all $\boldsymbol{\varphi}\in\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$. In order to do so, we prove the convergences

$$
(3.17) \qquad \mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) \to \mathbf{D}(\boldsymbol{v}) \text{ a.e. in } \Omega, \quad \boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} \text{ a.e. on } \Gamma_2,
\\ \text{and} \quad p^{\varepsilon_n} \to p \text{ a.e. in } \Omega,
$$

in the next subsection.

3.3. The almost everywhere convergence

Let us rewrite inequality (2.8) in the form

$$
Y^n := \int_\Omega\int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) + s(\mathbf{D}(\boldsymbol{v}) - \mathbf{D}(\boldsymbol{v}^{\varepsilon_n}))|^2)^{(r-2)/2} |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x},
\\ \frac{C_1}{2} Y^n \leqslant \int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : (\mathbf{D}(\boldsymbol{v}^{\varepsilon_n} - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x} + \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2.
$$

Taking $\boldsymbol{\varphi} := \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}$ in (3.2), $\xi := p^{\varepsilon_n}$ in (3.1), using (3.14), (3.15), and taking $\boldsymbol{\varphi} := \boldsymbol{v}$ in (3.16), we observe that

$$\begin{aligned}\limsup_{\varepsilon_n \searrow 0} \bigg(\int_\Omega & [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : (\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v}))\, \mathrm{d}\boldsymbol{x} \\ &+ \langle b_2(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n}) - b_2(\boldsymbol{v} \cdot \boldsymbol{n}), (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \cdot \boldsymbol{n} \rangle_{\Gamma_2} \bigg) \\ &= \limsup_{\varepsilon_n \searrow 0} \bigg(\int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})\, \mathrm{d}\boldsymbol{x} + \langle b_2(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n}), \boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{n} \rangle_{\Gamma_2} \bigg) \\ &\quad - \int_\Omega \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} - \left\langle \overline{b_2}, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\Gamma_2} \leqslant 0,\end{aligned}$$

which together with (2.16) yields (denoting by $o(1)$ a sequence vanishing as $\varepsilon_n \searrow 0$)

$$m(\|\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\|_{\gamma_2, \Gamma_2}) + \frac{C_1}{2} Y^n \leqslant \frac{\gamma_0^2}{2C_1} \|p^{\varepsilon_n} - p\|_2^2 + o(1). \tag{3.18}$$

Next, we set $\boldsymbol{\varphi}^n := \tilde{\mathcal{B}}(p^{\varepsilon_n} - p)$, $\|\boldsymbol{\varphi}^n\|_{1,2} \leqslant \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) \|p^{\varepsilon_n} - p\|_2$. Note that since $(p^{\varepsilon_n} - p) \rightharpoonup 0$ weakly in $\mathrm{L}^{r'}(\Omega)$, it follows that $\boldsymbol{\varphi}^n \rightharpoonup 0$ weakly in $\mathbf{W}^{1,r}(\Omega)$ and $\boldsymbol{\varphi}^n \to 0$ strongly in $\mathbf{L}^{\gamma_i}(\Gamma_i)$, $i = 1, 2$. Testing (3.2) with $\boldsymbol{\varphi}^n$, we obtain

$$\begin{aligned}\int_\Omega p^{\varepsilon_n}(p^{\varepsilon_n} - p)\, \mathrm{d}\boldsymbol{x} = & \int_\Omega \operatorname{div}(\boldsymbol{v}^{\varepsilon_n} \otimes \boldsymbol{v}^{\varepsilon_n}) \cdot \boldsymbol{\varphi}^n\, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^{\varepsilon_n})(\boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{\varphi}^n)\, \mathrm{d}\boldsymbol{x} \\ &+ \int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{\varphi}^n)\, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{v}^{\varepsilon_n}), \boldsymbol{\varphi}^n \rangle - \langle \boldsymbol{f}, \boldsymbol{\varphi}^n \rangle,\end{aligned}$$

from which it follows that

$$\|p^{\varepsilon_n} - p\|_2^2 = \int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : \mathbf{D}(\boldsymbol{\varphi}^n)\, \mathrm{d}\boldsymbol{x} + o(1).$$

This implies, by virtue of (2.9), (3.14), and (3.18), that

$$\begin{aligned}\|p^{\varepsilon_n} - p\|_2^2 &\leqslant C_2 \sqrt{Y^n} \|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + \gamma_0 \|p^{\varepsilon_n} - p\|_2 \|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + o(1) \\ &\leqslant \gamma_0 \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) \Big(1 + \frac{C_2}{C_1}\Big) \|p^{\varepsilon_n} - p\|_2^2 + o(1) \|p^{\varepsilon_n} - p\|_2 + o(1),\end{aligned}$$

which leads to

$$\Big(1 - \gamma_0 \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) \Big(1 + \frac{C_2}{C_1}\Big)\Big) \|p^{\varepsilon_n} - p\|_2^2 \leqslant o(1) \|p^{\varepsilon_n} - p\|_2 + o(1).$$

Due to the assumption $(2.22)_2$, (3.18), and (2.10), we finally observe that

$$\|p^{\varepsilon_n} - p\|_2 \to 0, \quad \|\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v})\|_r \to 0, \quad \text{and} \quad \|\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\|_{\gamma_2, \Gamma_2} \to 0,$$

which implies (3.17) and completes the proof of (i) of Theorem 2.5.

4. Uniqueness considerations

Take two possible weak solutions $(\boldsymbol{v}^1, p^1)$, $(\boldsymbol{v}^2, p^2)$. Subtracting (2.19) and denoting $\mathbf{S}^i := \mathbf{S}(p^i, \mathbf{D}(\boldsymbol{v}^i))$, $i = 1, 2$, we obtain (for every $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$)

$$\begin{aligned}(4.1)\qquad \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = \int_\Omega (p^1 - p^2)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi}\big\rangle \\ - \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Setting $\boldsymbol{\varphi} := \boldsymbol{v}^1 - \boldsymbol{v}^2$, we get (as $\operatorname{div}\boldsymbol{v}^i = 0$, $i = 1, 2$)

$$\begin{aligned}(4.2)\qquad \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x} = -\big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{v}^1 - \boldsymbol{v}^2\big\rangle \\ - \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Let us assume that (2.23) holds with $C_I K \leqslant K_1$, where C_I comes from the embedding inequality $\|\boldsymbol{u}\|_{\gamma_1,\Gamma_1} \leqslant C_I \|\boldsymbol{u}\|_{1,r}$. Then the right-hand side of (4.2) can be estimated using the embeddings $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{2r'}(\Omega)$, $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^{\gamma_1}(\Gamma_1)$, (2.17), and the monotonicity of b_2, as follows:

$$(4.3\text{a})\qquad \left|\int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot (\boldsymbol{v}^1 - \boldsymbol{v}^2)\,\mathrm{d}\boldsymbol{x}\right| \leqslant CK \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|^2_{1,r},$$

$$(4.3\text{b})\qquad -\big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{v}^1 - \boldsymbol{v}^2\big\rangle \leqslant C\lambda_1 \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|^2_{1,r}.$$

Again, in what follows, $C, K > 0$ stand for generic constants determined by the problem data. Here and later in this section, C is independent of $\boldsymbol{f}$, β_1, and β_2, i.e. it is not correlated to K. Applying this back to (4.2) and using (2.8), we thus obtain

$$(4.4)\qquad \frac{C_1}{2}\int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x} \leqslant \frac{\gamma_0^2}{2C_1}\|p^1 - p^2\|_2^2 + C(K + \lambda_1)\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|^2_{1,r}.$$

This together with (2.10), Korn's and Friedrichs' inequalities yields that for λ_1 and K small enough

$$(4.5)\qquad \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \leqslant C\|p^1 - p^2\|_2.$$

Next, using (2.9) and Hölder's inequality, we obtain for any $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$

$$\begin{aligned}(4.6)\qquad &\left|\int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\right| \\ &\leqslant C_2\Big(\int_\Omega I^{1,2}\,\mathrm{d}\boldsymbol{x}\Big)^{1/2}\|\mathbf{D}(\boldsymbol{\varphi})\|_2 + \gamma_0\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2 \\ &\overset{(4.4)}{\leqslant} \Big(\gamma_0\Big(1 + \frac{C_2}{C_1}\Big) + C\sqrt{K + \lambda_1}\Big)\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2.\end{aligned}$$

Let us set $\boldsymbol{\varphi} := \tilde{\mathcal{B}}(p^1 - p^2)$ in (4.1). Note that $\|\boldsymbol{\varphi}\|_{1,2} \leqslant \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2)\|p^1 - p^2\|_2$ and also that $\|\boldsymbol{\varphi}\|_{\gamma_1,\Gamma_1}, \|\boldsymbol{\varphi}\|_{\infty,\Gamma_2} \leqslant C\|p^1 - p^2\|_2$. We arrive at

$$\int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x} = \|p^1 - p^2\|_2^2 - \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi}\big\rangle \\ - \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x},$$

which in combination with (4.6) gives

$$\|p^1 - p^2\|_2^2 \leqslant \Big(\gamma_0\Big(1 + \frac{C_2}{C_1}\Big) + C\sqrt{K + \lambda_1}\Big)\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2 \\ + \big\langle \boldsymbol{b}(\boldsymbol{v}^1) - \boldsymbol{b}(\boldsymbol{v}^2), \boldsymbol{\varphi}\big\rangle + \int_\Omega \operatorname{div}(\boldsymbol{v}^1 \otimes \boldsymbol{v}^1 - \boldsymbol{v}^2 \otimes \boldsymbol{v}^2) \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}. \tag{4.7}$$

From (2.18), $(2.21)_3$, the embedding and (4.5) it follows that

$$\big\langle b_2(\boldsymbol{v}^1 \cdot \boldsymbol{n}) - b_2(\boldsymbol{v}^2 \cdot \boldsymbol{n}), \boldsymbol{\varphi} \cdot \boldsymbol{n}\big\rangle \leqslant C\lambda_2\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}\|p^1 - p^2\|_2 \\ \leqslant C\lambda_2\|p^1 - p^2\|_2^2, \tag{4.8}$$

provided that $C_I K \leqslant K_2$, with C_I from $\|\boldsymbol{u}\|_{r^*,\Gamma_2} \leqslant C_I\|\boldsymbol{u}\|_{1,r}$. Applying the same technique as in (4.3), namely the embeddings and (2.17), then using (4.5) and (4.8), we can collectively estimate the boundary and the convective term on the right-hand side of (4.7) by the expression $C(\lambda_1 + \lambda_2 + K)\|p^1 - p^2\|_2^2$ and obtain

$$\bigg(1 - \tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2)\Big(\gamma_0\Big(1 + \frac{C_2}{C_1}\Big)\Big) \\ - C(\sqrt{K + \lambda_1} + \lambda_1 + \lambda_2 + K)\bigg) \cdot \|p^1 - p^2\|_2^2 \leqslant 0. \tag{4.9}$$

Due to $(2.22)_2$, for λ_1, λ_2 and K small enough the coefficient on the left-hand side is positive and thus $(\boldsymbol{v}^1, p^1) = (\boldsymbol{v}^2, p^2)$.

Remark 4.1 (pressure is fixed by velocity). Let $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ be weak solutions to (P). Then, under the assumptions of Theorem 2.5, $p^1 = p^2$.

Proof. From (2.9) we observe that

$$\left|\int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi})\, \mathrm{d}\boldsymbol{x}\right| \leqslant \gamma_0\|p^1 - p^2\|_2\|\mathbf{D}(\boldsymbol{\varphi})\|_2 \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega).$$

Then we subtract (2.19), take a test function $\boldsymbol{\varphi} := \tilde{\mathcal{B}}(p^1 - p^2)$ and obtain

$$\|p^1 - p^2\|_2^2 \leqslant \gamma_0\tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2)\|p^1 - p^2\|_2^2.$$

Since by assumption $\gamma_0\tilde{C}_{\mathrm{div}}(\Omega, \Gamma_1, \Gamma_2, 2) < 1$, we conclude that $p^1 = p^2$. □

Remark 4.2. Note that the additional assumptions—namely the requirement of small data $\boldsymbol{f}$, β_1, β_2—stated in (iii) of Theorem 2.5, are due to the presence of the convective term and the nonlinear boundary terms, not due to the nonlinear viscosity.

Indeed, one can consider a Stokes-like system ($\mathrm{P_S}$)

$$-\operatorname{div}\mathbf{S} + \nabla p = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega$$

and the boundary terms

$$\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}) \quad \text{on } \Gamma.$$

The readers can verify themselves that the weak solution to ($\mathrm{P_S}$) exists and is unique even for large data.

5. Boundary conditions in applications

Although the assumptions (B1)–(B7) seem to be motivated mainly by PDE analysis, they cover important engineering applications; we mention three types of them in the sequel.

Artificial boundary. In numerical simulations, large or even unbounded domains arising from the physical model must be truncated and the boundary condition for artificial boundaries has to be provided. For example in [13], an application to the flow through a cascade of profiles with the outflow condition

$$-\mathbf{T}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{n})^-\boldsymbol{v} \tag{5.1}$$

is considered (see also Section 1). In [6], several b.c. including (5.1) were proposed (for unsteady incompressible Navier-Stokes equations) in order to perform long-time simulations at high Reynolds numbers. See also [4], [5], [7].

Note that $\boldsymbol{b}_1$ given by (5.1) meets (B1), (B2) with $\gamma_1 = 3$ and $\beta_1 = \|\boldsymbol{h}\|_{3/2,\Gamma_1}$. Note also that $\|\boldsymbol{b}_1(\boldsymbol{v}^1) - \boldsymbol{b}_1(\boldsymbol{v}^2)\|_{3/2,\Gamma_1} \leqslant \frac{1}{2}\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{3,\Gamma_1}(\|\boldsymbol{v}^1\|_{3,\Gamma_1} + \|\boldsymbol{v}^2\|_{3,\Gamma_1})$ allows to establish (B6) with any $\lambda_1 > 0$, provided $K_1 > 0$ is chosen sufficiently small.

Conditions involving Bernoulli's pressure. In some applications, the quantity $p + \frac{1}{2}|\boldsymbol{v}|^2$, referred to as the *total pressure* or the *Bernoulli pressure*, is used for prescribing the inflow/outflow boundary conditions on artificial boundaries (see e.g. [12], [14], [20], [33]). Note that this class of conditions

$$\Big(p + \frac{1}{2}|\boldsymbol{v}|^2\Big)\boldsymbol{n} - \mathbf{S}\boldsymbol{n} = \boldsymbol{h}(\boldsymbol{x}) \tag{5.2}$$

is covered by our theory. Similarly to (5.1), $\boldsymbol{b}_1$ given by (5.2) satisfies (B1), (B2) with $\gamma_1 = 3$ and $\beta_1 = \|\boldsymbol{h}\|_{3/2,\Gamma_1}$, and (B6) with any $\lambda_1 > 0$, provided that $K_1 > 0$ is sufficiently small.

However, it is questionable whether the *total pressure* is generally applicable, when seeking after proper boundary conditions for viscous flows. The authors of [20] note: "The total pressure is constant along streamlines in Euler flow and therefore is an important quantity in some high-Reynolds-number situations", but later they correctly point out that these conditions[3] "... are not satisfied by Poiseuille flow. Thus their poor performance is to be expected." In other words, we do not recommend (5.2) as a suitable outflow condition for artificial boundaries. At the same time, this emphasizes that (5.1) is intended to be used for outflow—not inflow—boundaries.

Porous wall. Boundary conditions of the type (1.3) are applicable to the flows where an inflow/outflow is possible through a porous wall (*filtration* boundary conditions). In most studies, for the flow through an isotropic porous medium the linear law of Darcy

$$-\nabla p = \frac{\mu}{k}\,\boldsymbol{v}$$

is considered (with k the permeability of the medium, $\boldsymbol{v}$ the volumetric velocity, μ the viscosity and p the pressure; body forces such as gravity are neglected here). As an analogy, when studying the flow where a part of the boundary is a thin porous wall (or membrane), one can prescribe the condition

$$-\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n} = p_{\mathrm{out}} + c_1\boldsymbol{v}\cdot\boldsymbol{n} \quad \text{with } c_1 \geqslant 0 \tag{5.3}$$

for the normal part of the velocity, see e.g. [34]. However, Darcy's law is valid only for slow flows. It can be in fact derived from the Stokes equation, i.e. neglecting the inertia of the fluid, see e.g. [31]. For higher Reynolds numbers, the experimental observations "did not allow to find a universally accepted formula" [31]. Nevertheless, the relation

$$-\nabla p = \frac{\mu}{k}\,\boldsymbol{v} + d_2|\boldsymbol{v}|\boldsymbol{v} + d_3|\boldsymbol{v}|^2\boldsymbol{v}, \quad \text{with } d_2, d_3 > 0, \tag{5.4}$$

was proposed more than a century ago in [15]. Here, the last two terms were added to make the equation fit the experimental results. Formula (5.4) with $d_3 = 0$ is well established as the Forchheimer equation; see e.g. [2] for a survey of both experimental and theoretical results prior to 1972, or [19], [31] for more recent references. The authors are not aware of any reference concerning the porous wall boundary condition

[3] considering the intuitive setting of $\boldsymbol{h}(\boldsymbol{x})$ constant across the channel, analogously to (1.2)

which would involve both the high velocity effects and the non-Newtonian fluids with pressure and/or shear rate dependent viscosities.

As an analogy of (5.4), the boundary condition of the type

$$-\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n} = p_{\mathrm{out}} + (c_1 + c_2|\boldsymbol{v}\cdot\boldsymbol{n}| + c_3|\boldsymbol{v}\cdot\boldsymbol{n}|^2)\boldsymbol{v}\cdot\boldsymbol{n} \quad \text{with } c_1, c_2, c_3 \geqslant 0 \tag{5.5}$$

seems to correspond to the physics better than (5.3). If $c_3 > 0$ then b_2 given by (5.5) meets (B3)–(B5) with $\gamma_2 = 4$ and e.g. with $\underline{\beta_2} = c_3/2$ and $\beta_2 = |\Gamma_2|(1/c_3)^3 + \|p_{\mathrm{out}}\|_{4/3,\Gamma_2}^{4/3}(1/c_3)^{1/3}$. Considering (5.5) with $c_3 = 0$, one has to assume $c_2 > \frac{1}{2}$ and verify (B3)–(B5) e.g. by setting $\underline{\beta_2} = \frac{1}{2}(c_2 - \frac{1}{2})$ and $\beta_2 = (c_2 - \frac{1}{2})^{-1/2}\|p_{\mathrm{out}}\|_{3/2,\Gamma_2}^{3/2}$. From Hölder's inequality we have

$$\begin{aligned} \|b_2(w) - b_2(z)\|_{1,\Gamma_2} \leqslant{}& c_1|\Gamma_2|^{1/r^{*\prime}}\|w - z\|_{r^*} + c_2(\|w\|_{r^{*\prime}} + \|z\|_{r^{*\prime}})\|w - z\|_{r^*} \\ &+ \frac{3}{2}c_3(\|w\|_{2r^{*\prime}}^2 + \|z\|_{2r^{*\prime}}^2)\|w - z\|_{r^*}. \end{aligned} \tag{5.6}$$

Note that $2r^{*\prime} < r^*$, since $r > 3d/(d+2)$. Thus, (B7) can be achieved for any $\lambda_2 > c_1|\Gamma_2|^{(r^*-1)/r^*}$, choosing $K_2 > 0$ sufficiently small.

Concerning the boundary conditions given on the tangential part of the velocity on a porous wall, the no-slip condition $(2.4)_1$ is chosen here as one of several possible choices. It was preferred mainly in order to keep the ideas simple, even though from the physical point of view there is no particular preference over kinds of the slip condition. Nevertheless, the no-slip condition can be reasonable either as an approximation or in cases justified by the particular application, see for instance [17], [18], [34].

6. Conclusion

The class of fluids with pressure and shear rate dependent viscosities together with mixed boundary conditions involving the pressure was studied. Under certain assumptions, it was shown that a weak solution exists and that this weak solution is unique if the data are small. In contrast to previous studies, no constraint on the pressure mean value is present in the formulation of the problem. The proof follows the ideas of [16], except for the treatment of the inflow/outflow boundary conditions. Finally, a brief survey on these boundary conditions fitting to our theory is presented together with their physical application.

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Finite element approximation of flow of fluids with shear-rate- and pressure-dependent viscosity

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In this paper we consider a class of incompressible viscous fluids whose viscosity depends on the shear rate and pressure. We deal with isothermal steady flow and analyse the Galerkin discretization of the corresponding equations. We discuss the existence and uniqueness of discrete solutions and their convergence to the solution of the original problem. In particular, we derive *a priori* error estimates, which provide optimal rates of convergence with respect to the expected regularity of the solution. Finally, we demonstrate the achieved results by numerical experiments. The fluid models under consideration appear in many practical problems, for instance, in elastohydrodynamic lubrication where very high pressures occur. Here we consider shear-thinning fluid models similar to the power-law/Carreau model. A restricted sublinear dependence of the viscosity on the pressure is allowed. The mathematical theory concerned with the self-consistency of the governing equations has emerged only recently. We adopt the established theory in the context of discrete approximations. To our knowledge, this is the first analysis of the finite element method for fluids with pressure-dependent viscosity. The derived estimates coincide with the optimal error estimates established recently for Carreau-type models, which are covered as a special case.

Keywords: non-Newtonian fluid; shear-rate- and pressure-dependent viscosity; finite element method; error analysis.

1. Introduction

The article is devoted to the finite element discretization of equations governing the steady flow of a class of incompressible fluids whose viscosity depends nonlinearly on the shear rate and pressure. We discuss the well-posedness of the discretized problem and derive *a priori* estimates for the discretization error.

The isothermal flow of an incompressible viscous fluid is typically described by the Navier–Stokes equations, which embody Newton's hypothesis that the viscosity—the ratio between the shear stress and the shear rate—is constant. Since the early formation of fluid mechanics it has been known that this assumption may not be applicable to all viscous flows. In past decades many non-Newtonian phenomena have become the subject of scientific interest. We will consider models with shear-dependent and pressure-dependent viscosity, which play an important role in many areas such as elastohydrodynamic lubrication, geology and glaciology (see, e.g., Hindmarsh, 1998; Bair & Gordon, 2006; Stemmer *et al.*, 2006; Schoof, 2007; Szeri, 2010 and the references given in Hron *et al.*, 2001). The viscosity of fluids in such applications varies considerably with the pressure, even by several orders of magnitude.

We study the steady isothermal flow of an homogeneous incompressible viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, governed by the following system of PDEs:

$$\left.\begin{aligned} -\operatorname{div} \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) + \nabla\pi &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{1.1}$$

where $\boldsymbol{v}$ is the velocity, π denotes the pressure (more specifically, the ratio of the mean normal stress and the density) and $\boldsymbol{f}$ represents the density of an applied body force. Here, $\boldsymbol{D}\boldsymbol{v}$ is the symmetric part of the velocity gradient. Note that we avoid mathematical difficulties related to the convective term by neglecting inertial forces in the first equation. We consider extra stress tensors $\boldsymbol{S}$ of the form

$$\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) = 2\eta(\pi, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{1.2}$$

where η is the generalized kinematic viscosity. Many details, examples, and an extensive discussion concerning the class of models (1.2) can be found in Málek & Rajagopal (2006, 2007).

We assume that the domain boundary $\partial\Omega$ is Lipschitz and consists of two parts, $\partial\Omega = \Gamma_D \cup \Gamma_P$, $|\Gamma_D| > 0$. Then, we complement the system (1.1) with the boundary conditions

$$\boldsymbol{v} = \boldsymbol{v}_D \ \text{ on } \Gamma_D, \tag{1.3}$$

$$-\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\mathbf{n} + \pi\mathbf{n} = \boldsymbol{b} \quad \text{on } \Gamma_P, \tag{1.4}$$

where $\mathbf{n}$ denotes the unit outer normal vector to $\partial\Omega$. We distinguish two cases.

(a) If $|\Gamma_P| = 0$ (i.e., the Dirichlet boundary conditions are prescribed on the whole boundary, $\Gamma_D = \partial\Omega$) then we additionally fix the level of pressure by requiring

$$\fint_\Omega \pi \, \mathrm{d}\boldsymbol{x} = \pi_0 \in \mathbb{R}. \tag{1.5}$$

For simplicity of notation[1] we assume $\pi_0 = 0$.

(b) If $|\Gamma_P| > 0$ then (1.4) suffices to fix the level of the pressure. This was shown in Lanzendörfer & Stebel (2011a,b); see also Lemma 2.9, Remark 2.11 and Theorem 3.2 below.

It is a special feature of piezoviscous fluids, in case (a), that through $\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})$, the number π_0 affects the whole solution, including the velocity field. Hence, the nonphysical constraint (1.5) comprises an important input parameter undeterminable by practical applications. In contrast, $\boldsymbol{b}$ in (1.4) represents the force acting on the domain boundary and reflects physically reasonable input data.

While the mathematical self-consistency of the shear-thinning or shear-thickening fluid models has been studied intensively since the 1960s, the rigorous analysis of those with pressure-dependent

[1] The theoretical methods and results of this paper are not restricted to the choice $\pi_0 = 0$.

viscosity has emerged only recently (see Málek & Rajagopal, 2006 for references). The well-posedness of problems in which the viscosity depends solely on the pressure, or grows with the pressure superlinearly, has not been resolved, except under severe restrictions on the data size or time interval. When the viscosity changes with the pressure too rapidly, the equations corresponding to steady flow lose their ellipticity. A breakthrough result appeared in Málek *et al.* (2002), where viscosities depending on both the pressure and the shear rate have been considered. The structure of the viscosity proposed therein has allowed for global and large data existence results for both steady and unsteady motions under various boundary conditions (see, e.g., Franta *et al.*, 2005; Bulíček *et al.*, 2007; Lanzendörfer, 2009; Lanzendörfer & Stebel, 2011a).

Our aim is to adopt the established mathematical theory in the framework of Galerkin discretizations. The finite element method has been studied extensively in the context of power-law/Carreau models, for which the viscosity depends only on the shear rate (see Baranger & Najib, 1990; Barrett & Liu, 1993, 1994; and the references therein). In particular, Hirn (2010) and Belenki *et al.* (2010) have recently derived optimal *a priori* error estimates in the shear-thinning case. However, no such analysis is available when the fluid's viscosity also depends on the pressure. To the best of our knowledge, the present paper provides the first analytical study of the finite element method in the context of fluids with shear-rate- and pressure-dependent viscosity.

This paper is devoted to the finite element approximation of the problem (1.1)–(1.5) where the extra stress tensor $\boldsymbol{S}$ is supposed to satisfy a certain p-structure; see Assumptions (A1)–(A2) below. For $p \in (1,2]$ we will show that the finite element solutions $(\boldsymbol{v}_h, \pi_h)$ exist, are determined uniquely, and that they converge to the weak solution $(\boldsymbol{v}, \pi)$ strongly in $\mathbf{W}^{1,p}(\Omega) \times \mathrm{L}^{p'}(\Omega)$, $p' := p/(p-1)$, for diminishing mesh size, h. Moreover, if the solution $(\boldsymbol{v}, \pi)$ satisfies the regularity condition

$$\int_\Omega (1 + |\boldsymbol{Dv}|)^{p-2} |\nabla \boldsymbol{Dv}|^2 \,\mathrm{d}\boldsymbol{x} < \infty \quad \text{and } \pi \in \mathrm{W}^{1,p'}(\Omega), \tag{1.6}$$

then an $\mathcal{O}(h)$ error bound for the velocity in $\mathbf{W}^{1,p}(\Omega)$ and an $\mathcal{O}(h^{2/p'})$ error bound for the pressure in $\mathrm{L}^{p'}(\Omega)$ will be established:

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leqslant ch, \quad \|\pi - \pi_h\|_{p'} \leqslant ch^{\frac{2}{p'}}.$$

These estimates will be derived by means of the well-known quasinorm technique, which was originally developed for the error analysis of the p-Laplace equation (see Barrett & Liu, 1994). Numerical experiments indicate that these estimates are optimal with respect to the supposed regularity. Moreover, the present paper also covers the case of Carreau-type models for which the *a priori* error estimates derived here coincide with those established in Belenki *et al.* (2010) and Hirn (2010).

The paper is organized as follows: in Section 2 we formulate basic assumptions, introduce tools, and define the problem and its discretization. Section 3 deals with the existence and uniqueness of the discrete solutions and their convergence to the weak solution of the problem. *A priori* error estimates are derived in Section 4 and are applied to the finite element discretization in Section 5. Finally, in Section 6 we demonstrate the theoretical results by numerical experiments.

2. Preliminaries

In this section we introduce the notation, we state our assumptions on the extra stress tensor, indicate how the stress tensor is related to N-functions and we show its resulting properties. Then, we introduce the weak formulation of the system (1.1)–(1.5) and its Galerkin discretization.

2.1 *Notation and function spaces*

The set of all positive real numbers is denoted by $\mathbb{R}^+$. Let $\mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\}$. The Euclidean scalar product of two vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d$ is denoted by $\boldsymbol{p} \cdot \boldsymbol{q}$, the scalar product of $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}$ is defined by $\boldsymbol{P} : \boldsymbol{Q} := \sum_{i,j=1}^d P_{ij} Q_{ij}$. We set $|\boldsymbol{Q}| := (\boldsymbol{Q} : \boldsymbol{Q})^{1/2}$. Often we use c as a generic constant, whose value may change from line to line but does not depend on important variables. We write $a \sim b$ if there exist positive constants c and C independent of all relevant quantities such that $cb \leqslant a \leqslant Cb$. Similarly, the notation $a \lesssim b$ is used for $a \leqslant Cb$.

For a measurable set $\omega \subset \Omega$, $|\omega|$ denotes its d-dimensional Lebesgue measure. For $\nu \in [1, \infty]$, $\mathrm{L}^\nu(\Omega)$ stands for the Lebesgue space and $\mathrm{W}^{m,\nu}(\Omega)$ for the Sobolev space of order m. The space $\mathrm{L}^\nu_0(\Omega)$ contains all $q \in \mathrm{L}^\nu(\Omega)$ with $⨍_\Omega q \,\mathrm{d}\boldsymbol{x} := \frac{1}{|\Omega|}\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0$. For $\nu > 1$ we use the notation $\mathrm{W}^{1,\nu}_0(\Omega)$ for the Sobolev space with vanishing traces on $\partial\Omega$. The $\mathrm{L}^\nu(\omega)$-norm is denoted by $\|\cdot\|_{\nu;\omega}$ and the $\mathrm{W}^{m,\nu}(\omega)$-norm is denoted by $\|\cdot\|_{m,\nu;\omega}$. The notation $(u, v)_\omega$ is used for the integral $\int_\omega uv \,\mathrm{d}\boldsymbol{x}$. In the case of $\omega = \Omega$, we usually omit the index Ω. Spaces of $\mathbb{R}^d$-valued functions are denoted with boldface type, though no distinction is made in the notation of norms and inner products; the norm in $\mathbf{W}^{m,\nu}(\Omega) \equiv [\mathrm{W}^{m,\nu}(\Omega)]^d$ is given by $\|\boldsymbol{w}\|_{m,\nu} = \big(\sum_{1\leqslant i\leqslant d}\sum_{0\leqslant|\alpha|\leqslant m}\|\partial^\alpha w_i\|_\nu^\nu\big)^{1/\nu}$, etc.

2.2 *Structural assumptions on the extra stress tensor*

Let $p > 1$, $\varepsilon > 0$, and $\gamma_0 \geqslant 0$ be given. We suppose that the extra stress tensor $\boldsymbol{S}$ belongs to the class (1.2) and satisfies the following structural assumptions.

(A1) There exist positive constants σ_0, σ_1 such that for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $q \in \mathbb{R}$ there holds

$$\sigma_0\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2 \leqslant \frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q}\otimes\boldsymbol{Q}) \leqslant \sigma_1\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2,$$

where $\mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\boldsymbol{P} \in \mathbb{R}^{d\times d};\, \boldsymbol{P} = \boldsymbol{P}^\top\}$ and $(\boldsymbol{Q}\otimes\boldsymbol{Q})_{ijkl} = Q_{ij}Q_{kl}$.

(A2) For all $\boldsymbol{P} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $q \in \mathbb{R}$ there holds

$$\left|\frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial q}\right| \leqslant \gamma_0\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{4}}.$$

REMARK 2.1 Models satisfying Assumptions (A1)–(A2) can approximate some real-world liquids within a certain range of shear rates and pressures (see Málek *et al.*, 2002; Málek & Rajagopal, 2006, 2007, for examples and applications; see also Remark 6.1). Note that both assumptions are rather restrictive with regard to the dependence of the viscosity on the pressure, which is usually considered as $\eta \sim \exp(\alpha\pi)$ in practical applications. The well-posedness of problems with superlinear dependence on the pressure is, however, an open problem, as, similarly, is the limiting case $\varepsilon = 0$. For a possible generalization of the theoretical results to unbounded viscosities see Bulíček *et al.* (2009). Most real-life fluids, that are under consideration, exhibit shear-thinning behaviour, which corresponds to exponents $p < 2$. Later, we will restrict ourselves to shear-thinning models. The case $p = 2$ will be included in the subsequent analysis, although we will mostly speak of shear-thinning fluids only. An exemplary model that satisfies (A1)–(A2) with $p = 2$ can be found in Málek & Rajagopal (2007).

We show how the stress tensor relates to N-functions. A continuous convex function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called an N-function if $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$, $\lim_{t\to 0+} \psi(t)/t = 0$ and $\lim_{t\to\infty} \psi(t)/t = \infty$. Consequently, the right derivative ψ' of ψ exists, is nondecreasing and satisfies $\psi'(0) = 0$, $\psi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \psi'(t) = \infty$. We define the complementary N-function ψ^* by $\psi^*(t) := \sup_{s\geqslant 0}(st - \psi(s))$ for all $t \geqslant 0$. If ψ' is strictly increasing then $(\psi^*)' = (\psi')^{-1}$. An important subclass of N-functions consists of those that satisfy the Δ_2-condition: ψ satisfies the Δ_2-condition if there exists $C > 0$ such that $\psi(2t) \leqslant C\psi(t)$ for all $t \geqslant 0$. Here, $\Delta_2(\psi)$ denotes the smallest such constant. Diening & Ettwein (2008, Lemma 32) provides the following Young-type inequality: for all $\delta > 0$ there exists $c_\delta > 0$, which depends only on $\Delta_2(\psi)$, $\Delta_2(\psi^*) < \infty$, such that for all $s, t \geqslant 0$ there holds

$$s\psi'(t) + \psi'(s)t \leqslant \delta\psi(s) + c_\delta\psi(t). \tag{2.1}$$

Let us consider the following simple examples: for $p > 1$ we introduce the convex function,

$$\varphi \in C(\mathbb{R}_0^+, \mathbb{R}_0^+), \quad \varphi(t) := \frac{1}{p}t^p. \tag{2.2}$$

Clearly, φ and φ^*, where $\varphi^*(t) = \frac{1}{p'}t^{p'}$, are N-functions satisfying the Δ_2-condition. For a given N-function ψ with $\Delta_2(\psi)$, $\Delta_2(\psi^*) < \infty$, we define the family of *shifted* functions $\{\psi_a\}_{a\geqslant 0}$ by

$$\psi_a(t) := \int_0^t \psi_a'(s)\,\mathrm{d}s \quad \text{with} \quad \psi_a'(t) := \psi'(a+t)\frac{t}{a+t}. \tag{2.3}$$

Then, Diening & Ettwein (2008, Lemma 23) ensures that $\{\psi_a\}_{a\geqslant 0}$ are again N-functions and satisfy the Δ_2-condition uniformly in $a \geqslant 0$ with Δ_2-constants depending only on $\Delta_2(\psi)$, $\Delta_2(\psi^*)$. Let us return to case (2.2): the family of shifted N-functions $\{\varphi_a\}_{a\geqslant 0}$ belongs to $C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and satisfies the Δ_2-condition uniformly in $a \geqslant 0$ with Δ_2-constants depending only on p. Using the definition of φ_a we easily conclude that

$$\min\{1, p-1\}(a+t)^{p-2} \leqslant \varphi_a''(t) \leqslant \max\{1, p-1\}(a+t)^{p-2} \tag{2.4}$$

and $\varphi_a'(t) \sim \varphi''(a+t)t \sim \varphi_a''(t)t$. Moreover, $\varphi_a(t) \sim \varphi_a'(t)t$ uniformly in $t, a \geqslant 0$. Due to (2.4) the inequalities of Assumption (A1) defining the (p, ε)-structure of $\boldsymbol{S}$ can be expressed equivalently in terms of the N-functions φ_ε.

2.3 *Basic properties of the extra stress tensor*

We express several consequences of Assumptions (A1)–(A2). Below, we formulate the results as generally as possible, although in the forthcoming sections we only treat the shear-thinning case. We introduce the function $\boldsymbol{F} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ by

$$\boldsymbol{F}(\boldsymbol{P}) := (\varepsilon + |\boldsymbol{P}|)^{\frac{p-2}{2}}\boldsymbol{P}, \tag{2.5}$$

where p and ε are the same as in Assumptions (A1)–(A2). The quantity $\boldsymbol{F}$ is closely related to the extra stress tensor $\boldsymbol{S}$ as shown by the following lemma.

LEMMA 2.2 For given $p \in (1, \infty)$ and $\varepsilon \in [0, \infty)$, let $\boldsymbol{S}$ satisfy (A1), let $\boldsymbol{F}$ be defined by (2.5), and let φ be defined by (2.2). Then, uniformly for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $q \in \mathbb{R}$, it holds that

$$(\boldsymbol{S}(q,\boldsymbol{P}) - \boldsymbol{S}(q,\boldsymbol{Q})) : (\boldsymbol{P}-\boldsymbol{Q}) \sim (\varepsilon + |\boldsymbol{P}| + |\boldsymbol{Q}|)^{p-2}|\boldsymbol{P}-\boldsymbol{Q}|^2$$

$$\sim \varphi_{\varepsilon+|\boldsymbol{P}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \sim |\boldsymbol{F}(\boldsymbol{P}) - \boldsymbol{F}(\boldsymbol{Q})|^2,$$

$$|\boldsymbol{S}(q,\boldsymbol{P}) - \boldsymbol{S}(q,\boldsymbol{Q})| \sim \varphi'_{\varepsilon+|\boldsymbol{P}|}(|\boldsymbol{P}-\boldsymbol{Q}|),$$

where the constants depend only on σ_0, σ_1 and p. In particular, they are independent of $\varepsilon \geqslant 0$. Moreover, the following estimates hold:

$$\boldsymbol{S}(q,\boldsymbol{Q}) : \boldsymbol{Q} \geqslant \frac{\sigma_0}{2p}(|\boldsymbol{Q}|^p - \varepsilon^p) \quad \text{and} \quad |\boldsymbol{S}(q,\boldsymbol{Q})| \leqslant \frac{\sigma_1}{p-1}|\boldsymbol{Q}|^{p-1}. \tag{2.6}$$

Proof. For (2.6) we refer to Málek *et al.* (1996, Lemma 1.19). All remaining estimates are proven in Diening *et al.* (2007) or Diening & Ettwein (2008). □

As a straightforward consequence of Assumptions (A1)–(A2) we also obtain the following result.

LEMMA 2.3 For given $p \in (1, \infty)$, $\varepsilon \in (0, \infty)$, and $\gamma_0 \in [0, \infty)$, let $\boldsymbol{S}$ satisfy (A1), (A2). For $\boldsymbol{P}_0$, $\boldsymbol{P}_1 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $s \in [0, 1]$ let us set $\boldsymbol{P}_s := \boldsymbol{P}_0 + s(\boldsymbol{P}_1 - \boldsymbol{P}_0)$. Then, for all $\boldsymbol{P}_0, \boldsymbol{P}_1 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\pi, q \in \mathbb{R}$ it holds that

$$\frac{\sigma_0}{2}\int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}_1 - \boldsymbol{P}_0|^2\,\mathrm{d}s \leqslant (\boldsymbol{S}(\pi,\boldsymbol{P}_1) - \boldsymbol{S}(q,\boldsymbol{P}_0)) : (\boldsymbol{P}_1 - \boldsymbol{P}_0) + \frac{\gamma_0^2}{2\sigma_0}|\pi - q|^2, \tag{2.7}$$

$$|\boldsymbol{S}(\pi,\boldsymbol{P}_1) - \boldsymbol{S}(q,\boldsymbol{P}_0)| \leqslant \sigma_1\int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}_1 - \boldsymbol{P}_0|\,\mathrm{d}s + \gamma_0\int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{4}}|\pi - q|\,\mathrm{d}s. \tag{2.8}$$

Proof. See, e.g., Bulíček *et al.* (2007, Lemma 1.4). □

In view of Lemma 2.3 we define the distance

$$d(\boldsymbol{v},\boldsymbol{u})^2 := \int_\Omega\int_0^1 \left(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{u} + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u})|^2\right)^{\frac{p-2}{2}} |\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x} \tag{2.9}$$

for all $\boldsymbol{v}, \boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$. We get the following corollary.

COROLLARY 2.4 For given $p \in (1, \infty)$, $\varepsilon \in (0, \infty)$ and $\gamma_0 \in [0, \infty)$, let $\boldsymbol{S}$ satisfy (A1), (A2). Let $d(\cdot,\cdot)$ be defined by (2.9). Then, for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ and $\pi, q \in \mathrm{L}^2(\Omega)$, there holds

$$\frac{\sigma_0}{2}d(\boldsymbol{v},\boldsymbol{w})^2 \leqslant (\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w}), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{w})_\Omega + \frac{\gamma_0^2}{2\sigma_0}\|\pi - q\|_2^2. \tag{2.10}$$

For each $\delta > 0$ there exists a positive constant c_δ depending only on σ_1 and δ, such that

$$(\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w}), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{w})_\Omega \leqslant c_\delta d(\boldsymbol{v},\boldsymbol{w})^2 + \delta\gamma_0^2\|\pi - q\|_2^2. \tag{2.11}$$

If $p \leqslant 2$ then, for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ and all sufficiently smooth functions π, q, there holds

$$\|\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(q, \boldsymbol{Dw})\|_2 \leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}, \boldsymbol{w}) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - q\|_2, \tag{2.12}$$

$$\|\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(q, \boldsymbol{Dw})\|_{p'} \leqslant c d(\boldsymbol{v}, \boldsymbol{w})^{\frac{2}{p'}} + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - q\|_{p'}, \tag{2.13}$$

where $c = c(p, \sigma_1)$ is a positive constant.

Proof. Clearly, (2.10) is a consequence of (2.7), whereas (2.11) follows from (2.8) and Young's inequality. Setting $\boldsymbol{D}_s := \boldsymbol{Dw} + s(\boldsymbol{Dv} - \boldsymbol{Dw})$, for $\nu \geqslant 1$ we infer from (2.8) and Minkowski's inequality that

$$\begin{aligned}\|\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(q, \boldsymbol{Dw})\|_\nu \leqslant \sigma_1 &\left(\int_\Omega \left| \int_0^1 (\varepsilon^2 + |\boldsymbol{D}_s|^2)^{\frac{p-2}{2}} |\boldsymbol{Dv} - \boldsymbol{Dw}| \, \mathrm{d}s \right|^\nu \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{\nu}} \\ &+ \gamma_0 \left(\int_\Omega \left| \int_0^1 (\varepsilon^2 + |\boldsymbol{D}_s|^2)^{\frac{p-2}{4}} |\pi - q| \mathrm{d}s \right|^\nu \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{\nu}}.\end{aligned} \tag{2.14}$$

We immediately deduce (2.12) from (2.14) with $\nu = 2$ and Jensen's inequality. In order to derive (2.13) we recall the following well-known result (see Acerbi & Fusco, 1989, Lemma 2.1):

$$\left(\varepsilon^2 + (|\boldsymbol{P}_1| + |\boldsymbol{P}_2|)^2\right)^\alpha \sim \int_0^1 \left(\varepsilon^2 + |\boldsymbol{P}_2 + s(\boldsymbol{P}_1 - \boldsymbol{P}_2)|^2\right)^\alpha \mathrm{d}s \quad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{2.15}$$

which holds true for each $\alpha > -1/2$ provided that $\varepsilon + |\boldsymbol{P}_1| + |\boldsymbol{P}_2| > 0$. Note that the constants in (2.15) depend only on α. Furthermore, we mention the trivial inequality

$$\frac{1}{2}(|\boldsymbol{P}_1| + |\boldsymbol{P}_2|) \leqslant |\boldsymbol{P}_1| + |\boldsymbol{P}_1 - \boldsymbol{P}_2| \leqslant 2(|\boldsymbol{P}_1| + |\boldsymbol{P}_2|) \quad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{2.16}$$

Using (2.15), (2.16), and the fact that $p \leqslant 2$, we conclude from (2.14) with $\nu = p'$ that

$$\begin{aligned}\|\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(q, \boldsymbol{Dw})\|_{p'} &\leqslant c\left(\int_\Omega \left(\varepsilon^2 + (|\boldsymbol{Dw}| + |\boldsymbol{Dv} - \boldsymbol{Dw}|)^2\right)^{\frac{p-2}{2}p'} |\boldsymbol{Dv} - \boldsymbol{Dw}|^{p'} \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p'}} \\ &\quad + \gamma_0 \left(\int_\Omega \left| \int_0^1 \left(\varepsilon^2 + |\boldsymbol{D}_s|^2\right)^{\frac{p-2}{4}} |\pi - q| \, \mathrm{d}s \right|^{p'} \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p'}} \\ &\leqslant c\left(\int_\Omega \left(\varepsilon^2 + (|\boldsymbol{Dw}| + |\boldsymbol{Dv} - \boldsymbol{Dw}|)^2\right)^{\frac{p-2}{2}} |\boldsymbol{Dv} - \boldsymbol{Dw}|^2 \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p'}} \\ &\quad + \gamma_0 \varepsilon^{\frac{p-2}{2}} \left(\int_\Omega |\pi - q|^{p'} \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p'}},\end{aligned}$$

where the constant c depends only on p and σ_1. This yields (2.13). □

We remark that the distance $d(\cdot,\cdot)$ is equivalent to the so-called *quasinorm*, which was introduced in Barrett & Liu (1993). Hence, all results below can also be expressed in terms of quasinorms. The following lemma indicates that $d(\cdot,\cdot)$ is also equivalent to the $\boldsymbol{F}$-distance.

LEMMA 2.5 For $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$, let $\boldsymbol{S}$ satisfy (A1). Let $d(\cdot,\cdot)$ be defined by (2.9) and let $\boldsymbol{F}$ be defined by (2.5). For all $\boldsymbol{v},\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and $\pi \in \mathrm{L}^2(\Omega)$, there holds

$$d(\boldsymbol{v},\boldsymbol{u})^2 \sim \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du})\|_2^2 \sim (\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(\pi,\boldsymbol{Du}), \boldsymbol{Dv} - \boldsymbol{Du})_\Omega. \tag{2.17}$$

All constants depend only on p, σ_0, σ_1.

Proof. See, e.g., Diening *et al.* (2007). The assertion follows from Lemma 2.2 and (2.15). □

The following lemma, whose proof can be found in Berselli *et al.* (2010), shows the connection between the quasinorms and Sobolev norms.

LEMMA 2.6 For $p \in (1,2]$ and $\varepsilon \in (0,\infty)$, let $\boldsymbol{S}$ satisfy (A1) and let $\boldsymbol{F}$ be defined by (2.5). Then, for all sufficiently smooth functions $\boldsymbol{v},\boldsymbol{u}$, and for $\nu \in [1,2]$, there holds

$$\|\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})\|_\nu^2 \lesssim \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du})\|_2^2 \|(\varepsilon + |\boldsymbol{Dv}| + |\boldsymbol{Du}|)^{2-p}\|_{\frac{\nu}{2-\nu}}, \tag{2.18}$$

where the constant depends only on p, σ_0, and σ_1. If $\nu = 2$ then $\frac{\nu}{2-\nu} = \infty$.

2.4 *Weak formulation*

The natural spaces for the velocity and pressure are given by

$$\boldsymbol{X}^p := \{\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega);\ \mathrm{tr}\,\boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_D\},$$

$$Q^p := \{q \in \mathrm{L}^{p'}(\Omega);\ \text{if } |\Gamma_P| = 0 \text{ then } \textstyle\int_\Omega q\,\mathrm{d}\boldsymbol{x} = 0\},$$

where $p' := p/(p-1)$. The following Korn inequality holds in $\boldsymbol{X}^p$ as long as $|\Gamma_D| > 0$.

LEMMA 2.7 (Korn's inequality). Let $\nu \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$, be a bounded domain and $\partial\Omega, \Gamma_D \in C^{0,1}$, where $\Gamma_D \subset \partial\Omega$ has nonzero $(d-1)$-dimensional measure. Then, there exists a constant $c_K := c_K(\Omega,\Gamma_D,\nu) > 0$ such that

$$c_K\|\boldsymbol{w}\|_{1,\nu} \leqslant \|\boldsymbol{Dw}\|_\nu \quad \forall\, \boldsymbol{w} \in \boldsymbol{X}^\nu.$$

Proof. The result can be found in Málek *et al.* (1996, Theorem 1.10, p. 196); although it is formulated for $\Gamma_D = \partial\Omega$ there, its proof covers the case $|\Gamma_D| > 0$. □

Let us summarize the general assumptions that will be used in the following sections.

ASSUMPTION 2.8 We suppose that

- $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, is a bounded domain, $\partial\Omega = \Gamma_D \cup \Gamma_P$ and $\partial\Omega, \Gamma_D, \Gamma_P \in C^{0,1}$, $|\Gamma_D| > 0$.
- For given $p \in (1,2]$, $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon_0 > 0$ arbitrary, and $\gamma_0 \in (0,\infty)$, Assumptions (A1)–(A2) hold true.

- The following data are given:

$$\boldsymbol{v}_0 \in \mathbf{W}^{1,p}(\Omega), \quad \operatorname{div} \boldsymbol{v}_0 = 0 \text{ a.e. in } \Omega, \quad \boldsymbol{v}_0 = \boldsymbol{v}_D \text{ on } \Gamma_D,$$

$$\boldsymbol{f} \in \mathbf{L}^{p'}(\Omega) \quad \text{and } \boldsymbol{b} \in \mathbf{L}^{(p^\#)'}(\Gamma_P), \quad \text{with } (p^\#)' := \tfrac{(d-1)p}{d(p-1)}.$$

Here, $p^\# := \frac{(d-1)p}{d-p}$ is such that $\operatorname{tr}(\mathrm{W}^{1,p}(\Omega)) \hookrightarrow \mathrm{L}^{p^\#}(\partial\Omega)$.

The weak formulation of system (1.1)–(1.5) reads:

(pS) find $(\boldsymbol{v}, \pi) \in (\boldsymbol{v}_0 + \boldsymbol{X}^p) \times Q^p$ (the weak solution) such that

$$(\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}, \boldsymbol{w})_{\Gamma_P} \quad \forall \boldsymbol{w} \in \boldsymbol{X}^p, \tag{2.19}$$

$$(\operatorname{div} \boldsymbol{v}, q)_\Omega = 0 \quad \forall q \in Q^p. \tag{2.20}$$

2.5 *Galerkin approximation*

For given $h > 0$ let $\boldsymbol{X}_h$, Y_h, be finite-dimensional spaces and

$$\boldsymbol{X}_h^p := \boldsymbol{X}_h \cap \boldsymbol{X}^p, \quad Q_h^p := Y_h \cap Q^p,$$

$$\boldsymbol{V}_h^p := \{\boldsymbol{w}_h \in \boldsymbol{X}_h^p; (\operatorname{div} \boldsymbol{w}_h, q_h)_\Omega = 0 \quad \forall q_h \in Q_h^p\}.$$

We will specify the spaces in the context of finite elements in Section 5, h will then stand for the mesh parameter. At this stage we only require that $\boldsymbol{X}_h^p$ and Q_h^p approximate $\boldsymbol{X}^p$ and Q^p in the following sense:

$$\lim_{h \searrow 0} \inf_{\boldsymbol{w}_h \in \boldsymbol{X}_h^p} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{1,p} = \lim_{h \searrow 0} \inf_{q_h \in Q_h^p} \|q - q_h\|_{p'} = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{X}^p \ \forall q \in Q^p. \tag{2.21}$$

The pure Galerkin approximation of (pS) consists in replacing the Banach spaces $\boldsymbol{X}^p$ and Q^p by their finite-dimensional subspaces $\boldsymbol{X}_h^p$ and Q_h^p:

($\mathbf{pS}_h$) find $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ (the discrete solution) such that

$$(\boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} \quad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p, \tag{2.22}$$

$$(\operatorname{div} \boldsymbol{v}_h, q_h)_\Omega = 0 \quad \forall q_h \in Q_h^p. \tag{2.23}$$

Here, $\boldsymbol{v}_{0,h}$ is any[2] appropriate approximation of the Dirichlet data that satisfies

$$(\operatorname{div} \boldsymbol{v}_{0,h}, q_h)_\Omega = 0 \quad \forall q_h \in Q_h^p \qquad \text{and} \qquad \lim_{h \searrow 0} \|\boldsymbol{v}_0 - \boldsymbol{v}_{0,h}\|_{1,p} = 0. \tag{2.24}$$

[2]For example, $\boldsymbol{v}_{0,h} \in \boldsymbol{X}_h$ is typical in the context of finite elements; but one can also take $\boldsymbol{v}_{0,h} = \boldsymbol{v}_0$.

2.6 *Inf–sup conditions*

The following observation plays an essential role in the further analysis.

LEMMA 2.9 Let Assumption 2.8 be satisfied. For any $\nu \in (1, \infty)$ there exists a constant $\beta(\nu)$ (depending on ν, Ω and Γ_P) such that

$$0 < \beta(\nu) \leqslant \inf_{q \in Q^\nu} \sup_{\boldsymbol{w} \in \boldsymbol{X}^\nu} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}. \tag{2.25}$$

In particular, there exists a constant $\beta_0(\nu)$ depending on ν and Ω such that

$$0 < \beta_0(\nu) \leqslant \inf_{q \in \mathrm{L}_0^{\nu'}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}_0^{1,\nu}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}. \tag{2.26}$$

If $|\Gamma_P| > 0$ then one possible choice of $\beta(\nu)$ is related to $\beta_0(\nu)$ through (2.27).

Proof. If $|\Gamma_P| = 0$ then $\boldsymbol{X}^\nu = \mathbf{W}_0^{1,\nu}(\Omega)$ and $Q^\nu = \mathrm{L}_0^{\nu'}(\Omega)$. Then, (2.25) and (2.26) are identical, well known and follow from the properties of the Bogovskii operator; see Remark 2.10. Let $|\Gamma_P| > 0$. Then, (2.25) can be derived from (2.26) (see, e.g., Haslinger & Stebel, 2011). For $q \in \mathrm{L}^{\nu'}(\Omega)$ arbitrary, we write $q = q_0 + (\fint_\Omega q \,\mathrm{d}\boldsymbol{x})$. Since $q_0 \in \mathrm{L}_0^{\nu'}(\Omega)$, there exists $\boldsymbol{w}_0 \in \mathbf{W}_0^{1,\nu}(\Omega)$, $\|\boldsymbol{w}_0\|_{1,\nu} = 1$ such that $\beta_0(\nu)\|q_0\|_{\nu'} \leqslant (q_0, \operatorname{div} \boldsymbol{w}_0)_\Omega = (q, \operatorname{div} \boldsymbol{w}_0)_\Omega$. Since $\Gamma_P \in C^{0,1}$, $|\Gamma_P| > 0$, there exists some $\boldsymbol{\xi} \in \boldsymbol{X}^\nu$ such that $\int_\Omega \operatorname{div} \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_P} \boldsymbol{\xi} \cdot \mathbf{n} \,\mathrm{d}\boldsymbol{x} = 1$. Taking

$$\boldsymbol{w} := \boldsymbol{w}_0 + \delta \operatorname{sign}(\fint_\Omega q \,\mathrm{d}\boldsymbol{x})\boldsymbol{\xi} \quad \text{with } \delta := \frac{\beta_0(\nu)|\Omega|^{1/\nu'}}{1 + |\Omega|^{1/\nu'}\|\operatorname{div} \boldsymbol{\xi}\|_\nu},$$

and using $\|q\|_{\nu'} \leqslant \|q_0\|_{\nu'} + |\Omega|^{1/\nu'}|\fint_\Omega q \,\mathrm{d}\boldsymbol{x}|$, we obtain

$$\begin{aligned}
(q, \operatorname{div} \boldsymbol{w})_\Omega &= (q, \operatorname{div} \boldsymbol{w}_0)_\Omega + \delta \operatorname{sign}(\fint_\Omega q \,\mathrm{d}\boldsymbol{x})\,(q_0, \operatorname{div} \boldsymbol{\xi})_\Omega + \delta|\fint_\Omega q \,\mathrm{d}\boldsymbol{x}|(1, \operatorname{div} \boldsymbol{\xi})_\Omega \\
&\geqslant \beta_0(\nu)\|q_0\|_{\nu'} - \delta\|q_0\|_{\nu'}\|\operatorname{div} \boldsymbol{\xi}\|_\nu + \delta|\fint_\Omega q \,\mathrm{d}\boldsymbol{x}| \\
&\geqslant \frac{\beta_0(\nu)}{1 + |\Omega|^{1/\nu'}\|\operatorname{div} \boldsymbol{\xi}\|_\nu}\|q\|_{\nu'}.
\end{aligned}$$

Also, $\boldsymbol{w} \in \boldsymbol{X}^\nu$, and $\|\boldsymbol{w}\|_{1,\nu} \leqslant 1 + \delta\|\boldsymbol{\xi}\|_{1,\nu}$, which finally gives (2.25) with

$$\beta(\nu) = \frac{\beta_0(\nu)}{1 + |\Omega|^{1/\nu'}\|\operatorname{div} \boldsymbol{\xi}\|_\nu + \beta_0(\nu)|\Omega|^{1/\nu'}\|\boldsymbol{\xi}\|_{1,\nu}}. \tag{2.27}$$

This completes the proof. □

REMARK 2.10 There exists a continuous linear operator $\mathcal{B} : \mathrm{L}_0^\nu(\Omega) \to \mathbf{W}_0^{1,\nu}(\Omega)$, referred to as the Bogovskii operator, such that $\operatorname{div}(\mathcal{B}f) = f$ in Ω and $\|\mathcal{B}f\|_{1,\nu} \leqslant C_{\mathrm{div}}(\Omega, \nu)\|f\|_\nu$ (see Bogovskii, 1980; Amrouche & Girault, 1994; Novotný & Straškraba, 2004). In the preceding studies (see Franta *et al.*, 2005; Lanzendörfer, 2009), the Bogovskii operator, instead of the inf–sup condition, was applied directly. For $|\Gamma_P| = 0$ one observes $C_{\mathrm{div}}(\Omega, 2) \geqslant \beta_0(2)^{-1}$. For $|\Gamma_P| > 0$ the modified operator $\widetilde{\mathcal{B}}f := \mathcal{B}(f - (\int_\Omega f \,\mathrm{d}\boldsymbol{x}) \operatorname{div} \boldsymbol{\xi}) + (\int_\Omega f \,\mathrm{d}\boldsymbol{x})\boldsymbol{\xi}$ was utilized (see Lanzendörfer & Stebel, 2011b, Lemma 2.4). Note from (2.27) that the corresponding constant (see *ibid.*) $\widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, \nu)$ equals $\beta(\nu)^{-1}$.

REMARK 2.11 Lemma 2.9 reveals, in terms of the spaces $\boldsymbol{X}^p$, Q^p, why the additional constraint (1.5) is requisite to fix the level of pressure if $\partial\Omega = \Gamma_D$. Note that $(1, \operatorname{div} \boldsymbol{w})_\Omega = 0$ for all $\boldsymbol{w} \in \mathbf{W}_0^{1,\nu}(\Omega)$ and thus, obviously,

$$\inf_{q \in \mathrm{L}^{\nu'}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}_0^{1,\nu}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}} = 0.$$

Below, we require for given $\nu \in (1, \infty)$ that the families of spaces $\{\boldsymbol{X}_h^\nu\}_{h>0}$, $\{Q_h^\nu\}_{h>0}$, satisfy the discrete inf–sup condition:

(IS$^\nu$) for given $\nu \in (1, \infty)$ there exists a constant $\tilde{\beta}(\nu)$ independent of h such that

$$0 < \tilde{\beta}(\nu) \leqslant \inf_{q \in Q_h^\nu} \sup_{\boldsymbol{w} \in \boldsymbol{X}_h^\nu} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}.$$

The availability of (IS$^\nu$) and the value of $\tilde{\beta}(\nu)$ depend on the choice of the spaces $\boldsymbol{X}_h$ and Y_h. In Section 5 we will deal with the construction of appropriate spaces. For the purposes of Theorem 3.3 we also require the following modification of (IS$^\nu$):

(IS$_0^\nu$) there exists a constant $\tilde{\beta}_0(\nu)$, independent of h, such that

$$0 < \tilde{\beta}_0(\nu) \leqslant \inf_{q \in Y_h \cap \mathrm{L}_0^{\nu'}(\Omega)} \sup_{\boldsymbol{w} \in \boldsymbol{X}_h \cap \mathbf{W}_0^{1,\nu}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}.$$

REMARK 2.12 If $|\Gamma_P| = 0$ then (IS$_0^\nu$) is exactly (IS$^\nu$). In general, (IS$_0^\nu$) need not be implied by (IS$^\nu$) and vice versa. Let us suppose for a while that both conditions hold true. Since (2.27) in Lemma 2.9 indicates[3] $\beta_0(\nu) \geqslant \beta(\nu)$ on the continuous level, we can expect $\tilde{\beta}_0(\nu) \geqslant \tilde{\beta}(\nu)$ for typical choices of $\boldsymbol{X}_h$, Y_h. In such a case, the additional requirement of (IS$_0^\nu$) will guarantee convergence results for a larger range of γ_0; see (3.7) in Theorem 3.3 and (3.18) in Corollary 3.6.

Later, we will use (IS$_0^2$) in conjunction with the following observation: let (IS$_0^2$) hold, let $|\Gamma_P| > 0$ and $p \in (1, 2)$. For arbitrary $q \in Q_h^p$, we write $q = q_0 + ⨍_\Omega q \,\mathrm{d}\boldsymbol{x}$, where[4] $q_0 \in Y_h \cap \mathrm{L}_0^2(\Omega)$. Since $\|q\|_2 \leqslant \|q_0\|_2 + |\Omega|^{1/2} |⨍_\Omega q \,\mathrm{d}\boldsymbol{x}|$, we obtain

$$\tilde{\beta}_0(2) \left(\|q\|_2 - |\Omega|^{1/2} |⨍_\Omega q \,\mathrm{d}\boldsymbol{x}| \right) \leqslant \sup_{\boldsymbol{w} \in \boldsymbol{X}_h^2} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}} \quad \forall\, q \in Q_h^p. \tag{2.28}$$

3. Well-posedness of the problem

Below, we show the existence of solutions to (**pS**$_h$) (discrete solutions), we discuss the conditions guaranteeing the uniqueness of solutions to both (**pS**$_h$) and (pS), and we finally establish the existence of a solution to (pS) (a weak solution) as the limit of the discrete solutions.

[3]We did not prove $\beta_0(\nu) \geqslant \beta(\nu)$; (2.27) merely gives a lower bound for $\beta(\nu)$, which is lower than $\beta_0(\nu)$.

[4]Here, we assume that constants belong to Y_h.

Note that the well-posedness of (pS) with a convective term included has already been resolved: for $\Gamma_D = \partial\Omega$ this was published in Franta *et al.* (2005) and Lanzendörfer (2009), while the case $|\Gamma_P| > 0$ was dealt with in Lanzendörfer & Stebel (2011a). In these works, the proof was done in a different way to here: first a quasicompressible approximation to (pS) was established (by the Galerkin method), and later it was shown that this approximation converges (on the continuous level) to the 'incompressible' solution to (pS). Here, since our concern lies with the finite element discretization, the weak solution is established directly as a limit of discrete solutions, where the discrete solutions satisfy the (discrete) incompressibility constraint (2.23). Many of the estimates used here will be employed also in the next section. Compared to the previous studies, we slightly relax the restriction on γ_0 and—since we neglect convection—our procedure allows for $p \in (1, 2]$. We begin with the well-posedness of $(\mathbf{pS}_h)$.

THEOREM 3.1 (Existence of discrete solutions). Let Assumption 2.8 hold. Let $\boldsymbol{X}_h^p$ and Q_h^p fulfil (IS^p) with $\tilde{\beta}(p) > 0$ arbitrary.

Then there exists a solution to $(\mathbf{pS}_h)$. Any such solution $(\boldsymbol{v}_h, \pi_h)$ satisfies the *a priori* estimate

$$\|\boldsymbol{v}_h\|_{1,p} + \|\boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \tilde{\beta}(p)\|\pi_h\|_{p'} \leqslant K. \tag{3.1}$$

The constant K depends only on $\Omega, \Gamma_D, p, \varepsilon_0, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$.

Proof. For any $\delta > 0$ (small) we consider the quasicompressible problem
$(\mathbf{pS}_h^\delta)$: find $(\boldsymbol{v}_h^\delta, \pi_h^\delta) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ such that

$$\left(\boldsymbol{S}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{w}_h\right)_\Omega - \left(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h\right)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{X}_h^p, \tag{3.2}$$

$$\delta\left(\pi_h^\delta, q_h\right)_\Omega + \left(\operatorname{div} \boldsymbol{v}_h^\delta, q_h\right)_\Omega = 0 \quad \forall\, q_h \in Q_h^p. \tag{3.3}$$

The inserted term $\delta(\pi_h^\delta, q_h)_\Omega$ ensures the coercivity of the equations with respect to the pressure and allows use of the Brouwer fixed-point theorem to establish the solution to $(\mathbf{pS}_h^\delta)$. Indeed, setting $\boldsymbol{w}_h := \boldsymbol{v}_h^\delta - \boldsymbol{v}_{0,h}$ and $q_h := \pi_h^\delta$, summing the equations and using Hölder's and Korn's inequality, $(2.24)_1$, the embedding $\mathrm{tr}(\mathbf{W}^{1,p}(\Omega)) \hookrightarrow \mathbf{L}^{p^\#}(\partial\Omega)$, the estimate

$$\left(\boldsymbol{S}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{v}_h^\delta - \boldsymbol{D}\boldsymbol{v}_{0,h}\right)_\Omega \geqslant \frac{\sigma_0}{2p}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^p - \frac{\sigma_1}{p-1}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^{p-1}\|\boldsymbol{D}\boldsymbol{v}_{0,h}\|_p - \frac{\sigma_0}{2p}|\Omega|\varepsilon^p$$

due to (2.6), and Young's inequality, we obtain the *a priori* bound

$$\delta\|\pi_h^\delta\|_2^2 + \|\boldsymbol{v}_h^\delta\|_{1,p}^p + \|\boldsymbol{S}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta)\|_{p'}^{p'} \leqslant C,$$

where $C > 0$ depends on $\Omega, \Gamma_D, p, \varepsilon_0, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$. In particular, C is independent of δ and h. Therefore, using (IS^p) and (3.2), we observe that

$$\tilde{\beta}(p)\|\pi_h^\delta\|_{p'} \leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^p} \frac{(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}} \leqslant C,$$

with $C > 0$ and $\tilde{\beta}(p) > 0$, independent of δ and h. The same arguments applied to $(\mathbf{pS}_h)$ prove (3.1).

The uniform bounds above and the fact that $\boldsymbol{X}_h^p$ and Q_h^p are of finite dimension imply that there is

$(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ such that (for some sequence $\delta_n \searrow 0$)

$$\boldsymbol{v}_h^{\delta_n} \to \boldsymbol{v}_h \quad \text{in } \mathbf{W}^{1,p}(\Omega),$$

$$\pi_h^{\delta_n} \to \pi_h \quad \text{in } \mathrm{L}^{p'}(\Omega),$$

$$\boldsymbol{S}\left(\pi_h^{\delta_n}, \boldsymbol{D}\boldsymbol{v}_h^{\delta_n}\right) \to \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h) \quad \text{in } \mathrm{L}^{p'}(\Omega)^{d\times d}.$$

Consequently, $(\boldsymbol{v}_h, \pi_h)$ is a solution to $(\mathbf{pS}_h)$. □

Note that the constant K in (3.1) does not depend on h since $\|\boldsymbol{v}_{0,h}\|_{1,p} \leqslant 2\|\boldsymbol{v}_0\|_{1,p}$ for $h \leqslant h_0$. According to Theorem 3.1, discrete solutions exist regardless of Assumption (A2). However, uniqueness of a solution can only be shown by means of (A2) under a smallness assumption on γ_0 as depicted by

THEOREM 3.2 (Uniqueness). Let the assumptions of Theorem 3.1 hold. If (IS^2) is satisfied and

$$\gamma_0 < \tilde{\beta}(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}, \tag{3.4}$$

then the solution to $(\mathbf{pS}_h)$ is determined uniquely.

Similarly, there is at most one solution to (pS) provided that Assumption 2.8 is satisfied and

$$\gamma_0 < \beta(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}.$$

Proof. We prove the uniqueness of a solution to $(\mathbf{pS}_h)$; the other result is analogous. Let $(\boldsymbol{v}_h^i, \pi_h^i)$, $i = 1, 2$, be two solutions to $(\mathbf{pS}_h)$. Then, we realize that

$$\left(\boldsymbol{S}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{S}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{w}_h\right)_\Omega = (\pi_h^1 - \pi_h^2, \operatorname{div} \boldsymbol{w}_h)_\Omega \quad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p.$$

In particular, choosing $\boldsymbol{w}_h := \boldsymbol{v}_h^1 - \boldsymbol{v}_h^2$, we observe

$$\left(\boldsymbol{S}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{S}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{v}_h^1 - \boldsymbol{D}\boldsymbol{v}_h^2\right)_\Omega = 0,$$

and we thus obtain from (2.10) that

$$d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2)^2 \leqslant \frac{\gamma_0^2}{\sigma_0^2} \left\|\pi_h^1 - \pi_h^2\right\|_2^2. \tag{3.5}$$

Hence, (IS^2) and (2.12) yield the following estimate

$$\begin{aligned}
\tilde{\beta}(2)\left\|\pi_h^1 - \pi_h^2\right\|_2 &\leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^2} \frac{(\pi_h^1 - \pi_h^2, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} \\
&\leqslant \left\|\boldsymbol{S}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{S}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2)\right\|_2 \\
&\leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} d\left(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2\right) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \left\|\pi_h^1 - \pi_h^2\right\|_2,
\end{aligned} \tag{3.6}$$

which together with (3.5) and (3.4) leads to $\pi_h^1 = \pi_h^2$ a.e. in Ω and to $d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) = 0$. But this completes the proof because (2.17), (2.18) and the *a priori* bound (3.1) ensure that $\|\boldsymbol{D}\boldsymbol{v}_h^1 - \boldsymbol{D}\boldsymbol{v}_h^2\|_p^2 \leqslant C\, d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2)^2 = 0$. Since $|\Gamma_D| > 0$, Lemma 2.7 yields $\boldsymbol{v}_h^1 = \boldsymbol{v}_h^2$ a.e. in Ω. □

THEOREM 3.3 (Convergence of discrete solutions). Let the assumptions of Theorem 3.1 hold, let the discrete spaces $\{(\boldsymbol{X}_h^p, Q_h^p)\}_{h>0}$ satisfy (2.21) and let $\{\boldsymbol{v}_{0,h}\}_{h>0}$ satisfy (2.24). In addition, let (IS_0^2) hold and let γ_0 fulfill

$$\gamma_0 < \tilde{\beta}_0(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}. \tag{3.7}$$

Then the solutions to $(\mathbf{pS}_h)$ converge to a solution to (pS) as follows:

$$(\boldsymbol{v}_{h_n}, \pi_{h_n}) \to (\boldsymbol{v}, \pi) \quad \text{strongly in } \mathbf{W}^{1,p}(\Omega) \times \mathrm{L}^{p'}(\Omega) \quad \text{for some } h_n \searrow 0. \tag{3.8}$$

In addition, if the solution to (pS) is unique, then the whole sequence $\{(\boldsymbol{v}_h, \pi_h)\}_{h>0}$ tends to $(\boldsymbol{v}, \pi)$.

REMARK 3.4 Note that $\tilde{\beta}_0(2)$ appears in (3.7) even in the case $|\Gamma_P| > 0$. In general, this guarantees convergence for a larger range of γ_0 compared to, e.g., (3.4); see Remark 2.12.

Proof of Theorem 3.3. Theorem 3.1 ensures that solutions $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ to $(\mathbf{pS}_h)$ exist and satisfy the *a priori* estimate (3.1). Hence, there exist $(\boldsymbol{v}, \pi) \in (\boldsymbol{v}_0 + \boldsymbol{X}^p) \times Q^p$ and $\overline{\boldsymbol{S}} \in \mathrm{L}^{p'}(\Omega)^{d\times d}$ such that for a sequence $h_n \searrow 0$ there holds

$$\boldsymbol{v}_{h_n} \rightharpoonup \boldsymbol{v} \quad \text{weakly in } \mathbf{W}^{1,p}(\Omega), \tag{3.9}$$

$$\pi_{h_n} \rightharpoonup \pi \quad \text{weakly in } \mathrm{L}^{p'}(\Omega), \tag{3.10}$$

$$\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) \rightharpoonup \overline{\boldsymbol{S}} \quad \text{weakly in } \mathrm{L}^{p'}(\Omega)^{d\times d}. \tag{3.11}$$

Obviously, the weak limits satisfy (2.20) and

$$(\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}, \boldsymbol{w})_{\Gamma_P} \quad \forall\, \boldsymbol{w} \in \boldsymbol{X}^p. \tag{3.12}$$

Here, we have used the density (2.21). Subtracting (3.12) and (2.22), we observe

$$(\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{w}_{h_n})_\Omega = (\pi_{h_n} - \pi, \operatorname{div} \boldsymbol{w}_{h_n})_\Omega \quad \forall\, \boldsymbol{w}_{h_n} \in \boldsymbol{X}_{h_n}^p. \tag{3.13}$$

Then (3.13) with $\boldsymbol{w}_h := \boldsymbol{v}_{h_n} - \boldsymbol{v}_{0,h_n}$ implies

$$(\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega = (\pi_{h_n} - \pi, \operatorname{div}(\boldsymbol{v}_{h_n} - \boldsymbol{v}_{0,h_n}))_\Omega$$
$$+ (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}_{0,h_n})_\Omega + (\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}), \boldsymbol{D}\boldsymbol{v}_{0,h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega - (\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega.$$

Using (2.24), (2.23) and (2.20), we realize that

$$(\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega = (\pi, \operatorname{div}(\boldsymbol{v} - \boldsymbol{v}_{h_n}))_\Omega$$
$$+ (\pi, \operatorname{div}(\boldsymbol{v}_{0,h_n} - \boldsymbol{v}_0))_\Omega + (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega$$
$$+ (\overline{\boldsymbol{S}} - \boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}), \boldsymbol{D}\boldsymbol{v})_\Omega + (\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{v}_{0,h_n})_\Omega - (\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega.$$

Recalling (3.9)–(3.11) and using (2.24) we conclude that

$$(\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega = \mathrm{o}(1), \quad h_n \searrow 0, \tag{3.14}$$

where o(1) denotes an arbitrary sequence that tends to zero for $h_n \searrow 0$.

Furthermore, from (2.18), (3.1), (2.10) and (3.14) we deduce (cf. (3.5))

$$C\|\boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}\|_p^2 \leqslant d(\boldsymbol{v}_{h_n}, \boldsymbol{v})^2 \leqslant \frac{\gamma_0^2}{\sigma_0^2}\|\pi_{h_n} - \pi\|_2^2 + \mathrm{o}(1) \tag{3.15}$$

for some $C > 0$ independent of h_n. We suppose for a while that

$$\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 \leqslant \|\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + \mathrm{o}(1). \tag{3.16}$$

Then combining (3.16) and (2.12), we arrive at

$$\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 \leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}_{h_n}, \boldsymbol{v}) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi_{h_n} - \pi\|_2 + \mathrm{o}(1), \quad h_n \searrow 0.$$

Using (3.15) and assumption (3.7), we conclude $\|\pi_{h_n} - \pi\|_2 \leqslant \mathrm{o}(1)$. Consequently, (3.15) also yields $\|\boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}\|_p \leqslant \mathrm{o}(1)$, which finally implies that

$$\pi_{h_n} \to \pi \text{ a.e. in } \Omega \quad \text{and} \quad \boldsymbol{D}\boldsymbol{v}_{h_n} \to \boldsymbol{D}\boldsymbol{v} \text{ a.e. in } \Omega.$$

This allows us to apply Vitali's lemma and to identify $\overline{\boldsymbol{S}}$,

$$\int_\Omega \boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) : \boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) : \boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\boldsymbol{S}} : \boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \quad \forall\, \boldsymbol{w} \in \boldsymbol{X}^p.$$

Therefore, it only remains to show (3.16). Define $\tilde{\boldsymbol{w}}_{h_n} \in \boldsymbol{X}^2_{h_n}$, $\|\tilde{\boldsymbol{w}}_{h_n}\|_{1,2} = 1$, such that

$$\sup_{\boldsymbol{w}_{h_n} \in \boldsymbol{X}^2_{h_n}} \frac{(\pi_{h_n} - \pi, \operatorname{div} \boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} = (\pi_{h_n} - \pi, \operatorname{div} \tilde{\boldsymbol{w}}_{h_n})_\Omega.$$

Then, there exists $\tilde{\boldsymbol{w}} \in \boldsymbol{X}^2$ such that (for a not-relabelled subsequence) $\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}} \rightharpoonup 0$ weakly in $\boldsymbol{X}^2$ and[5] $\|\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}}\|_{1,2} \leqslant 1$. Hence, using (3.13) and (3.11) we obtain

$$\begin{aligned}(\pi_{h_n} - \pi, \operatorname{div} \tilde{\boldsymbol{w}}_{h_n})_\Omega &= (\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{S}}, \boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n} - \boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega + \mathrm{o}(1) \\ &= (\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n} - \boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega + \mathrm{o}(1) \\ &\leqslant \|\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + \mathrm{o}(1), \qquad h_n \searrow 0.\end{aligned}$$

[5] Indeed, for n large enough, $\|\tilde{\boldsymbol{w}}\|_{1,2}^2 \leqslant 2(\tilde{\boldsymbol{w}}_{h_n}, \tilde{\boldsymbol{w}})_{1,2;\Omega}$, which implies $\|\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}}\|_{1,2}^2 \leqslant \|\tilde{\boldsymbol{w}}_{h_n}\|_{1,2}^2$ $(= 1)$.

Recalling (2.28) and using that $\fint_\Omega \pi_{h_n} - \pi \,\mathrm{d}\boldsymbol{x} \to 0$, we deduce that for any $q_{h_n} \in Q^p_{h_n}$,

$$\begin{aligned}
\tilde{\beta}_0(2)\|\pi_{h_n} - q_{h_n}\|_2 &\leqslant \sup_{\boldsymbol{w}_{h_n}\in \boldsymbol{X}^2_{h_n}} \frac{(\pi_{h_n} - q_{h_n}, \operatorname{div}\boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} + \tilde{\beta}_0(2)|\Omega|^{1/2}\left|\fint_\Omega \pi_{h_n} - q_{h_n}\,\mathrm{d}\boldsymbol{x}\right| \\
&\leqslant \sup_{\boldsymbol{w}_{h_n}\in \boldsymbol{X}^2_{h_n}} \frac{(\pi_{h_n} - \pi, \operatorname{div}\boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} + \|\pi - q_{h_n}\|_2 + C\left|\fint_\Omega \pi_{h_n} - q_{h_n}\,\mathrm{d}\boldsymbol{x}\right| \\
&\leqslant \|\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + C\,\|\pi - q_{h_n}\|_2 + \mathrm{o}(1), \qquad h_n \searrow 0,
\end{aligned}$$

with $C > 0$ independent of h_n. Using the density of $\{Q^p_{h_n}\}$ in Q^p, we finally assert (3.16),

$$\begin{aligned}
\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 &\leqslant \tilde{\beta}_0(2) \inf_{q_{h_n}\in Q^p_{h_n}} \left\{\|\pi_{h_n} - q_{h_n}\|_2 + \|q_{h_n} - \pi\|_2\right\} \\
&\leqslant \|\boldsymbol{S}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + \mathrm{o}(1), \qquad h_n \searrow 0.
\end{aligned}$$

This completes the proof. □

Theorem 3.3 guarantees the existence of a solution to (pS) provided that we have a suitable family of discrete spaces $\{\boldsymbol{X}^p_h, Q^p_h\}_{h>0}$. The proper existence result is formulated in Corollary 3.6. In the following lemma we construct such a family of discrete spaces that satisfies (IS^p) and (IS$^2_0$) with a constant $\tilde{\beta}_0(2)$, which is almost equal to $\beta_0(2)$.

LEMMA 3.5 Let Ω, Γ_D, Γ_P, p be as in Assumption 2.8. Then, for any $\delta > 0$ (small), there exists a family of finite-dimensional spaces $\{\boldsymbol{X}_{h_n}\}$, $\{Y_{h_n}\}$, $h_n \searrow 0$ that satisfy (2.21) and fulfill (IS^p) and (IS$^2_0$) with

$$\tilde{\beta}(p) \geqslant \beta(p) - \delta \quad \text{and} \quad \tilde{\beta}_0(2) \geqslant \beta_0(2) - \delta. \tag{3.17}$$

Proof. Consider arbitrary $h_n \searrow 0$, $n = 1, 2, \ldots$. Since $\mathbf{W}^{1,2}_0(\Omega)$, $\boldsymbol{X}^p$, Q^p are separable Banach spaces with the bases $\{\bar{\boldsymbol{w}}_n\}_{n=1}^\infty$, $\{\boldsymbol{w}_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$, respectively, and since $\mathbf{W}^{1,2}_0(\Omega) \subset \boldsymbol{X}^p$, we can define the Galerkin spaces by $\tilde{\boldsymbol{X}}_m := \operatorname{span}\{\bar{\boldsymbol{w}}_i, \boldsymbol{w}_i\}_{i=1}^m$ and $\tilde{Y}_n := \operatorname{span}\{q_i\}_{i=1}^n$, clearly allowing for (2.21). In order to ensure (3.17) we only need to choose suitable pairs of the spaces, i.e., to any discrete pressure space we have to assign a rich enough discrete velocity space. We show this only for (IS$^2_0$) and (3.17)$_2$, the inclusion of (IS^p) is obvious.

Due to (2.21) and Lemma 2.9, for any $q \in \mathrm{L}^2_0(\Omega)$ there exists $k(q)$ such that

$$\beta_0(2) - \delta \leqslant \sup_{\boldsymbol{w}\in\tilde{\boldsymbol{X}}_{k(q)}\cap\mathbf{W}^{1,2}_0(\Omega)} \frac{(q, \operatorname{div}\boldsymbol{w})_\Omega}{\|q\|_2\|\boldsymbol{w}\|_{1,2}}.$$

We choose minimal such $k(q)$. For n fixed, define $m(n) := \sup\{k(q) : q \in \tilde{Y}_n \cap \mathrm{L}^2_0(\Omega)\}$. It is easy to see that $Y_{h_n} := \tilde{Y}_n$ and $\boldsymbol{X}_{h_n} := \tilde{\boldsymbol{X}}_{m(n)}$ satisfy (IS$^2_0$) and (3.17). It remains to prove that $m(n)$ is finite. This is shown by contradiction: let $m(n)$ be infinite. Then, we find a sequence $q_j \in \tilde{Y}_n \cap \mathrm{L}^2_0(\Omega)$, $\|q_j\|_2 = 1$, $j = 1, 2, \ldots$, such that $k(q_j) > j$ and

$$\sup_{\boldsymbol{w}\in\tilde{\boldsymbol{X}}_j\cap\mathbf{W}^{1,2}_0(\Omega)} \frac{(q_j, \operatorname{div}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}} < \beta_0(2) - \delta.$$

Since $\tilde{Y}_n$ is of finite dimension, we find some $\tilde{q} \in \tilde{Y}_n \cap \mathrm{L}^2_0(\Omega)$, $\|\tilde{q}\|_2 = 1$, and a subsequence $j_i > i$ such that $\|q_{j_i} - \tilde{q}\|_2 < \delta/2$ for $i = 1, 2, \dots$. But then, the inequality

$$\sup_{\boldsymbol{w} \in \tilde{\boldsymbol{X}}_i \cap \mathbf{W}^{1,2}_0(\Omega)} \frac{(\tilde{q}, \operatorname{div} \boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}} < \beta_0(2) - \delta/2$$

holds for any $i = 1, 2, \dots$, which combined with the density (2.21) and Lemma 2.9 gives the desired contradiction. □

COROLLARY 3.6 (Existence of solutions). Let Assumption 2.8 hold and

$$\gamma_0 < \beta_0(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}. \tag{3.18}$$

Then, there exists a solution to (pS). Moreover, any solution to (pS) fulfils the *a priori* estimate

$$\|\boldsymbol{v}\|_{1,p} + \|\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\|_{p'} + \beta(p)\|\pi\|_{p'} \leqslant K. \tag{3.19}$$

The constant K depends only on Ω, Γ_D, p, ε_0, σ_0, σ_1, $\|\boldsymbol{f}\|_{p'}$, $\|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_0\|_{1,p}$.

Proof. The *a priori* estimate (3.19) follows analogously to the proof of (3.1). The existence of a solution results from Theorems 3.1 and 3.3 and Lemma 3.5. □

4. *A priori* error estimates

In this section we aim to derive *a priori* estimates for the error in the approximation $\boldsymbol{v} - \boldsymbol{v}_h$ and $\pi - \pi_h$. For the remainder of this paper let us use the convention that $(\boldsymbol{v}, \pi)$ and $(\boldsymbol{v}_h, \pi_h)$ denote the solution to (pS) and ($\mathbf{pS}_h$), respectively, whose existence and uniqueness was shown in the previous section. The main results are given by Corollaries 4.3 and 4.4, which state *a priori* error estimates in the form of a best approximation result.

LEMMA 4.1 Let Assumption 2.8 hold. For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that for all $\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}^p_h)$ and $r_h \in Q^p_h$ there holds

$$d(\boldsymbol{v}, \boldsymbol{v}_h) \leqslant c_\delta(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}) + \left(\frac{1}{\sigma_0} + \delta\right)\gamma_0\|\pi - \pi_h\|_2,$$

where the constant c_δ also depends on p, ε_0, σ_0, σ_1, Γ_D, Ω, $\|\boldsymbol{f}\|_{p'}$, $\|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_0\|_{1,p}$.

Proof. Let $(\boldsymbol{u}_h, r_h)$ be an arbitrary element of $(\boldsymbol{v}_{0,h} + \boldsymbol{V}^p_h) \times Q^p_h$. From (pS) and ($\mathbf{pS}_h$) it follows that

$$(\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega = (\pi - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\pi - r_h, \operatorname{div} \boldsymbol{w}_h)_\Omega$$

for all $\boldsymbol{w}_h \in \boldsymbol{V}^p_h$. This, with $\boldsymbol{w}_h := (\boldsymbol{u}_h - \boldsymbol{v}_h) \in \boldsymbol{V}^p_h$, implies

$$\begin{aligned}(\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h)_\Omega &= (\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega \\ &\quad + (\pi - r_h, \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega =: I_1 + I_2.\end{aligned}$$

Applying (2.10) we conclude that

$$\frac{\sigma_0}{2} d(\boldsymbol{v}, \boldsymbol{v}_h)^2 \leqslant I_1 + I_2 + \frac{\gamma_0^2}{2\sigma_0}\|\pi - \pi_h\|_2^2. \tag{4.1}$$

It remains to estimate I_1 and I_2. First of all, we split the term I_1 in the following way,

$$I_1 = (\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(\pi_h, \boldsymbol{Du}_h), \boldsymbol{Dv} - \boldsymbol{Du}_h)_\Omega$$

$$+ (\boldsymbol{S}(\pi_h, \boldsymbol{Du}_h) - \boldsymbol{S}(\pi_h, \boldsymbol{Dv}_h), \boldsymbol{Dv} - \boldsymbol{Du}_h)_\Omega =: I_3 + I_4.$$

Due to (2.11) and Lemma 2.5, for each $\delta_1 > 0$ there exists $c_{\delta_1} > 0$ such that

$$I_3 \leqslant c_{\delta_1} d(\boldsymbol{v}, \boldsymbol{u}_h)^2 + \delta_1 \gamma_0^2 \|\pi - \pi_h\|_2^2 \leqslant c_{\delta_1} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 + \delta_1 \gamma_0^2 \|\pi - \pi_h\|_2^2.$$

Let φ be defined by (2.2). In order to get an upper bound for I_4, we apply Lemma 2.2 and Young's inequality (2.1) for the shifted N-functions, φ_a, taking into account that the Δ_2-constants of φ_a, $(\varphi_a)^*$ depend only on p and do not depend on the shift-parameter $a \geqslant 0$. Hence, for any $\delta_2 > 0$ we obtain

$$\begin{aligned} I_4 &\leqslant c \int_\Omega \varphi'_{\varepsilon + |\boldsymbol{Du}_h|}(|\boldsymbol{Du}_h - \boldsymbol{Dv}_h|) |\boldsymbol{Dv} - \boldsymbol{Du}_h| \, \mathrm{d}\boldsymbol{x} \\ &\leqslant \delta_2 \int_\Omega \varphi_{\varepsilon + |\boldsymbol{Du}_h|}(|\boldsymbol{Du}_h - \boldsymbol{Dv}_h|) \, \mathrm{d}\boldsymbol{x} + c_{\delta_2} \int_\Omega \varphi_{\varepsilon + |\boldsymbol{Du}_h|}(|\boldsymbol{Dv} - \boldsymbol{Du}_h|) \, \mathrm{d}\boldsymbol{x} \\ &\sim \delta_2 \|\boldsymbol{F}(\boldsymbol{Du}_h) - \boldsymbol{F}(\boldsymbol{Dv}_h)\|_2^2 + c_{\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 \\ &\leqslant \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2, \end{aligned}$$

where we have also used Lemma 2.5. Collecting the estimates above we arrive at

$$I_1 \leqslant c_{\delta_1, \delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 + \delta_1 \gamma_0^2 \|\pi - \pi_h\|_2^2 + \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2. \tag{4.2}$$

Next we estimate I_2. Using Korn's & Young's inequality, applying Lemma 2.6 with $\nu = p$, recalling the uniform *a priori* bounds (3.1) and (3.19), we deduce that for each $\delta_3 > 0$ there exists $c_{\delta_3} > 0$ so that

$$\begin{aligned} I_2 &\leqslant |(\pi - r_h, \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega| \leqslant c \|\pi - r_h\|_{p'} \|\boldsymbol{Du}_h - \boldsymbol{Dv}_h\|_p \\ &\leqslant \delta_3 \big(\|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \|\boldsymbol{Dv} - \boldsymbol{Dv}_h\|_p^2 \big) + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\ &\leqslant \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \delta_3 c \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Dv}_h)\|_2^2 \|\varepsilon + |\boldsymbol{Dv}| + |\boldsymbol{Dv}_h|\|_p^{2-p} + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\ &\leqslant \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_3} \|\pi - r_h\|_{p'}^2. \end{aligned} \tag{4.3}$$

Combining the estimates (4.1), (4.2) and (4.3), we conclude that

$$\begin{aligned} \frac{\sigma_0}{2} d(\boldsymbol{v}, \boldsymbol{v}_h)^2 &\leqslant \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_1, \delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 + \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 \\ &\quad + c_{\delta_3} \|\pi - r_h\|_{p'}^2 + \left(\frac{1}{2\sigma_0} + \delta_1 \right) \gamma_0^2 \|\pi - \pi_h\|_2^2. \end{aligned}$$

Multiplying this by $2/\sigma_0$ and taking the square root, we easily complete the proof. $\square$

Lemma 4.1 enables us to estimate the pressure error in the L^2-norm.

THEOREM 4.2 Let Assumption 2.8 hold. Let the discrete spaces fulfil (IS^2) and let the parameters meet condition (3.4): $\gamma_0 < \tilde{\beta}(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0+\sigma_1}$. Then, there exists a constant $c > 0$, which depends only on $p, \varepsilon, \gamma_0, \sigma_0, \sigma_1, \tilde{\beta}(2), \Gamma_D, \Omega, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}, \|\boldsymbol{v}_0\|_{1,p}$, so that the pressure error is bounded in $L^2(\Omega)$ by

$$\|\pi - \pi_h\|_2 \leqslant c \inf_{\boldsymbol{u}_h \in \boldsymbol{v}_{0,h}+\boldsymbol{V}_h^p} (\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p) + c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'}.$$

Proof. Let $(\boldsymbol{u}_h, r_h)$ be an arbitrary element of $(\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$. Then, (pS) and ($\mathbf{pS}_h$) imply

$$(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega + (r_h - \pi, \operatorname{div} \boldsymbol{w}_h)_\Omega \tag{4.4}$$

for all $\boldsymbol{w}_h \in \boldsymbol{X}_h^p$. Using ($IS^2$) and (4.4), we deduce (compare with (3.6))

$$\tilde{\beta}(2)\|r_h - \pi_h\|_2 \leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^2} \frac{(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} \leqslant \|\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|r_h - \pi\|_2.$$

Applying (2.12) and Lemma 4.1 we conclude that for each $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\begin{aligned}
\tilde{\beta}(2)\|r_h - \pi_h\|_2 &\leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}, \boldsymbol{v}_h) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2 \\
&\leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} c_\delta (\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}) \\
&\quad + \sigma_1 \varepsilon^{\frac{p-2}{2}} \left(\frac{1}{\sigma_0} + \delta\right) \gamma_0 \|\pi - \pi_h\|_2 + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2.
\end{aligned}$$

Using Minkowski's inequality and $L^{p'}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \leqslant 2$ we arrive at

$$\begin{aligned}
\|\pi - \pi_h\|_2 &\leqslant c_\delta (\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}) \\
&\quad + \tilde{\beta}(2)^{-1} \sigma_1 \varepsilon^{\frac{p-2}{2}} \left(\frac{1}{\sigma_0} + \delta\right) \gamma_0 \|\pi - \pi_h\|_2 + \tilde{\beta}(2)^{-1} \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2.
\end{aligned}$$

Recalling (3.4), and choosing $\delta > 0$ sufficiently small, we can absorb all terms, that include the pressure error, into the left-hand side. Hence, we get the desired result. □

COROLLARY 4.3 Let the assumptions of Theorem 4.2 be satisfied. Then, the error of approximation of the velocity field is bounded by

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant c \inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h}+\boldsymbol{V}_h^p)} (\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p) + c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'}. \tag{4.5}$$

Proof. The estimate follows from Lemma 2.5, Lemma 4.1 and Theorem 4.2. □

COROLLARY 4.4 Let the assumptions of Theorem 4.2 hold. In addition, let (IS^p) hold and

$$\gamma_0 < \tilde{\beta}(p)\varepsilon^{\frac{2-p}{2}}. \tag{4.6}$$

Then, the error of approximation of the pressure field is bounded in $\mathrm{L}^{p'}(\Omega)$ by

$$\|\pi - \pi_h\|_{p'} \leqslant c\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in Q_h^p} \|r_h - \pi\|_{p'}. \tag{4.7}$$

Proof. The estimate is again based on the inf–sup inequality (IS^p). Using (IS^p), (4.4), Hölder's inequality, (2.13) and (2.17), for arbitrary $r_h \in Q_h^p$ we obtain the estimate

$$\begin{aligned}
\tilde{\beta}(p)\|r_h - \pi_h\|_{p'} &\leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^p} \frac{(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}} \\
&\leqslant \|\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \|r_h - \pi\|_{p'} \\
&\leqslant c\, \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_{p'} + \|r_h - \pi\|_{p'}.
\end{aligned}$$

Due to assumption (4.6) this completes the proof. □

In practice, one never obtains the solution $(\boldsymbol{v}_h, \pi_h)$ to problem ($\mathbf{pS}_h$) exactly. Instead, one obtains its approximation $(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$, satisfying

$$(\boldsymbol{S}(\tilde{\pi}_h, \boldsymbol{D}\tilde{\boldsymbol{v}}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\tilde{\pi}_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} + \langle \boldsymbol{e}, \boldsymbol{w}_h\rangle \quad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p,$$

$$(\operatorname{div} \tilde{\boldsymbol{v}}_h, q_h)_\Omega = \langle g, q_h\rangle \quad \forall q_h \in Q_h^p,$$

where $\boldsymbol{e} \in (\boldsymbol{X}_h^p)^*$, $g \in (Q_h^p)^*$ and the brackets $\langle\cdot,\cdot\rangle$ denote the corresponding duality pairings. Here, $\boldsymbol{e} = \boldsymbol{e}(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ and $g = g(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ represent some additional error which includes, e.g., the residual associated with the approximate solution to the nonlinear algebraic problem or the error due to numerical integration. However, provided that one is able to estimate $\boldsymbol{e}$ and g, then one can derive error estimates for $\boldsymbol{v} - \tilde{\boldsymbol{v}}_h$ and $\pi - \tilde{\pi}_h$ similar to those derived above by following the same procedure. For instance, denoting $|\langle \boldsymbol{e}, \boldsymbol{w}_h\rangle| \leqslant E\,\|\boldsymbol{w}_h\|_{1,p}$ and $|\langle g, q_h\rangle| \leqslant G\,\|q_h\|_2$ (with E, G independent of h and assuming, say, $E, G \leqslant 1$, such that $\|\boldsymbol{D}\tilde{\boldsymbol{v}}_h\|_p$ remains reasonably bounded), one can show (cf. (4.5), (4.7)):

$$\begin{aligned}
\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2 \leqslant c &\inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)} (\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p) \\
&+ c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'} + c(E + G),
\end{aligned}$$

$$\|\pi - \tilde{\pi}_h\|_{p'} \leqslant c\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in Q_h^p} \|r_h - \pi\|_{p'} + cE.$$

5. Finite element approximation

In this section we consider some finite element approximations of (pS) that satisfy the abstract theory of the previous sections. We assume that, for ease of exposition, Ω is a polygonal/polyhedral domain and that $\mathbb{T}_h$ is a shape regular decomposition of Ω into d-dimensional simplices (or quadrilaterals/hexahedra) so that $\overline{\Omega} = \bigcup_{K \in \mathbb{T}_h} \overline{K}$. By h_K we denote the diameter of an element $K \in \mathbb{T}_h$; the mesh parameter h represents the maximum diameter of the elements, i.e., $h := \max\{h_K;\ K \in \mathbb{T}_h\}$. We assume that $\mathbb{T}_h$ is nondegenerate (see Brenner & Scott, 1994). Hence, the neighbourhood S_K of $K \in \mathbb{T}_h$, which denotes the union of all elements in $\mathbb{T}_h$ touching K, fulfils $|K| \sim |S_K|$ with constants independent of h. Furthermore, the number of elements in S_K is uniformly bounded with respect to $K \in \mathbb{T}_h$.

Let X_h and Y_h be appropriate finite element spaces defined on $\mathbb{T}_h$ that satisfy $X_h \subset \mathrm{W}^{1,\infty}(\Omega)$ and $Y_h \subset \mathrm{L}^\infty(\Omega)$. We recall that the finite element spaces for the velocity and pressure are given by $\boldsymbol{X}_h^p := \boldsymbol{X}_h \cap \boldsymbol{X}^p$, $\boldsymbol{X}_h = [X_h]^d$ and $Q_h^p := Y_h \cap Q^p$. In order to ensure approximation properties and the discrete inf–sup conditions, we need to specify the choice of spaces.

ASSUMPTION 5.1 (Approximation property of X_h and Y_h). We assume that X_h contains the set of linear polynomials on Ω. Moreover, we suppose that there exists a linear projection $\boldsymbol{j}_h : \mathbf{W}^{1,1}(\Omega) \to \boldsymbol{X}_h$ and an interpolation operator $i_h : \mathrm{W}^{1,1}(\Omega) \to Y_h$ such that

(1) $\boldsymbol{j}_h$ preserves zero boundary values on Γ_D, such that $\boldsymbol{j}_h(\boldsymbol{X}^p) \subset \boldsymbol{X}_h^p$,

(2) $\boldsymbol{j}_h$ is locally $\mathrm{W}^{1,1}$-stable in the sense that there exists $c > 0$ (independent of h) such that

$$\fint_K |\boldsymbol{j}_h \boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \leqslant c \fint_{S_K} |\boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} + c \fint_{S_K} h_K |\nabla \boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \quad \forall\, \boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega),\ \ \forall\, K \in \mathbb{T}_h, \tag{5.1}$$

where S_K denotes a local neighbourhood of K (as defined above),

(3) $\boldsymbol{j}_h$ preserves divergence[6] in the Y_h^*-sense, i.e.,

$$(\operatorname{div} \boldsymbol{w}, q_h)_\Omega = (\operatorname{div} \boldsymbol{j}_h \boldsymbol{w}, q_h)_\Omega \quad \forall\, \boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega),\ \ \forall\, q_h \in Y_h, \tag{5.2}$$

(4) i_h preserves mean values, i.e., $i_h(Q^p) \subset Q_h^p$, and, for any $\nu \geqslant 1$, i_h satisfies

$$\|q - i_h q\|_\nu \leqslant c h \|q\|_{1,\nu} \quad \forall\, q \in \mathrm{W}^{1,\nu}(\Omega). \tag{5.3}$$

Later, we will suppose that functions in X_h satisfy the following global inverse inequality.

ASSUMPTION 5.2 (Inverse property of X_h). For $\nu, \mu \in [1, \infty]$ and $0 \leqslant m \leqslant l$ there holds

$$\|w_h\|_{l,\nu} \leqslant C h^{m-l+\min(0, \frac{d}{\nu} - \frac{d}{\mu})} \|w_h\|_{m,\mu} \quad \forall\, w_h \in X_h. \tag{5.4}$$

Assumption 5.2 usually requires that the mesh is quasiuniform in the sense of Brenner & Scott (1994). Assumption 5.1 is similar to Assumption 2.21 in Belenki *et al.* (2010). Clearly, the existence of $\boldsymbol{j}_h$ and i_h, as in Assumption 5.1, depends on the choice of the finite element pairing X_h/Y_h.

[6] Note that in case of $|\Gamma_P| > 0$ this implies $\int_{\Gamma_P} \boldsymbol{w} \cdot \mathbf{n} \,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_P} (\boldsymbol{j}_h \boldsymbol{w}) \cdot \mathbf{n} \,\mathrm{d}\boldsymbol{x}$, which requires that the triangulation matches Γ_P appropriately.

- The construction of an operator $\boldsymbol{j}_h$, that satisfies Assumption 5.1 (1)–(3), is well known for some particular finite elements, including the Crouzeix–Raviart and MINI element (see Belenki *et al.*, 2010). If $\Gamma_D \neq \partial\Omega$, Assumption 5.1 (1) requires that the triangulation matches Γ_D appropriately (compare with Scott & Zhang, 1990).
- Assumption 5.1 (2) is standard in the context of interpolation in Sobolev–Orlicz spaces (see Diening & Růžička, 2007). For standard finite elements, it is well known that the Scott–Zhang interpolation operator satisfies (5.1) (see Scott & Zhang, 1990). It is crucial that from (5.1) one can derive the local stability result

$$\fint_K \psi(|\nabla \boldsymbol{j}_h \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \leqslant c \fint_{S_K} \psi(|\nabla \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \quad \forall\, \boldsymbol{w} \in \mathbf{W}^{1,\psi}(\Omega), \ \ \forall\, K \in \mathbb{T}_h, \tag{5.5}$$

  which is valid for arbitrary N-functions, ψ, with $\Delta_2(\psi) < \infty$. Here, $\mathbf{W}^{1,\psi}(\Omega)$ is the classical Sobolev–Orlicz space and the constant c depends only on $\Delta_2(\psi)$. For details we refer to Diening & Růžička (2007).
- For standard finite elements, i_h may be chosen as the L^2-projection onto Y_h,

$$(i_h q, q_h)_\Omega = (q, q_h)_\Omega \quad \forall\, q_h \in Y_h, \ \ \forall\, q \in \mathrm{L}^1(\Omega). \tag{5.6}$$

  Indeed, it is shown in Crouzeix & Thomée (1987) that the L^2-projection is L^ν-stable and even $\mathrm{W}^{1,\nu}$-stable for any $\nu \in [1,\infty]$, and, consequently, the L^2-projection fulfills (5.3). The results of Crouzeix & Thomée (1987) are derived for finite element spaces Y_h based on simplices, $Y_h := \{w \in \mathrm{C}(\overline{\Omega});\, w|_K \in \mathbb{P}_r(K) \text{ for all } K \in \mathbb{T}_h\}$, where $\mathbb{P}_r(K)$ denotes the space of polynomials on K of degree less than or equal to r. Moreover, setting $q_h = 1$ in (5.6), we deduce that i_h preserves mean values. Hence, $i_h(Q^p) \subset Q_h^p$.

Next, we depict important consequences of Assumption 5.1.

LEMMA 5.3 Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 5.1 (2). Then, for all $K \in \mathbb{T}_h$ and $\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ there holds

$$\fint_K |\boldsymbol{F}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} \leqslant c h_K^2 \fint_{S_K} |\nabla \boldsymbol{F}(\boldsymbol{D}\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} \tag{5.7}$$

provided that $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{w}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$. The constant c depends only on p.

Proof. The proof is based on the Orlicz-stability (5.5). We refer to Belenki *et al.* (2010) and Hirn (2010). □

Moreover, the assumptions on $\boldsymbol{j}_h$ imply the discrete versions of the inf–sup inequality.

LEMMA 5.4 Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 5.1 (1)–(3). Then, for $\nu \in (1,\infty)$ the discrete inf–sup inequality (IS^ν) is satisfied.

Proof. Since $\mathbb{T}_h$ is nondegenerate, the local stability result (5.5) (with $\psi(t) := t^\nu$) leads to the global $\mathrm{W}^{1,\nu}$-stability inequality, $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu} \leqslant C_s\|\boldsymbol{w}\|_{1,\nu}$ for all $\boldsymbol{w} \in \boldsymbol{X}^\nu$, where $\nu \in (1,\infty)$ and the stability constant C_s does not depend on h. Thus, the continuous inf–sup inequality (2.25) and Assumption 5.1

imply that for arbitrary $q_h \in Q_h^\nu \subset Q^\nu$ it holds

$$\|q_h\|_{\nu'} \leqslant \beta(\nu)^{-1} \sup_{\boldsymbol{w}\in \boldsymbol{X}^\nu} \frac{(q_h, \operatorname{div} \boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}} = \beta(\nu)^{-1} \sup_{\boldsymbol{w}\in \boldsymbol{X}^\nu} \frac{(q_h, \operatorname{div} \boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}}$$

$$\leqslant \beta(\nu)^{-1} C_s \sup_{\boldsymbol{w}\in \boldsymbol{X}^\nu} \frac{(q_h, \operatorname{div} \boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu}} \leqslant \tilde{\beta}(\nu)^{-1} \sup_{\boldsymbol{w}_h\in \boldsymbol{X}_h^\nu} \frac{(q_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,\nu}},$$

where $\tilde{\beta}(\nu) := \beta(\nu)/C_s$ is independent of h. □

REMARK 5.5 Let us briefly discuss the case of unstable discretizations. For instance, one can consider the equal-order d-linear $\mathbb{Q}_1/\mathbb{Q}_1$ element (based on quadrilateral/hexahedral grids), which uses continuous isoparametric d-linear shape functions for both the velocity and the pressure approximation. In this case the discrete inf–sup condition is violated. For p-Stokes systems, for which the generalized viscosity depends only on the shear rate, Hirn proposed a stabilization technique based on the local projection stabilization (LPS) method that leads to optimal convergence results (see Hirn, 2010). Whether the stabilization method can be applied to the equal-order discretization of (pS) is subject of current research.

Next, we state our *a priori* error estimates that quantify the convergence of the finite element method. For this, the regularity $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$ of the solution $\boldsymbol{v}$ is required. This condition is equivalent to $(1.6)_1$ (see Berselli *et al.*, 2010). We mention that (1.6) is available for sufficiently smooth data at least in the space-periodic setting in two space dimensions (see Bulíček & Kaplický, 2008).

COROLLARY 5.6 Let the assumptions of Theorem 4.2 hold. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h satisfying Assumption 5.1. Moreover, we additionally assume the regularity of the weak solution

$$\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d} \quad \text{and} \quad \pi \in \mathrm{W}^{1,p'}(\Omega),$$

and we set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded in terms of the maximum mesh size h as follows:

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant C_{\boldsymbol{v}} h, \quad \|\pi - \pi_h\|_2 \leqslant C_\pi h. \tag{5.8}$$

Additionally assume (4.6): $\gamma_0 < \tilde{\beta}(p)\varepsilon^{\frac{2-p}{2}}$. Then, the pressure error in $\mathrm{L}^{p'}(\Omega)$ is bounded by

$$\|\pi - \pi_h\|_{p'} \leqslant C_\pi' h^{\frac{2}{p'}}. \tag{5.9}$$

The constants $C_{\boldsymbol{v}}, C_\pi, C_\pi' > 0$ depend only on $p, \varepsilon, \gamma_0, \sigma_0, \sigma_1, \tilde{\beta}(2), \Gamma_D, \Omega, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$, $\|\boldsymbol{v}_0\|_{1,p}, \|\nabla \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2, \|\pi\|_{1,p'}$ and C_π' additionally depends on $\tilde{\beta}(p)$.

Proof. According to Lemma 5.4, the discrete inf–sup inequalities (IS^2), (IS^p) hold true. Hence, the desired error estimates follow from Theorem 4.2, Corollaries 4.3 and 4.4, and the interpolation properties of $\boldsymbol{j}_h$ and i_h. More precisely, the velocity is given by $\boldsymbol{v} = \boldsymbol{v}_0 + \hat{\boldsymbol{v}}$ for some $\hat{\boldsymbol{v}} \in \boldsymbol{X}^p$. Since $\hat{\boldsymbol{v}}$ is divergence-free, the interpolant $\boldsymbol{j}_h\hat{\boldsymbol{v}}$ fulfills $(\operatorname{div} \boldsymbol{j}_h\hat{\boldsymbol{v}}, q_h)_\Omega = 0$ for all $q_h \in Q_h^p$. Hence, $\boldsymbol{j}_h\hat{\boldsymbol{v}} \in \boldsymbol{V}_h^p$ and $\boldsymbol{j}_h\boldsymbol{v} = \boldsymbol{j}_h\boldsymbol{v}_0 + \boldsymbol{j}_h\hat{\boldsymbol{v}} \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)$. Consequently, we can set $\boldsymbol{u}_h := \boldsymbol{j}_h\boldsymbol{v}$ and $r_h := i_h\pi$ in Theorem 4.2 and Corollary 4.3. Using Lemma 2.6 with $\nu := p$, the global $\mathrm{W}^{1,p}$-stability of $\boldsymbol{j}_h$ (which follows from (5.5) with $\psi(t) = t^p$ and the nondegeneracy of $\mathbb{T}_h$), the *a priori* bound (3.19), and the interpolation properties (5.7) and (5.3), we easily conclude (5.8). Finally, (5.9) follows from Corollary 4.4 and (5.8). □

REMARK 5.7 Using (2.18), (3.1) and (3.19), we deduce from Corollary 5.6 that

$$\|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h\|_p \leqslant c\,\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant ch. \tag{5.10}$$

Hence, we also obtain an *a priori* error estimate in $\mathbf{W}^{1,p}(\Omega)$.

If $d = 2$ then the $\mathrm{W}^{1,p'}$-regularity assumption for the pressure can be avoided and confined to $\pi \in \mathrm{W}^{1,2}(\Omega)$ provided that the velocity additionally satisfies $\boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega)$. Note that in the case of space-periodic boundary conditions, $C^{1,\alpha}$-regularity of $\boldsymbol{v}$ has been proven in Bulíček & Kaplický (2008). The following corollary represents a variant of Corollary 5.6 that is motivated by our subsequent numerical experiments.

COROLLARY 5.8 Let $d = 2$. Let the hypothesis of Theorem 4.2 hold true and let Assumption 5.2 be satisfied. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h as in Assumption 5.1. Moreover, we assume that the solution $(\boldsymbol{v}, \pi)$ satisfies the additional regularity

$$\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}, \quad \boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega) \quad \text{and} \quad \pi \in \mathrm{W}^{1,2}(\Omega).$$

We set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded as follows:

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant C_{\boldsymbol{v}}h, \quad \|\pi - \pi_h\|_2 \leqslant C_\pi h. \tag{5.11}$$

Assume additionally (4.6) and the $\mathrm{W}^{1,2}$-stability of i_h. Then, there holds

$$\|\pi - \pi_h\|_{p'} \leqslant C'_\pi h^{\frac{2}{p'}}. \tag{5.12}$$

The constants $C_{\boldsymbol{v}}$, C_π, $C'_\pi > 0$ depend only on p, ε, γ_0, σ_0, σ_1, $\tilde{\beta}(2)$, Γ_D, Ω, $\|\nabla\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{1,\infty}$ and C'_π additionally depends on $\tilde{\beta}(p)$.

Proof. Under the supposed regularity, (5.11) and (5.12) are not surprising: since $\boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega)$ and $\varepsilon > 0$, the generalized viscosity, η, remains bounded from below and above so that system (1.1) can basically be interpreted as a Stokes system. We only need to show that $\boldsymbol{v}_h$ is uniformly bounded in $\mathbf{W}^{1,\infty}(\Omega)$: first of all, we mention that the projection $\boldsymbol{j}_h$ is $\mathrm{W}^{1,\infty}$-stable. Indeed, similarly to Scott & Zhang (1990), it can be shown that $\boldsymbol{j}_h$ is locally $\mathbf{W}^{1,1}$-stable, i.e., there holds $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,1;K} \lesssim \|\boldsymbol{w}\|_{1,1;S_K}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega)$ and $K \in \mathbb{T}_h$. Moreover, since $X_h(K)$ is finite dimensional, there holds $|\nabla^i\boldsymbol{j}_h\boldsymbol{w}(\boldsymbol{y})| \lesssim \fint_K |\nabla^i\boldsymbol{j}_h\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}$, $i \in \{0,1\}$, for all $\boldsymbol{y} \in K$ and $K \in \mathbb{T}_h$. Due to the nondegeneracy of $\mathbb{T}_h$ it follows that $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;K} \lesssim \|\boldsymbol{w}\|_{1,\infty;S_K}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,\infty}(\Omega)$. This yields $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;\Omega} \lesssim \|\boldsymbol{w}\|_{1,\infty;\Omega}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,\infty}(\Omega)$. Using the inverse inequality (5.4) with $d = 2$, the $\mathrm{W}^{1,\infty}$-stability of $\boldsymbol{j}_h$, Korn's Lemma 2.7, and Lemma 2.6 with $\nu = 2$, we can estimate $\boldsymbol{v}_h$ in $\mathbf{W}^{1,\infty}(\Omega)$ as follows:

$$\begin{aligned}
\|\boldsymbol{v}_h\|_{1,\infty} &\leqslant \|\boldsymbol{v}_h - \boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} + \|\boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} \\
&\leqslant c\left[h^{-1}\|\boldsymbol{v}_h - \boldsymbol{j}_h\boldsymbol{v}\|_{1,2} + \|\boldsymbol{v}\|_{1,\infty}\right] \\
&\leqslant c\left[h^{-1}\|\boldsymbol{D}\boldsymbol{v}_h - \boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{1,\infty}\right] \\
&\leqslant c\left[h^{-1}\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2(\varepsilon_0 + \|\nabla\boldsymbol{v}_h\|_\infty + \|\nabla\boldsymbol{v}\|_\infty)^{\frac{2-p}{2}} + \|\boldsymbol{v}\|_{1,\infty}\right].
\end{aligned} \tag{5.13}$$

Similarly to the way in which we have derived (4.5), via Lemma 2.6 with $\nu = 2$ we can infer the error estimate

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2+\|\pi-\pi_h\|_2 \lesssim \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2+(\varepsilon_0+\|\nabla\boldsymbol{j}_h\boldsymbol{v}\|_\infty+\|\nabla\boldsymbol{v}_h\|_\infty)^{\frac{2-p}{2}}\|\pi-i_h\pi\|_2.$$

Using the properties of $\boldsymbol{j}_h$ and i_h we consequently arrive at (w.l.o.g. $\varepsilon_0 \geqslant 1$)

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|\pi - \pi_h\|_2 \leqslant Ch(\varepsilon_0 + \|\nabla\boldsymbol{v}\|_\infty + \|\nabla\boldsymbol{v}_h\|_\infty)^{\frac{2-p}{2}}, \tag{5.14}$$

where the constant C depends on $\|\nabla\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2$ and $\|\pi\|_{1,2}$. Combining (5.13) and (5.14) we conclude

$$\|\boldsymbol{v}_h\|_{1,\infty} \leqslant C = C(\|\nabla\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2, \|\pi\|_{1,2}, \|\boldsymbol{v}\|_{1,\infty}).$$

The constant C also depends on $p,\ \varepsilon,\ \varepsilon_0,\ \gamma_0,\ \sigma_0,\ \sigma_1,\ \tilde{\beta}(2),\ \Omega$, but it is independent of h. Thus, (5.14) yields the desired error estimates (5.11). It remains to prove the pressure estimate in $\mathrm{L}^{p'}(\Omega)$. Interpolating $\mathrm{L}^{p'}(\Omega)$ between $\mathrm{L}^2(\Omega)$ and $\mathrm{W}^{1,2}(\Omega)$, using (5.3) and the $\mathrm{W}^{1,2}$-stability of i_h, for $p > \frac{2d}{d+2}$ and $\lambda := \frac{d}{2} - \frac{d}{p'}$ we obtain the estimate

$$\|\pi - i_h\pi\|_{p'} \leqslant c\,\|\pi - i_h\pi\|_{1,2}^{\lambda}\|\pi - i_h\pi\|_2^{1-\lambda} \leqslant ch^{1+\frac{d}{p'}-\frac{d}{2}}\|\pi\|_{1,2}. \tag{5.15}$$

Thus, for $d = 2$ the estimate (5.12) follows from the combination of (4.7), (5.11) and (5.15). □

6. Numerical examples

In this section we present some numerical examples, which illustrate the *a priori* error estimates of Corollary 5.6. Here, the following model is used:

$$\eta(\pi, |\boldsymbol{D}\boldsymbol{v}|^2) := \eta_0\left(\delta_1 + \delta_2(\delta_3 + \exp(\alpha\pi))^{-s} + \delta_4|\boldsymbol{D}\boldsymbol{v}|^2\right)^{\frac{p-2}{2}}, \tag{6.1}$$

where $s, \alpha, \delta_1, \ldots, \delta_4 \geqslant 0$.

REMARK 6.1 Similarly to, e.g., Málek *et al.* (2002), it can be shown that model (6.1) satisfies Assumptions (A1)–(A2), e.g., with $\varepsilon^2 := \delta_1/\delta_4$, $\sigma_0 := \eta_0\delta_4^{(p-2)/2}(p-1)(1+\delta_2\delta_3^{-s}/\delta_1)^{(p-2)/2}$, $\sigma_1 := \eta_0\delta_4^{(p-2)/2}$ and $\gamma_0 := \eta_0\delta_4^{(p-4)/4}s\alpha\frac{2-p}{2}\delta_2^{p/4}\delta_3^{-sp/4}$.

Problem (pS) was discretized with the following finite elements based on quadrilateral meshes: the first-order $\mathbb{Q}_2/\mathbb{Q}_0$ elements, the second-order $\mathbb{Q}_2/\mathbb{Q}_1$ and $\mathbb{Q}_2/\mathbb{P}_{-1}$ elements and the bilinear $\mathbb{Q}_1/\mathbb{Q}_1$ elements (see Gresho & Sani, 2000, or Sani *et al.*, 1981). The latter element pair is not stable, thus we used the LPS-type stabilization method presented in Hirn (2010); it is worth mentioning that in all examples the stabilization method was less sensitive with respect to the stabilization parameter. The algebraic equations were solved by Newton's method, the linear subproblems by the GMRES method. All computations were performed by means of the software package (Gascoigne, 2006) and/or the software developed by Hron *et al.* (2003). In the following numerical experiments we depict the rates of convergence with respect to the number of cells (under global mesh refinement). For ease of presentation we use the shortcuts $E_{\boldsymbol{v}}^{\boldsymbol{F}} := \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2$, $E_{\boldsymbol{v}}^{1,\nu} := \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\nu}$, $E_{\boldsymbol{v}}^{\nu} := \|\boldsymbol{v} - \boldsymbol{v}_h\|_\nu$ and $E_\pi^\nu := \|\pi - \pi_h\|_\nu$.

Example 1: In a square domain $\Omega := (-0.5, 0.5) \times (-0.5, 0.5)$ the exact solution to (pS) is given by $\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1}(x_2, -x_1)^\top$ and $\pi(\boldsymbol{x}) := |\boldsymbol{x}|^b x_1 x_2$ for $a, b \in \mathbb{R}$. Problem ($\mathbf{pS}_h$) was then solved[7] for $\boldsymbol{f} := -\operatorname{div} \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) + \nabla\pi$. The parameters a and b were chosen so that $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$ and $\pi \in \mathrm{W}^{1,2}(\Omega)$; this requirement amounts to the conditions $a > 1$ and $b > -2$. Since $\|\nabla \boldsymbol{v}\|_\infty$ is bounded for $a > 1$, according to Corollary 5.8, the requirement $\pi \in \mathrm{W}^{1,2}(\Omega)$ is sufficient to ensure the optimal rate of convergence (note that Corollary 5.6 would require $\pi \in \mathrm{W}^{1,p'}(\Omega)$ with $p' > 2$). We set $a = 1.01$ and $b = -1.99$. Hence, as soon as (3.4) is satisfied, we expect $E_{\boldsymbol{v}}^{\boldsymbol{F}} = \mathcal{O}(h)$, $E_\pi^2 = \mathcal{O}(h)$, and $E_\pi^{p'} = \mathcal{O}(h^{2/p'})$, for finite elements satisfying Assumption 5.1.

The parameters of the model (6.1) were set to $\delta_1 := 10^{-8}$, $s := 2/(2-p)$ and $\eta_0 = \delta_2 = \delta_3 = \delta_4 := 1$ in this example. Then, Remark 6.1 implies $\gamma_0 = \alpha$ and, hence, (3.4) is ensured at least for $\alpha < \tilde{\beta}(2)\delta_1^{(2-p)/4} \frac{(p-1)(1+1/\delta_1)^{(p-2)/2}}{(p-1)(1+1/\delta_1)^{(p-2)/2}+1}$, i.e., by virtue of $\delta_1 \ll 1$, (3.4) is satisfied for $\alpha \ll 1$. In this particular example, for the stated parameters, we have numerically observed the expected convergence rates for $\alpha \in [0, 8]$ approximately. For greater α, Newton's method did not converge any more. One may ask whether assumption (3.4) could be relaxed[8] and in particular, whether the estimates (5.9) and (5.10) remain valid in the degenerate case, $\varepsilon \searrow 0$. Here, it is worth noting that in the case of Carreau-type models (i.e., $\gamma_0 \equiv 0$), error estimates similar to (5.9) and (5.10) actually hold true and are numerically validated also for $\varepsilon = 0$ (see Belenki *et al.*, 2010; Hirn, 2010). For fluids with pressure-dependent viscosity, though, the behaviour for $\varepsilon \searrow 0$ remains an open question. In what follows we set $\alpha := 1$.

For the stable first-order $\mathbb{Q}_2/\mathbb{Q}_0$ elements the convergence rates for different values of $p \in (1, 2)$ are presented in Table 1(a–c). We realize that the numerical results agree with the presented theory very well. In particular, the example reflects that the rate of convergence for the pressure in $\mathrm{L}^{p'}(\Omega)$ depends on the choice of p as predicted by the estimate (5.12). Apart from that, we observed that the experimental order of convergence declines as soon as $a < 1$ or $b < -2$. This indicates that the derived *a priori* error estimates are optimal with respect to the regularity of the solution. We also observe that the error $E_{\boldsymbol{v}}^p$ behaves like $\mathcal{O}(h^2)$. This raises hope that a duality argument similar to the one described in Brenner & Scott (1994) may be applicable here. In Table 1(d–i) we present the observed convergence rates for the element pairs $\mathbb{Q}_1/\mathbb{Q}_1$, $\mathbb{Q}_2/\mathbb{Q}_1$ and $\mathbb{Q}_2/\mathbb{P}_{-1}$. In this example they basically coincide with those obtained for $\mathbb{Q}_2/\mathbb{Q}_0$.

Example 2: pressure drop problem. In order to confirm the results in a realistic flow configuration, we consider a planar flow between two steady parallel plates, driven by the difference of pressure between inlet and outlet. Here, $\Omega = (0, 1.64) \times (0, 0.41)$ and the homogeneous Dirichlet boundary condition is prescribed on the upper and lower edge, while we set $\boldsymbol{b} := 0.8\,\mathbf{n}$ on the inflow (left) boundary, and $\boldsymbol{b} := \mathbf{0}$ on the outflow (right) boundary. Moreover, we additionally require[9] there that $\boldsymbol{v} = (\boldsymbol{v}\cdot\mathbf{n})\mathbf{n}$, i.e., the streamlines are orthogonal to the inflow and outflow boundary (compare with Heywood *et al.*, 1996). Note that if the viscosity did not vary with the pressure, this setting would lead to a unidirectional flow (Poiseuille flow) of the form $\boldsymbol{v} = (v_1(x_2), 0)^\top$ and $\pi = \pi(x_1)$. Since the viscosity depends on the pressure, however, this need not be the case; e.g., there is no such unidirectional solution for the Barus

[7] Both $\Gamma_P = \emptyset$ (with $\fint_\Omega \pi \,\mathrm{d}\boldsymbol{x}$ prescribed) and Γ_P chosen as one of the square edges were tested as the boundary conditions.

[8] However, this observation does not allow us to *claim* that (3.4) could be relaxed. The solution to Example 1 is given a priori while $\boldsymbol{f}$ is defined accordingly. In particular, the solution always exists, whatever the values of α and γ_0 are. Moreover, the above estimate for γ_0 takes into account *all* $\pi \in \mathbb{R}$, $|\boldsymbol{D}\boldsymbol{v}| \geqslant 0$, and may be far from describing the behaviour of the viscosity in a neighbourhood of the given solution.

[9] This requirement is achieved by altering the definition of the space $\boldsymbol{X}^p$ (see, e.g., Lanzendörfer & Stebel, 2011b).

TABLE 1 *Numerical verification of the a priori error estimates*

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	0.98	1.83	0.82	0.74
4^5	1.01	1.89	0.85	0.77
4^6	1.02	1.92	0.88	0.79
4^7	1.01	1.93	0.90	0.80
4^8	1.01	1.96	0.91	0.81
Expected	1	—	1	0.82

(a) $p = 1.7$, $\mathbb{Q}_2/\mathbb{Q}_0$

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	0.97	1.85	0.82	0.65
4^5	1.00	1.91	0.85	0.66
4^6	1.00	1.95	0.88	0.67
4^7	1.01	1.96	0.90	0.67
4^8	1.01	1.96	0.91	0.67
Expected	1	—	1	0.67

(b) $p = 1.5$, $\mathbb{Q}_2/\mathbb{Q}_0$

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	0.90	1.90	0.82	0.19
4^5	0.95	1.95	0.85	0.19
4^6	0.98	1.97	0.88	0.19
4^7	0.98	1.99	0.90	0.19
4^8	0.98	2.00	0.91	0.19
Expected	1	—	1	0.18

(c) $p = 1.1$, $\mathbb{Q}_2/\mathbb{Q}_0$

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^5	1.00	2.17	1.00	0.83
4^6	1.00	2.17	1.00	0.83
4^7	1.00	2.17	1.00	0.82
4^8	1.00	2.16	1.00	0.83
4^9	1.00	2.16	1.00	0.83

(d) $p = 1.7$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^5	0.99	2.49	1.00	0.46
4^6	0.99	2.48	1.00	0.46
4^7	0.99	2.45	1.00	0.46
4^8	1.00	2.41	1.00	0.47
4^9	1.00	2.36	1.00	0.47

(e) $p = 1.3$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^5	0.99	2.70	0.99	0.19
4^6	0.99	2.66	1.00	0.19
4^7	0.99	2.56	1.00	0.19
4^8	1.00	2.44	1.00	0.19
4^9	1.00	2.30	1.01	0.19

(f) $p = 1.1$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	—	—	—	—
4^5	1.01	2.33	1.01	0.68
4^6	1.01	2.33	1.01	0.67
4^7	1.00	2.32	1.01	0.67
4^8	1.00	2.31	1.01	0.67
4^9	1.00	2.29	1.01	0.67

(g) $p = 1.5$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

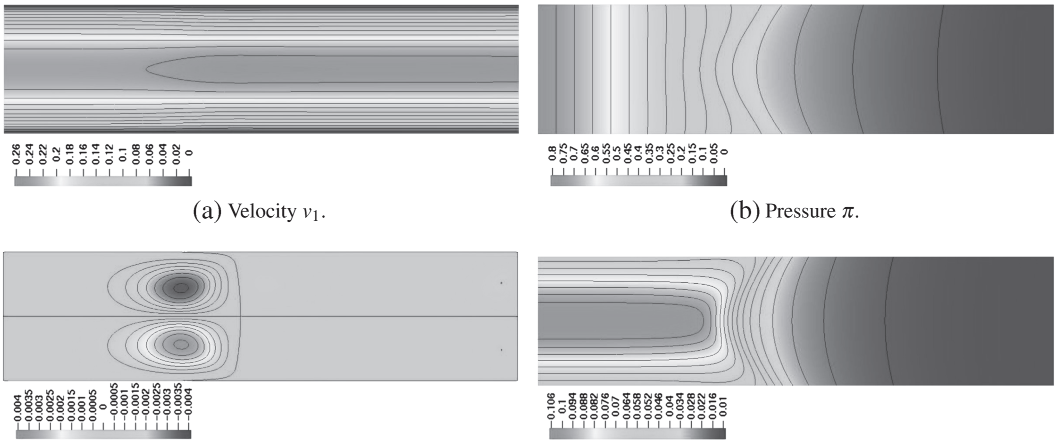
No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	1.02	2.33	1.01	0.68
4^5	1.01	2.32	1.01	0.68
4^6	1.02	2.33	1.01	0.68
4^7	1.02	2.30	1.01	0.68
4^8	1.02	2.25	1.01	0.68
4^9	—	—	—	—

(h) $p = 1.5$, $\mathbb{Q}_2/\mathbb{Q}_1$

No. of cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	1.02	2.30	1.01	0.68
4^5	1.02	2.27	1.01	0.68
4^6	1.02	2.26	1.01	0.68
4^7	1.01	2.23	1.01	0.68
4^8	1.02	2.10	1.01	0.67
4^9	—	—	—	—

(i) $p = 1.5$, $\mathbb{Q}_2/\mathbb{P}_{-1}$

model, $\eta = \eta_0 \exp(\alpha\pi)$, as shown in Hron *et al.* (2001). Here, we consider the model (6.1), provided with $\eta_0 := 0.005$, $p = 1.5$, $s := \frac{2}{2-p}$, $\delta_1 := 5 \times 10^{-6}$, $\delta_2 = \delta_3 := 1$, $\delta_4 := 10^{-5}$, and $\alpha := 10$. The resulting velocity, pressure and viscosity fields are shown in Fig. 1. For moderate and low pressures (in the midlength and in the right-hand part of the domain) this model approximates the Barus model, while for higher pressures (in the left-hand part of the domain) the behaviour is that of the Carreau model. In Table 2 we present the observed convergence rates for the different finite element pairs. Since the exact solution is unknown, we have used the finite element approximation computed on a grid of 4^{10} cells as the reference solution. Looking at Table 2, we observe good agreement with the derived estimates. While E_{π}^{2} behaves as $\mathcal{O}(h)$ in the case of the $\mathbb{Q}_2/\mathbb{Q}_0$ discretization, the higher-order element pairs, including the $\mathbb{Q}_1/\mathbb{Q}_1$ discretization, lead to better convergence rates.



(a) Velocity v_1. (b) Pressure π. (c) Velocity v_2. (d) Viscosity η.

FIG. 1. Pressure drop problem, $p = 1.5$.

TABLE 2 *Numerical verification of the error estimates: pressure drop problem*

No. of cells	(a) $E_{\boldsymbol{v}}^{1,p}$	(a) $E_{\boldsymbol{v}}^{p}$	(a) E_{π}^{2}	(b) $E_{\boldsymbol{v}}^{1,p}$	(b) $E_{\boldsymbol{v}}^{p}$	(b) E_{π}^{2}	(c) $E_{\boldsymbol{v}}^{1,p}$	(c) $E_{\boldsymbol{v}}^{p}$	(c) E_{π}^{2}	(d) $E_{\boldsymbol{v}}^{1,p}$	(d) $E_{\boldsymbol{v}}^{p}$	(d) E_{π}^{2}
4^4	0.99	1.95	1.00	2.29	3.44	2.19	2.16	3.19	1.92	—	—	—
4^5	0.99	1.98	1.01	2.51	3.78	2.24	2.19	3.15	1.96	1.00	1.97	1.94
4^6	1.02	1.96	1.03	2.46	3.69	2.08	2.14	3.04	1.99	1.00	2.00	2.04
4^7	1.08	2.02	1.16	2.25	3.26	2.06	†	†	†	1.01	2.01	1.98
4^8	—	—	—	—	—	—	—	—	—	1.02	2.06	1.89
Expected	1	—	1									

(a) $\mathbb{Q}_2/\mathbb{Q}_0$ (b) $\mathbb{Q}_2/\mathbb{Q}_1$ (c) $\mathbb{Q}_2/\mathbb{P}_{-1}$ (d) $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

† In this case we were not able to solve the algebraic problem to the accuracy sufficient to improve the discrete solution on finer meshes. Note that $E_{\boldsymbol{v}}^{p}/\|\boldsymbol{v}\|_p \sim 10^{-7}$ at this level of refinement.

7. Conclusions

We have shown the convergence of the finite element method in the context of fluids with shear-rate- and pressure-dependent viscosity. The convergence of the method has been quantified by the *a priori* error estimates of Corollary 5.6. These error estimates have been demonstrated practically by numerical experiments. All results in the present paper also cover the case of Carreau-type models. In this case the error estimates of Corollary 5.6 coincide with the optimal error estimates for Carreau-type models which have been established in Belenki *et al.* (2010) and Hirn (2010).

The numerical experiments indicate that the problems are well posed for a wider class of models than required by the assumptions. This is encouraging for further investigation since the assumptions are rather restrictive from the point of view of practical applications.

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4.2 Shape optimization in applications

A typical example of industrial application is the optimization of the shape of a dividing header (also called headbox) inside the paper making machine. The dividing header is located at the so-called wet end of the large machine (see Figure 4.1). The role of the header is to distribute the mixture of water,

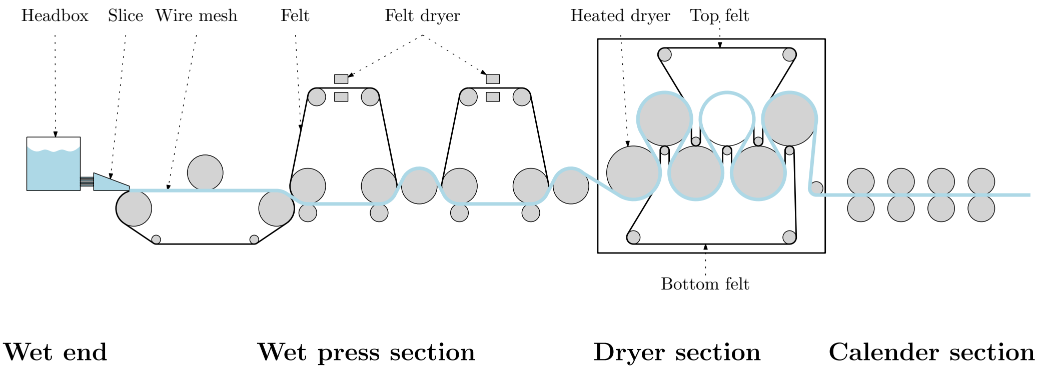

Figure 4.1: Simplified scheme of components of a paper-making machine.

wood fibers and additives onto a wire screen *evenly* so that the resulting paper is homogeneous and has the same properties in all parts. The distribution of the fibers is strongly affected by the flow regime inside the header. A natural way of controlling the flow properties is by adjusting the shape of the header. In order to simplify the design and minimize the amount of expensive experimental work, the problem has been solved by numerical simulation of the appropriate shape optimization problem.

In the papers [12, 24] we attempt to deal with the problem in a rigorous way, starting by precise formulation of the fluid flow model and the shape optimization problem, establishing the existence and uniqueness results for the problem as well as for its finite-element approximation, convergence of the approximations and finally showing the results of example computations. We present the reprint of the second part of this series, dealing with the numerical analysis and computation.

The flow in the header is turbulent, which we take into account by an algebraic turbulence model. In particular, the stress tensor has the form

$$\mathbb{T} = -p\mathbb{I} + \mu_0 \mathbb{D}\boldsymbol{v} + \mu_t(|\mathbb{D}\boldsymbol{v}|)\mathbb{D}\boldsymbol{v}, \tag{4.1}$$

where μ_0 is the viscosity of the fluid and $\mu_t(|\mathbb{D}\boldsymbol{v}|)$ the turbulent viscosity in the form

$$\mu_t(|\mathbb{D}\boldsymbol{v}|) = \varrho l_{m,\alpha}^2 |\mathbb{D}\boldsymbol{v}|. \tag{4.2}$$

The mixing length $l_{m,\alpha}$ is an experimentally determined function depending on the distance from the boundary.

The contribution of the papers is based on the following results:

- Mathematical analysis of the state problem, namely the steady-state Navier-Stokes equations with the algebraic turbulence model. Due to (4.2), the system of equations has a similar form like the equations for non-Newtonian fluids with shear-dependent viscosity. However, here the viscosity depends also on spacial variable and may vanish on the boundary. This is a major obstacle in the analysis and requires a careful rigorous formulation, which is based on non-standard weighted Sobolev spaces.

- Existence of an optimal shape (both in the continuous and the discrete case) is based on the uniform estimates of solutions to the fluid flow problem, which are proved independent of the geometry of the flow domain. This requires the use of some inequalities and auxiliary mathematical tools with attention to their dependence on the geometry of the domain. The lack of a density result in the weighted Sobolev spaces is overcome by formulating an augmented shape optimization problem.

- The algorithms for the numerical solution are supported by the existence and convergence results for the discrete problems, hence it is guaranteed under which assumptions the computations lead to meaningful results.

Reprint

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Shape Optimization for Navier–Stokes Equations with Algebraic Turbulence Model: Numerical Analysis and Computation

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Abstract We study the shape optimization problem for the paper machine headbox which distributes a mixture of water and wood fibers in the paper making process. The aim is to find a shape which a priori ensures the given velocity profile on the outlet part. The mathematical formulation leads to the optimal control problem in which the control variable is the shape of the domain representing the header, the state problem is represented by the generalized Navier-Stokes system with nontrivial boundary conditions. This paper deals with numerical aspects of the problem.

Keywords Optimal shape design · Paper machine headbox · Incompressible non-Newtonian fluid · Algebraic turbulence model

1 Introduction

The first component in the paper making process is the headbox which is located at the wet end of a paper machine. The headbox shape and the fluid flow phenomena taking place there largely determine the quality of the produced paper. The first flow passage in the headbox is a dividing manifold, called the header. It is designed to distribute the fibre suspension on the wire so that the produced paper has an optimal basis weight and fibre orientation across the whole width of a paper machine. The

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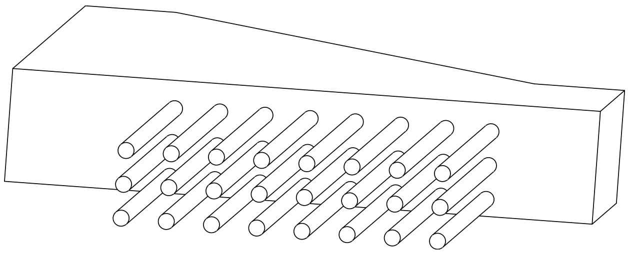


Fig. 1 The header

aim is to find an optimal shape for the back wall of the header so that the outlet flow rate distribution from the headbox results in an optimal paper quality.

The paper making pulp is a mixture of wood fibres, water, filler clays and various chemicals at concentration of 1% solids to 99% water by weight. In the large-scale simulation it seems reasonable to model this complex mixture as a single continuum, with the fluid being an incompressible liquid described by the Navier-Stokes equations.

The turbulence character of the flow in the header is a desirable phenomenon in the paper making process. Typically, the input Reynolds number is about 10^6. In the modelling of turbulence, one usually uses the averaging procedure, which requires additionally a closure formula for the so-called Reynolds tensor. Since the flow in the header is steady and it is expected that the geometry of the domain changes only in the part of the boundary, we use a classical algebraic model introduced by Prandtl [22], see Sect. 2.3.

Figure 1 shows the geometry of the header. The inlet is on the left and the so-called recirculation on the right hand side. Typically about 10% of the fluid flows out through the recirculation. The main outlet is performed by a number (usually several hundreds or thousands) of small tubes. This fact presents a difficulty in the numerical simulation and thus the complicated geometry of the tube bank is replaced by an effective medium using an approximate homogenization technique (see [12]).

This work was motivated by some previous papers: The fluid flow model which is used here has been derived and studied numerically in [12]. The shape optimization problem has also been solved numerically and the results are presented in [13], see also [14]. The above cited papers are of formal character, skipping completely existence results. The mathematical justification of the model is done in [15]. The present paper is focused on a discretization, convergence analysis and numerical realization of the shape optimization problem.

Numerical solution of shape optimization problems is usually realized by means of gradient based minimization methods, which requires to perform the sensitivity analysis. There are two approaches in computational sensitivity analysis, namely *differentiate-then-discretize* and *discretize-then-differentiate*. Both of them are accepted by the optimization community and both have their pro and con (see e.g. [11], Sect. 2.9 on p. 57). Since the differentiation and the discretization do not commute, results are different on a given discrete level. It is known that the discretization of the continuous shape gradient provides the true gradient neither of the continuous cost functional nor of its discretization. This may cause serious difficulties in numerical minimization. Also the rigorous mathematical derivation of the shape gradient is usually very demanding, (see e.g. [23]). Since our paper is devoted to the discretization and convergence analysis, it is very natural to use the discretize-then-differentiate approach which works with the true gradient of the discrete cost functional. To get it one can employ tools of the automatic differentiation.

The text is organized as follows. In Sect. 2 we present the complete model and the known existence results. An approximation of the fluid flow model and of the shape optimization problem is studied in Sects. 3 and 4, respectively. Finally, Sect. 5 describes an implementation and presents results of several model examples.

2 Description of the Model

In this section we define the mathematical model of the flow in the header and the shape optimization problem, and recall the main existence results. For its justification and for proofs we refer to [3].

We start by specifying the geometry of the problem.

2.1 Admissible Domains

Let $L_1, L_2, L_3 > 0$, $\alpha_{max} \geq H_1 \geq H_2 \geq \alpha_{min} > 0$, $\gamma > 0$ be given and suppose that $\alpha \in \mathcal{U}_{ad}$, where

$$\mathcal{U}_{ad} = \Big\{\alpha \in C^{0,1}([0, L]);\ \alpha_{min} \leq \alpha \leq \alpha_{max},$$
$$\alpha_{|[0,L_1]} = H_1,\ \alpha_{|[L_1+L_2,L]} = H_2, |\alpha'| \leq \gamma \text{ a.e. in } [0, L]\Big\}. \tag{1}$$

Here $L = L_1 + L_2 + L_3$. With any function $\alpha \in \mathcal{U}_{ad}$ we associate the domain $\Omega(\alpha)$, see Fig. 2:

$$\Omega(\alpha) = \Big\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2; 0 < x_1 < L, 0 < x_2 < \alpha(x_1)\Big\} \tag{2}$$

and introduce the system of admissible domains

$$\mathcal{O} = \big\{\Omega;\ \exists\, \alpha \in \mathcal{U}_{ad} :\ \Omega = \Omega(\alpha)\big\}.$$

Further we will need the domains $\widehat{\Omega} = (0, L) \times (0, \alpha_{max})$ and $\Omega_0 = ((0, L_1) \times (0, H_1)) \cup ((0, L) \times (0, \alpha_{min})) \cup ((L_1 + L_2, L) \times (0, H_2))$. Notice that $\Omega_0 \subset \Omega \subset \widehat{\Omega}$ for all $\Omega \in \mathcal{O}$.

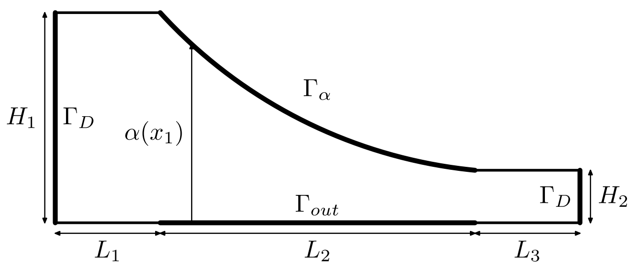


Fig. 2 Geometry of $\Omega(\alpha)$ and parts of the boundary $\partial\Omega(\alpha)$

Clearly $\Omega(\alpha) \in \mathcal{C}^{0,1}$ for all $\alpha \in \mathcal{U}_{ad}$. We will denote the parts of the boundary $\partial\Omega(\alpha)$ as follows (see Fig. 2):

$$\begin{aligned}
\Gamma_D &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha);\, x_1 = 0 \text{ or } x_1 = L\Big\},\\
\Gamma_{out} &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha);\, L_1 \le x_1 \le L_1 + L_2,\, x_2 = 0\Big\},\\
\Gamma_\alpha &= \Big\{\boldsymbol{x} \in \partial\Omega(\alpha);\, L_1 \le x_1 \le L_1 + L_2,\, x_2 = \alpha(x_1)\Big\},\\
\Gamma_f &= \partial\Omega(\alpha) \setminus \big(\Gamma_D \cup \Gamma_{out} \cup \Gamma_\alpha\big).
\end{aligned}$$

The components Γ_D, Γ_{out} and Γ_f do not depend on $\alpha \in \mathcal{U}_{ad}$.

2.2 Formulation of the Shape Optimization Problem

Let $\tilde{\Gamma} \subset \Gamma_{out}$ and $v_{opt} \in L^2(\tilde{\Gamma})$ be a given function representing the desired velocity profile at the outlet. We are interested in the problem

$$\begin{aligned}
&\min \quad J(\alpha, \boldsymbol{v}, p) := \int_{\tilde{\Gamma}} |v_2 - v_{opt}|^2\\
&\text{s.t.} \quad \alpha \in \mathcal{U}_{ad},\\
&\qquad (\boldsymbol{v}, p) \text{ solves (3)–(4) in } \Omega(\alpha).
\end{aligned}$$

Since $\tilde{\Gamma}$ is fixed, it is obvious that J does not depend explicitly on α. Further we will consider only a class of weak solutions to (3)–(4) which will be specified in what follows.

2.3 Classical Formulation of the State Problem

The fluid motion in $\Omega(\alpha)$ is described by the generalized Navier–Stokes system

$$\left.\begin{aligned}
-\operatorname{div}\ \mathbb{T}(p, \mathbb{D}(\boldsymbol{v})) + \rho \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) &= \boldsymbol{0}\\
\operatorname{div}\ \boldsymbol{v} &= 0
\end{aligned}\right\} \quad \text{in } \Omega(\alpha). \tag{3}$$

Here $\boldsymbol{v}$ means the velocity, p the pressure, ρ is the density of the fluid and the stress tensor $\mathbb{T}$ is defined by the following formulae:

$$\begin{aligned}
&\mathbb{T}(p, \mathbb{D}(\boldsymbol{v})) = -p\mathbb{I} + 2\mu(|\mathbb{D}(\boldsymbol{v})|)\mathbb{D}(\boldsymbol{v}),\\
&\mu(|\mathbb{D}(\boldsymbol{v})|) := \mu_0 + \mu_t(|\mathbb{D}(\boldsymbol{v})|) = \mu_0 + \rho l_{m,\alpha}^2 |\mathbb{D}(\boldsymbol{v})|,
\end{aligned}$$

where $\mu_0 > 0$ is a constant laminar viscosity and $\mu_t(|\mathbb{D}(\boldsymbol{v})|)$ stands for a turbulent viscosity. The function $l_{m,\alpha}$ represents a mixing length in the algebraic model of turbulence and it has the following form (see [13] for more details):

$$l_{m,\alpha}(\boldsymbol{x}) = \frac{1}{2}\alpha(x_1)\left(0.14 - 0.08\left(1 - \frac{2d_\alpha(\boldsymbol{x})}{\alpha(x_1)}\right)^2 - 0.06\left(1 - \frac{2d_\alpha(\boldsymbol{x})}{\alpha(x_1)}\right)^4\right),$$

where $d_\alpha(\boldsymbol{x}) = \min\{x_2, \alpha(x_1) - x_2\}, \boldsymbol{x} \in \Omega(\alpha)$.

The equations are completed by the following boundary conditions:

$$\begin{aligned}
&\boldsymbol{v} = \boldsymbol{0} && \text{on } \Gamma_f \cup \Gamma_\alpha, \\
&\boldsymbol{v} = \boldsymbol{v}_D && \text{on } \Gamma_D, \\
&\boldsymbol{v} \cdot \boldsymbol{\tau} = v_1 = 0 && \text{on } \Gamma_{out}, \\
&T_{22} := \mathbb{T}\boldsymbol{\nu} \cdot \boldsymbol{\nu} = -\sigma |v_2| v_2 && \text{on } \Gamma_{out},
\end{aligned} \tag{4}$$

where $\boldsymbol{\nu}, \boldsymbol{\tau}$ stands for the unit normal and tangential vector to Γ_{out}, respectively and $\sigma > 0$ is a given suction coefficient. The condition $(4)_4$ was suggested in [12] by a numerical study of the pipe flow. It can be also considered as a condition describing a porous wall, which can be derived from the Forchheimer equation, an analogy of the Darcy equation for high velocity flows [1, 16].

By a classical solution we mean any velocity field $\boldsymbol{v} \in (\mathcal{C}^2(\Omega(\alpha)))^2 \cap (\mathcal{C}^1(\overline{\Omega}(\alpha)))^2$ and a pressure $p \in \mathcal{C}^1(\Omega(\alpha)) \cap \mathcal{C}(\overline{\Omega}(\alpha))$ satisfying (3) and (4).

2.4 Weak Formulation of the State Problem

Throughout the paper we will assume that there exists a function $\boldsymbol{v}_0 \in (W^{1,3}(\Omega_0))^2$ which satisfies the Dirichlet boundary conditions in the sense of traces, i.e.

$$\boldsymbol{v}_{0|\Gamma_D} = \boldsymbol{v}_D, \qquad \boldsymbol{v}_{0|\partial\Omega_0 \setminus (\Gamma_D \cup \Gamma_{out})} = \boldsymbol{0}, \qquad \boldsymbol{v}_0 \cdot \boldsymbol{\tau}_{|\Gamma_{out}} = 0$$

and, in addition, $\operatorname{div} \boldsymbol{v}_0 = 0$ in Ω_0. We extend $\boldsymbol{v}_0$ by zero on $\widehat{\Omega} \setminus \overline{\Omega}_0$ so that $\boldsymbol{v}_0 \in (W^{1,3}(\widehat{\Omega}))^2$ and $\operatorname{div} \boldsymbol{v}_0 = 0$ in $\widehat{\Omega}$ (the extended function $\boldsymbol{v}_0$ will be denoted by the same symbol). Observe that such $\boldsymbol{v}_0$ is independent of $\alpha \in \mathcal{U}_{ad}$.

The norm in the space $W^{k,p}(\Omega(\alpha))$ will be denoted by $\|\cdot\|_{k,p,\Omega(\alpha)}$ in what follows. If $k = 0$, then notation $\|\cdot\|_{p,\Omega(\alpha)}$ will be used. For any $\alpha \in \mathcal{U}_{ad}$ we denote

$$\begin{aligned}
\mathcal{V}_0(\alpha) = \Big\{ &\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{C}_0^\infty(\Omega(\alpha)) \times \mathcal{C}^\infty(\overline{\Omega}(\alpha)); \\
&\operatorname{dist}(\operatorname{supp}(\varphi_2), \partial\Omega(\alpha) \setminus \Gamma_{out}) > 0 \Big\}
\end{aligned}$$

and define the spaces for the velocity

$$W(\alpha) = \overline{(\mathcal{C}^\infty(\overline{\Omega}(\alpha)))^2}^{\|\cdot\|_\alpha}, \qquad W_0(\alpha) = \overline{\mathcal{V}_0(\alpha)}^{\|\cdot\|_\alpha}, \tag{5}$$

where the closure is taken in the norm

$$\begin{aligned}
&\|\boldsymbol{v}\|_\alpha := \|\boldsymbol{v}\|_{1,2,\Omega(\alpha)} + \|M_\alpha \mathbb{D}(\boldsymbol{v})\|_{3,\Omega(\alpha)} + \|\operatorname{div} \boldsymbol{v}\|_{3,\Omega(\alpha)}, \\
&M_\alpha(\boldsymbol{x}) := \big(l_{m,\alpha}(\boldsymbol{x})\big)^{2/3}, \quad \boldsymbol{x} \in \overline{\Omega}(\alpha).
\end{aligned}$$

Finally, let

$$W_{\boldsymbol{v}_0}(\alpha) = \big\{\boldsymbol{v} \in W(\alpha);\ \boldsymbol{v} - \boldsymbol{v}_0 \in W_0(\alpha)\big\}.$$

We say that $\boldsymbol{v} \in W(\alpha)$ satisfies the stable boundary conditions $(4)_{1-3}$ in the weak sense iff $\boldsymbol{v} \in W_{\boldsymbol{v}_0}(\alpha)$.

Lemma 1 *$W(\alpha)$ and $W_0(\alpha)$ are separable reflexive Banach spaces.*

Definition 1 Define the operator $A_\alpha : W(\alpha) \to (W(\alpha))^*$ by the formula

$$\langle A_\alpha(\boldsymbol{v}), \boldsymbol{w}\rangle_\alpha := 2\rho \int_{\Omega(\alpha)} M_\alpha^3 |\mathbb{D}(\boldsymbol{v})|\mathbb{D}(\boldsymbol{v}) : \mathbb{D}(\boldsymbol{w}); \quad \boldsymbol{v}, \boldsymbol{w} \in W(\alpha).$$

Here $\langle \cdot, \cdot \rangle_\alpha$ denotes the duality pairing between $(W(\alpha))^*$ and $W(\alpha)$.

We are ready to give a weak formulation of the state problem. In what follows we will use the Einstein summation convention, i.e. $a_i b_i := \sum_{i=1}^n a_i b_i$. Further we denote $(f, g)_\alpha := \int_{\Omega(\alpha)} fg$, provided that $fg \in L^1(\Omega(\alpha))$.

Definition 2 A pair $(\boldsymbol{v}, p) \in W(\alpha) \times L^{\frac{3}{2}}(\Omega(\alpha))$ is said to be a weak solution of the state problem $(\mathcal{P}(\alpha))$ iff

(i) $\boldsymbol{v} \in W_{\boldsymbol{v}_0}(\alpha)$;
(ii) for every $\boldsymbol{\varphi} \in W_0(\alpha)$ it holds:

$$\begin{aligned}
&2\mu_0(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi}))_\alpha + \rho\left(v_j \frac{\partial v_i}{\partial x_j}, \varphi_i\right)_\alpha + \langle A_\alpha(\boldsymbol{v}), \boldsymbol{\varphi}\rangle_\alpha \\
&\quad + \sigma \int_{\Gamma_{out}} |v_2| v_2 \varphi_2 - (p, \operatorname{div} \boldsymbol{\varphi})_\alpha = 0; \qquad (6)
\end{aligned}$$

(iii) for every $\psi \in L^{\frac{3}{2}}(\Omega(\alpha))$ it holds: $(\psi, \operatorname{div} \boldsymbol{v})_\alpha = 0$.

Convention In the sections, where we will deal with the state problem on a fixed domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$, the letter α in the argument will be often omitted. Thus we will write $\Omega := \Omega(\alpha)$, $W := W(\alpha)$, $A := A_\alpha$, $(\cdot, \cdot) := (\cdot, \cdot)_\alpha$ etc. without causing confusion.

2.5 Existence of a Weak Solution

Recall that the function $\boldsymbol{v}_0$ is now defined in the whole $\widehat{\Omega}$ and it does not depend on $\alpha \in \mathcal{U}_{ad}$. This fact will be used further in order to establish estimates which are independent of $\alpha \in \mathcal{U}_{ad}$.

Theorem 2 *Let*

$$\sigma > \frac{\rho}{2}. \qquad (7)$$

Then

(i) *for every $\alpha \in \mathcal{U}_{ad}$ there exists at least one weak solution of $(\mathcal{P}(\alpha))$;*
(ii) *there exists a constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ such that for any weak solution $(\boldsymbol{v}, p)$ of $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, the following estimate holds:*

$$\|\nabla \boldsymbol{v}\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v})|\|_{3,\Omega}^3 + \|v_2\|_{3,\Gamma_{out}}^3 + \|p\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \le C_E. \qquad (8)$$

In addition, the constant C_E does not depend on $\alpha \in \mathcal{U}_{ad}$;

(iii) *if $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ are two weak solutions of $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, then $p^1 = p^2$. Moreover, for $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ small enough (independently of $\alpha \in \mathcal{U}_{ad}$) there exists a unique weak solution.*

For the proof we refer to [3].

2.6 Existence of an Optimal Shape

Note that the assumption of Theorem 2, which guarantees the existence of at least one weak solution to the state problem $(\mathcal{P}(\alpha))$, does not depend on a particular choice of $\Omega(\alpha) \in \mathcal{O}$. In what follows we assume that this assumption is satisfied. Further let

$$\widehat{W}(\alpha) := \left\{ \boldsymbol{v} \in \left(W^{1,2}(\Omega(\alpha)) \right)^2;\ \operatorname{div} \boldsymbol{v} \in L^3(\Omega(\alpha)),\ M_\alpha |\mathbb{D}(\boldsymbol{v})| \in L^3(\Omega(\alpha)) \right\}$$

and define

$$\widehat{W}_{\boldsymbol{v}_0}(\alpha) := \Big\{ \boldsymbol{v} \in \widehat{W}(\alpha);\ \boldsymbol{v} \text{ satisfies the Dirichlet conditions } (4)_1\text{–}(4)_3 \text{ on } \partial\Omega(\alpha) \Big\}.$$

Remark 1 It holds that $W_{\boldsymbol{v}_0}(\alpha) \subseteq \widehat{W}_{\boldsymbol{v}_0}(\alpha)$. The question arises, if these spaces are identical. This is in fact the density problem. For the moment we do not know the answer.

The lack of the mentioned density property leads us to modify the definition of the state problem as follows:

Definition 3 (Augmented state problem $(\widehat{\mathcal{P}}(\alpha))$) Let $\alpha \in \mathcal{U}_{ad}$. A pair $(\boldsymbol{v}, p) := (\boldsymbol{v}(\alpha), p(\alpha)) \in \widehat{W}_{\boldsymbol{v}_0}(\alpha) \times L^{\frac{3}{2}}(\Omega(\alpha))$ is said to be a solution of the augmented state problem $(\widehat{\mathcal{P}}(\alpha))$ iff

- $(\boldsymbol{v}, p)$ satisfies (ii) and (iii) of Definition 2;
- $(\boldsymbol{v}, p)$ satisfies the estimate (8).

Clearly any solution of $(\mathcal{P}(\alpha))$ becomes a solution of $(\widehat{\mathcal{P}}(\alpha))$ too. Moreover the statement of Theorem 2 can be applied to $(\widehat{\mathcal{P}}(\alpha))$ as well; in particular we have the same criterion for uniqueness.

Definition 4 (Augmented shape optimization problem $(\widehat{\mathbb{P}})$) Let us define the set

$$\widehat{\mathcal{G}} := \{(\alpha, \boldsymbol{v}, p);\ \alpha \in \mathcal{U}_{ad},\ (\boldsymbol{v}, p) \text{ is a solution of } (\widehat{\mathcal{P}}(\alpha))\}.$$

A triple $(\alpha^*, \boldsymbol{v}^*, p^*) \in \widehat{\mathcal{G}}$ is said to be a solution of the augmented shape optimization problem $(\widehat{\mathbb{P}})$ iff

$$J(\alpha^*, \boldsymbol{v}^*, p^*) \leq J(\alpha, \boldsymbol{v}, p) \quad \forall (\alpha, \boldsymbol{v}, p) \in \widehat{\mathcal{G}}.$$

Next we introduce convergence of a sequence of domains.

Definition 5 Let $\{\Omega(\alpha_n)\}$, $\alpha_n \in \mathcal{U}_{ad}$ be a sequence of domains. We say that $\{\Omega(\alpha_n)\}$ converges to $\Omega(\alpha)$, shortly $\Omega(\alpha_n) \rightsquigarrow \Omega(\alpha)$, iff $\alpha_n \rightrightarrows \alpha$ in $[0, L]$.

As a direct consequence of the Arzelà–Ascoli theorem we see that the system $\mathcal{O}$ is compact with respect to convergence introduced in Definition 5.

In [3] we proved the following stability result for the solutions $\{(\boldsymbol{v}_n, p_n)\}$ of $(\widehat{\mathcal{P}}(\alpha_n))$.

Theorem 3 *Let $(\boldsymbol{v}(\alpha_n), p(\alpha_n))$ be solutions to $(\widehat{\mathcal{P}}(\alpha_n))$, $n = 1, 2, \ldots$ and $\alpha \in \mathcal{U}_{ad}$ satisfy*

$$\alpha_n \rightrightarrows \alpha \quad \textit{in } [0, L], \ n \to \infty.$$

Then there exists $\widehat{\boldsymbol{v}} \in (W^{1,2}(\widehat{\Omega}))^2$, $\widehat{p} \in L^{\frac{3}{2}}(\widehat{\Omega})$ and a subsequence of $\{(\tilde{\boldsymbol{v}}_n, \tilde{p}_n)\}$ (denoted by the same symbol) such that

$$\begin{aligned} &\tilde{\boldsymbol{v}}_n \rightharpoonup \widehat{\boldsymbol{v}} && \textit{in } \left(W^{1,2}(\widehat{\Omega})\right)^2, \\ &\tilde{M}_{\alpha_n}\mathbb{D}(\tilde{\boldsymbol{v}}_n) \rightharpoonup \tilde{M}_{\alpha}\mathbb{D}(\widehat{\boldsymbol{v}}) && \textit{in } \left(L^3(\widehat{\Omega})\right)^{2\times 2}, \\ &\tilde{p}_n \rightharpoonup \widehat{p} && \textit{in } L^{\frac{3}{2}}(\widehat{\Omega}), \ n \to \infty, \end{aligned} \tag{9}$$

where the symbol $\tilde{}$ stands for the zero extension of a function from the domain of its definition to $\widehat{\Omega}$. In addition, denoting $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)}$ and $p(\alpha) := \widehat{p}_{|\Omega(\alpha)}$, then $(\boldsymbol{v}(\alpha), p(\alpha))$ solves $(\widehat{\mathcal{P}}(\alpha))$.

Corollary 4 *Problem $(\widehat{\mathbb{P}})$ has a solution.*

3 Approximation of the Flow Problem

In this section we describe the finite-element approximation of $(\mathcal{P}(\alpha))$ and analyze its properties such as the existence of discrete solutions and their convergence to a solution of the original problem.

Let $\tilde{\mathcal{U}}_{ad} \subset \mathcal{U}_{ad}$ be a set of all piecewise linear functions $\alpha \in \mathcal{U}_{ad}$. Throughout this section we will assume that $\alpha \in \tilde{\mathcal{U}}_{ad}$ is *fixed* (hence the symbol α will be often dropped), so that $\Omega := \Omega(\alpha)$ is a *polygonal* domain.

Let $\{\mathcal{T}_h\}$, $h \to 0+$ be a family of triangulations of Ω and h be the norm of $\mathcal{T}_h$. Throughout the section we will assume that the following conditions are satisfied:

(A1) the family $\{\mathcal{T}_h\}$ is *uniformly regular* with respect to h: there is $\theta_0 > 0$ such that $\theta(h) \geq \theta_0\ \forall h > 0$, where $\theta(h)$ is the minimal interior angle of all triangles from $\mathcal{T}_h$;

(A2) the family $\{\mathcal{T}_h\}$ is *consistent* with the decomposition of $\partial\Omega$ into Γ_{out} and $\partial\Omega \setminus \Gamma_{out}$.

In what follows we will assume that $W_{0h} \subset W_0$ and $L_h \subset L^{\frac{3}{2}}(\Omega)$ are finite dimensional spaces.

Definition 6 We say that (W_{0h}, L_h) satisfy the inf-sup condition (also the Babuška-Brezzi condition), if there exists a constant $C_{BB} > 0$ independent of h and $\alpha \in \tilde{\mathcal{U}}_{ad}$ s.t.

$$\inf_{q \in L_h} \sup_{\boldsymbol{w} \in W_{0h}} \frac{(q, \operatorname{div} \boldsymbol{w})}{\|q\|_{\frac{3}{2}} \|\boldsymbol{w}\|_\alpha} \geq C_{BB}. \tag{10}$$

Let us emphasize that we require the constant C_{BB} to be independent of $\alpha \in \tilde{\mathcal{U}}_{ad}$, which will be important in the shape optimization part. While in the literature there are many examples of inf-sup stable elements for the situation when velocity is prescribed on the whole boundary $\partial\Omega$, the choice of (W_{0h}, L_h) satisfying (10) may not be obvious.

Denote

$$L_0^q(\Omega) := \left\{ \psi \in L^q(\Omega);\ \int_\Omega \psi = 0 \right\}.$$

Lemma 5 *Assume that there exists $\boldsymbol{\varphi} \in W_{0h}$ such that $\int_{\Gamma_{out}} \boldsymbol{\varphi} \cdot \boldsymbol{\nu} > 0$. Let $V_h \subset W_{0h} \cap W_0^{1,2}(\Omega)^2$, $Q_h \subset L_h \cap L_0^{\frac{3}{2}}(\Omega)$ be finite dimensional spaces satisfying*

$$\inf_{q \in Q_h} \sup_{\boldsymbol{w} \in V_h} \frac{(q, \operatorname{div} \boldsymbol{w})}{\|q\|_{\frac{3}{2}} \|\boldsymbol{w}\|_\alpha} \geq C \tag{11}$$

with a constant $C > 0$ independent of h and $\alpha \in \tilde{\mathcal{U}}_{ad}$. Then (10) holds true.

Proof Let us pick arbitrary $\boldsymbol{\varphi} \in W_{0h}$ such that $\|\boldsymbol{\varphi}\|_\alpha = 1$ and $\beta := \int_{\Gamma_{out}} \boldsymbol{\varphi} \cdot \boldsymbol{\nu} > 0$. For any $\tilde{q} \in L_h$ we can write $\tilde{q} = q + c$, where $q \in Q_h$ and $c \in \mathbb{R}$. Moreover it can be easily shown that

$$\|\tilde{q}\|_{\frac{3}{2}} \leq \|q\|_{\frac{3}{2}} + |c||\Omega_0|^{\frac{2}{3}}. \tag{12}$$

From (11) we see that there exists $\boldsymbol{w} \in V_h$, $\|\boldsymbol{w}\|_\alpha = 1$, such that

$$(q, \operatorname{div} \boldsymbol{w}) \geq \frac{C}{2} \|q\|_{\frac{3}{2}}.$$

Setting $\tilde{\boldsymbol{w}} := \boldsymbol{w} + \frac{C}{4}(\operatorname{sgn} c)\boldsymbol{\varphi}$, using Hölder's inequality and (12) we obtain:

$$\begin{aligned}
(\tilde{q}, \operatorname{div} \tilde{\boldsymbol{w}}) &= (q, \operatorname{div} \boldsymbol{w}) + \frac{C}{4} \operatorname{sgn} c (q, \operatorname{div} \boldsymbol{\varphi}) + \frac{C}{4} \beta |c| \\
&\geq \frac{C}{4} \left(\|q\|_{\frac{3}{2}} + \beta |c| \right) \geq \frac{C}{4} \min \left\{ 1, \frac{\beta}{|\Omega_0|^{\frac{2}{3}}} \right\} \|\tilde{q}\|_{\frac{3}{2}},
\end{aligned}$$

which completes the proof. $\square$

It is possible to take usual finite element spaces V_h and Q_h which are inf-sup stable in the norms of $W_0^{1,3}(\Omega)$ and $L_0^{\frac{3}{2}}(\Omega)$ (such as the Taylor-Hood elements, see e.g. [2] for particular examples). Since the norm $\|\cdot\|_{1,3}$ is stronger than $\|\cdot\|_\alpha$, (11) holds true. Based on Lemma 5, one can easily obtain W_{0h}, L_h satisfying (10).

Definition 7 A pair $(\boldsymbol{v}_h, p_h) \in W \times L_h$ is said to be a solution of the discrete state problem $(\mathcal{P}_h(\alpha))$ iff

(i) $\boldsymbol{v}_h - \boldsymbol{v}_0 \in W_{0h}$,
(ii) for every $\boldsymbol{\varphi}_h \in W_{0h}$ it holds:

$$
\begin{aligned}
&2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho\left(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}\right) + \frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)\\
&\quad + (|\operatorname{div}\boldsymbol{v}_h|\operatorname{div}\boldsymbol{v}_h, \operatorname{div}\boldsymbol{\varphi}_h) + \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h\rangle + \sigma\int_{\Gamma_{out}} |v_{h2}|v_{h2}\varphi_{h2}\\
&\quad - (p_h, \operatorname{div}\boldsymbol{\varphi}_h) = 0,
\end{aligned} \tag{13}
$$

(iii) for every $\psi_h \in L_h$ it holds: $(\psi_h, \operatorname{div}\boldsymbol{v}_h) = 0$.

Let us point out that in contrast to $(\mathcal{P}(\alpha))$, problem $(\mathcal{P}_h(\alpha))$ contains the additional terms $\frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)$ and $(|\operatorname{div}\boldsymbol{v}_h|\operatorname{div}\boldsymbol{v}_h, \operatorname{div}\boldsymbol{\varphi}_h)$ in order to obtain a uniform estimate for the discrete solutions. In the continuous case, these terms vanish due to the divergence free velocity. However, (iii) of $(\mathcal{P}_h(\alpha))$ does not guarantee that $\operatorname{div}\boldsymbol{v}_h = 0$ a.e. in Ω.

3.1 Existence of a Discrete Solution

We will use a technique that is similar to the one presented in [3], Sect. 2.4, to prove that $(\mathcal{P}_h(\alpha))$, $\alpha \in \tilde{\mathcal{U}}_{ad}$ possesses a solution.

Theorem 6 *Let $\sigma > \frac{\rho}{2}$ and the Babuška-Brezzi condition* (10) *be satisfied. Then for every $h > 0$*

(i) *there exists a solution of $(\mathcal{P}_h(\alpha))$;*
(ii) *any solution $(\boldsymbol{v}_h, p_h)$ of $(\mathcal{P}_h(\alpha))$ admits the estimate*

$$
\|\nabla\boldsymbol{v}_h\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega}^3 + \|\operatorname{div}\boldsymbol{v}_h\|_{3,\Omega}^3 + \|v_{h2}\|_{3,\Gamma_{out}}^3 + \|p_h\|_{\frac{3}{2},\Omega}^{\frac{3}{2}} \le C_E, \tag{14}
$$

where the constant $C_E := C_E(\mu_0, \rho, \sigma, \|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}) > 0$ is the same as in Theorem 2, in particular independent of h and $\alpha \in \tilde{\mathcal{U}}_{ad}$;
(iii) *for $\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ small (independently of h), the solution is unique;*
(iv) *p_h is uniquely determined by $\boldsymbol{v}_h$.*

Theorem 6 will be proven in three steps. First we will deal with the existence of $\boldsymbol{v}_h$, then for given $\boldsymbol{v}_h$ we will establish p_h and finally uniqueness of $\boldsymbol{v}_h$ and p_h will be discussed.

Proof Let us define the mapping $\mathrm{div}_h : W_{0h} \to L_h^*$ as follows:

$$\langle \mathrm{div}_h\, \boldsymbol{w}_h, \psi_h \rangle := \int_\Omega \psi_h \,\mathrm{div}\, \boldsymbol{w}_h \quad \forall \boldsymbol{w}_h \in W_{0h}, \psi_h \in L_h$$

and denote $V_h := \ker \mathrm{div}_h$. We want to find $\boldsymbol{v}_h \in W$, such that

(i) $\boldsymbol{v}_h - \boldsymbol{v}_0 \in V_h$,
(ii) for every $\boldsymbol{\varphi}_h \in V_h$ it holds:

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho\left(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}\right) + \frac{\rho}{2}((\mathrm{div}\, \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)$$
$$+ (|\,\mathrm{div}\, \boldsymbol{v}_h|\,\mathrm{div}\, \boldsymbol{v}_h, \mathrm{div}\, \boldsymbol{\varphi}_h) + \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h\rangle + \sigma \int_{\Gamma_{out}} |v_{h2}|v_{h2}\varphi_{h2} = 0. \quad (15)$$

It is readily seen that for $\boldsymbol{\varphi}_h \in V_h$, (13) and (15) coincide. Using the technique of [3], Lemma 8, one can prove the existence of $\boldsymbol{v}_h$ by means of a priori estimates and the Brouwer fixed point theorem. Moreover, the estimate

$$\|\nabla \boldsymbol{v}_h\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega}^3 + \|\,\mathrm{div}\, \boldsymbol{v}_h\|_{3,\Omega}^3 + \|v_{h2}\|_{3,\Gamma_{out}}^3 \le C_E, \quad (16)$$

holds with a constant $C_E > 0$ independent of $h > 0$ and $\alpha \in \tilde{\mathcal{U}}_{ad}$.

Now let us define the functional $B_h \in W_{0h}^*$:

$$\langle B_h, \boldsymbol{\varphi}_h\rangle := 2\mu_0(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{\varphi}_h)) + \rho\left(v_{hj}\frac{\partial v_{hi}}{\partial x_j}, \varphi_{hi}\right) + \frac{\rho}{2}((\mathrm{div}\, \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{\varphi}_h)$$
$$+ (|\,\mathrm{div}\, \boldsymbol{v}_h|\,\mathrm{div}\, \boldsymbol{v}_h, \mathrm{div}\, \boldsymbol{\varphi}_h)$$
$$+ \langle A(\boldsymbol{v}_h), \boldsymbol{\varphi}_h\rangle + \sigma \int_{\Gamma_{out}} |v_{h2}|v_{h2}\varphi_{h2}, \quad \forall \boldsymbol{\varphi}_h \in W_{0h}. \quad (17)$$

In virtue of (15), we see that $B_h \in (V_h)^\circ$. From the well known properties of linear mappings in finite dimensional spaces it follows that

$$(V_h)^\circ = (\ker \mathrm{div}_h)^\circ = \mathcal{R}(\mathrm{div}_h').$$

Here V_h° is the polar set of V_h and $\mathrm{div}_h' : L_h \mapsto W_{0h}^*$ is the adjoint of div_h (see e.g. [9]). The last equality yields the existence of $p_h \in L_h$ satisfying $\mathrm{div}_h' p_h = B_h$, meaning that

$$\langle \mathrm{div}_h' p_h, \boldsymbol{\varphi}_h\rangle = (p_h, \mathrm{div}\, \boldsymbol{\varphi}_h) = \langle B_h, \boldsymbol{\varphi}_h\rangle$$

for every $\boldsymbol{\varphi}_h \in W_{0h}$. Using this, (10), (16) and (17) we obtain:

$$\|p_h\|_{\frac{3}{2}} \le C,$$

where $C > 0$ is a constant independent of h and $\alpha \in \tilde{\mathcal{U}}_{ad}$.

Uniqueness of $\boldsymbol{v}_h$ can be proven in the same way as in [3], Lemma 13. To prove (iv), let us assume that $(\boldsymbol{v}_h, p_h^1)$ and $(\boldsymbol{v}_h, p_h^2)$ are two solutions of $(\mathcal{P}_h(\alpha))$.

Then, if we insert $(\boldsymbol{v}_h, p_h^1)$ and $(\boldsymbol{v}_h, p_h^2)$ into (13) and subtract the respective equations, we obtain:

$$\forall \boldsymbol{\varphi}_h \in W_{0h} \quad (p_h^1 - p_h^2, \operatorname{div} \boldsymbol{\varphi}_h) = 0. \tag{18}$$

From (10) it follows that $p_h^1 = p_h^2$ a.e. in Ω. □

3.2 Convergence of Discrete Solutions

In this section we will study the relation between $(\boldsymbol{v}_h, p_h)$ and $(\boldsymbol{v}, p)$ for $h \to 0+$.

Convention Here and in what follows we will use the same symbol for an original sequence and its subsequences.

Theorem 7 *Let the assumptions of Theorem 6 be satisfied and let* $\{W_{0h}\}_{h>0}$, $\{L_h\}_{h>0}$ *be dense in* W_0 *and* $L^{\frac{3}{2}}(\Omega)$, *respectively. Then for any sequence* $\{(\boldsymbol{v}_h, p_h)\}$ *of solutions to* $(\mathcal{P}_h(\alpha))$ *there exists a subsequence and a limit pair* $(\boldsymbol{v}, p) \in W_{\boldsymbol{v}_0} \times L^{\frac{3}{2}}(\Omega)$ *such that*

$$\boldsymbol{v}_h \to \boldsymbol{v} \quad \text{in } W, \tag{19a}$$

$$p_h \rightharpoonup p \quad \text{in } L^{\frac{3}{2}}(\Omega),\ h \to 0+ \tag{19b}$$

and $(\boldsymbol{v}, p)$ *is a solution of* $(\mathcal{P}(\alpha))$, $\alpha \in \tilde{\mathcal{U}}_{ad}$.

For the proof of this theorem we will need the following auxiliary result which can be established using Lemma 1.19 in [20].

Lemma 8 (Some properties of A_α, $\alpha \in \mathcal{U}_{ad}$)

(i) A_α *is monotone in* $W(\alpha)$ *in the following sense*:

$$\langle A_\alpha(\boldsymbol{v}) - A_\alpha(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_\alpha \geq C\|M_\alpha \mathbb{D}(\boldsymbol{v} - \boldsymbol{w})\|_{3,\Omega}^3 \quad \forall \boldsymbol{v}, \boldsymbol{w} \in W(\alpha),$$

where $C > 0$ *is independent of* α.

(ii) A_α *is continuous in* $W(\alpha)$.

Proof of Theorem 7 The existence of $\boldsymbol{v} \in W$, $\overline{A} \in W^*$, $\overline{d} \in L^{\frac{3}{2}}(\Omega)$ satisfying

$$\boldsymbol{v}_h \rightharpoonup \boldsymbol{v} \quad \text{in } W, \tag{20}$$

$$\boldsymbol{v}_h \rightharpoonup \boldsymbol{v} \quad \text{in } W^{1,2}(\Omega), \tag{21}$$

$$M\mathbb{D}(\boldsymbol{v}_h) \rightharpoonup M\mathbb{D}(\boldsymbol{v}) \quad \text{in } L^3(\Omega),$$

$$A(\boldsymbol{v}_h) \rightharpoonup \overline{A} \quad \text{in } W^*, \tag{22}$$

$$|\operatorname{div} \boldsymbol{v}_h| \operatorname{div} \boldsymbol{v}_h \rightharpoonup \overline{d} \quad \text{in } L^{\frac{3}{2}}(\Omega), \tag{23}$$

$$p_h \rightharpoonup p \quad \text{in } L^{\frac{3}{2}}(\Omega),$$

follows from (14).

Next we prove that $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω. Let $\psi \in L^{\frac{3}{2}}(\Omega)$. Then there is a sequence $\{\psi_h\}$, $\psi_h \in L_h$, such that $\psi_h \to \psi$ in $L^{\frac{3}{2}}(\Omega)$. From this and (21) we obtain:

$$0 = (\psi_h, \operatorname{div} \boldsymbol{v}_h) \to (\psi, \operatorname{div} \boldsymbol{v}), \tag{24}$$

so that $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω.

Now we make the limit passage in (13). Let $\boldsymbol{\varphi} \in W_0$. Then there is a sequence $\{\boldsymbol{\varphi}_h\}$, $\boldsymbol{\varphi}_h \in W_{0h}$ such that

$$\boldsymbol{\varphi}_h \to \boldsymbol{\varphi} \quad \text{in } W_0. \tag{25}$$

Similarly to the proof of Lemma 9 in [3], we will use the compact imbeddings in the respective spaces, (19), (24) and (25) to pass to the limit with $h \to 0+$ in the standard terms, which together with (22) and (23) yield:

$$(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{\varphi})) + \left(v_j \frac{\partial v_i}{\partial x_j}, \varphi_i\right) + (\overline{d}, \operatorname{div} \boldsymbol{\varphi}) + \langle \overline{A}, \boldsymbol{\varphi}\rangle + \int_{\Gamma_{out}} |v_2| v_2 \varphi_2 - (p, \operatorname{div} \boldsymbol{\varphi}) = 0 \tag{26}$$

for every $\boldsymbol{\varphi} \in W_0$ (here we put $2\mu_0 = \rho = \sigma = 1$ for simplicity).

Now we use monotonicity of $|\operatorname{div} \boldsymbol{v}| \operatorname{div} \boldsymbol{v}$ and A to show that

$$(\overline{d}, \operatorname{div} \boldsymbol{\varphi}) + \langle \overline{A}, \boldsymbol{\varphi}\rangle = \langle A(\boldsymbol{v}), \boldsymbol{\varphi}\rangle \quad \text{for every } \boldsymbol{\varphi} \in W.$$

Indeed, let $\boldsymbol{\varphi} \in W$. Then

$$\begin{aligned}
0 &\le (|\operatorname{div} \boldsymbol{v}_h| \operatorname{div} \boldsymbol{v}_h - |\operatorname{div} \boldsymbol{\varphi}| \operatorname{div} \boldsymbol{\varphi}, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{\varphi})) + \langle A(\boldsymbol{v}_h) - A(\boldsymbol{\varphi}), \boldsymbol{v}_h - \boldsymbol{\varphi}\rangle \\
&= -(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) - \left(v_{hj} \frac{\partial v_{hi}}{\partial x_j}, v_{hi} - v_{0i}\right) \\
&\quad - \frac{1}{2}((\operatorname{div} \boldsymbol{v}_h)(\boldsymbol{v}_h - \boldsymbol{v}_0), \boldsymbol{v}_h - \boldsymbol{v}_0) \\
&\quad - \int_{\Gamma_{out}} |v_{h2}| v_{h2}(v_{h2} - v_{02}) + (|\operatorname{div} \boldsymbol{v}_h| \operatorname{div} \boldsymbol{v}_h, \operatorname{div}(\boldsymbol{v}_0 - \boldsymbol{\varphi})) \\
&\quad - (|\operatorname{div} \boldsymbol{\varphi}| \operatorname{div} \boldsymbol{\varphi}, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{\varphi})) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_0 - \boldsymbol{\varphi}\rangle \\
&\quad - \langle A(\boldsymbol{\varphi}), \boldsymbol{v}_h - \boldsymbol{\varphi}\rangle,
\end{aligned} \tag{27}$$

making use of (13) and the fact that $\boldsymbol{v}_h - \boldsymbol{v}_0 \in W_{0h}$. Letting $h \to 0+$ and using lower semicontinuity of $\|\mathbb{D}(\boldsymbol{v}_h)\|_{2,\Omega}$ and continuity of the remaining terms we obtain:

$$\begin{aligned}
0 &\le -(\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v} - \boldsymbol{v}_0)) - \left(v_j \frac{\partial v_i}{\partial x_j}, v_i - v_{0i}\right) - \int_{\Gamma_{out}} |v_2| v_2 (v_2 - v_{02}) \\
&\quad + (\overline{d}, \operatorname{div}(\boldsymbol{v}_0 - \boldsymbol{\varphi})) - (|\operatorname{div} \boldsymbol{\varphi}| \operatorname{div} \boldsymbol{\varphi}, \operatorname{div}(\boldsymbol{v} - \boldsymbol{\varphi})) \\
&\quad + \langle \overline{A}, \boldsymbol{v}_0 - \boldsymbol{\varphi}\rangle - \langle A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi}\rangle.
\end{aligned} \tag{28}$$

From (26) and (28) we arrive at the inequality

$$0 \le (\overline{d} - |\operatorname{div} \boldsymbol{\varphi}| \operatorname{div} \boldsymbol{\varphi}, \operatorname{div}(\boldsymbol{v} - \boldsymbol{\varphi})) + \langle \overline{A} - A(\boldsymbol{\varphi}), \boldsymbol{v} - \boldsymbol{\varphi} \rangle, \tag{29}$$

which holds for any $\boldsymbol{\varphi} \in W$. Choosing $\boldsymbol{\varphi} := \boldsymbol{v} \pm \lambda \boldsymbol{\psi}$, $\lambda > 0$, $\boldsymbol{\psi} \in W$ and dividing by λ we obtain for $\lambda \to 0+$:

$$(\overline{d}, \operatorname{div} \boldsymbol{\psi}) + \langle \overline{A}, \boldsymbol{\psi} \rangle = \langle A(\boldsymbol{v}), \boldsymbol{\psi} \rangle.$$

From this and (26) we see that $(\boldsymbol{v}, p)$ solves $(\mathcal{P}(\alpha))$.

To prove strong convergence of $\boldsymbol{v}_h$ to $\boldsymbol{v}$ we use (i) in Lemma 8:

$$\begin{aligned}
&C \left(\|\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})\|_{2,\Omega}^2 + \|M|\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})|\|_{3,\Omega}^3 + \|\operatorname{div} \boldsymbol{v}_h\|_{3,\Omega}^3 \right) \\
&\quad \le (\mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})) + \langle A(\boldsymbol{v}_h) - A(\boldsymbol{v}), \boldsymbol{v}_h - \boldsymbol{v} \rangle + \|\operatorname{div} \boldsymbol{v}_h\|_{3,\Omega}^3 \\
&\quad = (\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_h - \boldsymbol{v}_0 \rangle + (|\operatorname{div} \boldsymbol{v}_h| \operatorname{div} \boldsymbol{v}_h, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{v}_0)) \\
&\qquad + (\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_0 - \boldsymbol{v})) - (\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v})) \\
&\qquad + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_0 - \boldsymbol{v} \rangle - \langle A(\boldsymbol{v}), \boldsymbol{v}_h - \boldsymbol{v} \rangle.
\end{aligned} \tag{30}$$

The expression

$$(\mathbb{D}(\boldsymbol{v}_h), \mathbb{D}(\boldsymbol{v}_h - \boldsymbol{v}_0)) + \langle A(\boldsymbol{v}_h), \boldsymbol{v}_h - \boldsymbol{v}_0 \rangle + (|\operatorname{div} \boldsymbol{v}_h| \operatorname{div} \boldsymbol{v}_h, \operatorname{div}(\boldsymbol{v}_h - \boldsymbol{v}_0))$$

on the right hand side of (30) can be replaced using (13). Then due to weak convergence of $\{\boldsymbol{v}_h\}$ and $\{p_h\}$ the right hand side of (30) vanishes for $h \to 0+$, which yields (19a). □

4 Approximation of the Shape Optimization Problem

4.1 Parameterization of the Discrete Shapes

We now introduce two types of discretized domains: a *discrete design* and *discrete computational domain*. The boundary Γ_α of the discrete design domain is realized by a smooth, piecewise quadratic Bézier function. The optimal discrete design domain is the main output of the computational process according to which a designer makes decisions. On the other hand, our finite element method requires a polygonal computational domain.

Let $\varkappa > 0$ be a discretization parameter, $\Delta_\varkappa : L_1 = a_0 < a_1 < \cdots < a_n = L_1 + L_2$ be an equidistant partition of $[L_1, L_1 + L_2]$, $a_i = L_1 + \frac{i}{n} L_2$, $n = n(\varkappa) = \frac{L_2}{\varkappa}$ and $a_{i-1/2}$ be the midpoint of $[a_{i-1}, a_i]$, $i = 1, \dots, n$. Further let $A_{i-1/2} = (a_{i-1/2}, \alpha_i)$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$ be the *design nodes*, $A_i = \frac{1}{2}(A_{i-1/2} + A_{i+1/2})$ be the midpoint of the segment $[A_{i-1/2}, A_{i+1/2}]$, $i = 1, \dots, n-1$, $A_0 = (a_0, H_1)$, and $A_n = (a_n, H_2)$, see Fig. 3. We introduce the set

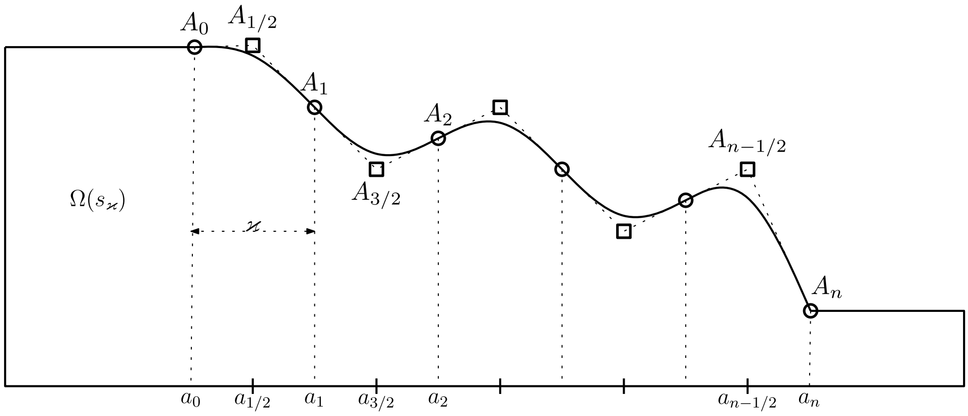


Fig. 3 Approximation of the boundary of $\Omega(\alpha)$

$$\begin{aligned}\mathcal{U}^{\varkappa} := \Big\{ & s_\varkappa \in C([0,L]);\ s_{\varkappa|[0,L_1]} = H_1,\ s_{\varkappa|[L_1+L_2,L]} = H_2,\\ & s_{\varkappa|[a_{i-1},a_i]} \text{ is a quadratic Bézier function}\\ & \text{determined by } \{A_{i-1}, A_{i-1/2}, A_i\},\ i = 1,\dots,n\Big\}.\end{aligned}$$

In order to define a family of admissible shapes locally realized by Bézier functions, it is necessary to specify $\alpha_i \in \mathbb{R}$ defining the position of the design nodes $A_{i-1/2}$, $i = 1,\dots,n$. With the partition $\Delta_\varkappa$ we associate the set $\boldsymbol{U} \subset \mathbb{R}^n$:

$$\begin{aligned}\boldsymbol{U} = \Big\{ & \boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n;\ \alpha_{min} \le \alpha_i \le \alpha_{max},\ i = 1,\dots,n;\\ & \frac{|\alpha_{i+1} - \alpha_i|}{\varkappa} \le \gamma,\ i = 1,\dots,n-1;\ \frac{2|\alpha_1 - H_1|}{\varkappa} \le \gamma,\ \frac{2|\alpha_n - H_2|}{\varkappa} \le \gamma \Big\},\end{aligned}$$

where $\gamma > 0$ is the same as in (1). The family of the *admissible discrete design domains* is now represented by

$$\mathcal{O}_\varkappa = \{\Omega(s_\varkappa);\ s_\varkappa \in \mathcal{U}^{\varkappa}_{ad}\},$$

where

$$\begin{aligned}\mathcal{U}^{\varkappa}_{ad} = \{ & s_\varkappa \in \mathcal{U}^{\varkappa};\ \text{the design nodes } A_{i-1/2} = (a_{i-1/2}, \alpha_i),\ i = 1,\dots,n,\\ & \text{are such that } \boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \boldsymbol{U}\}.\end{aligned}$$

Due to properties of the Bézier functions it holds that $\mathcal{U}^{\varkappa}_{ad} \subset \mathcal{U}_{ad}$.

We now turn to the definition of the computational domains. To this end we introduce another family of partitions $\{\Delta_h\}$, $h \to 0+$, of $[L_1, L_1 + L_2]$ (not necessarily equidistant), whose norm will be denoted by h. Next we will suppose that $h \to 0+$ iff $\varkappa \to 0+$. Let $r_h s_\varkappa$ be the piecewise linear Lagrange interpolant of $s_\varkappa \in \mathcal{U}^{\varkappa}_{ad}$ on Δ_h. The computational domain related to $\Omega(s_\varkappa)$ will be represented by $\Omega(r_h s_\varkappa)$; i.e. the curved side $\Gamma_{s_\varkappa}$, being the graph of $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, is replaced by its piecewise linear Lagrange approximation $r_h s_\varkappa$ on Δ_h. The system of *computational domains* will be denoted by $\mathcal{O}_{\varkappa h}$ in what follows:

$$\mathcal{O}_{\varkappa h} := \{\Omega(r_h s_\varkappa);\ s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}\}.$$

Since $\Omega(r_h s_\varkappa)$ is already polygonal, one can construct its triangulation $\mathcal{T}_h(s_\varkappa)$ with the norm $h > 0$ and depending on $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$.

Convention The domain $\Omega(r_h s_\varkappa)$ with a given triangulation $\mathcal{T}_h(s_\varkappa)$ will be denoted by $\Omega_h(s_\varkappa)$ in what follows.

4.2 Formulation of the Discrete Problem

Let us define the set

$$\mathcal{G}_{\varkappa h} := \{(s_\varkappa, \boldsymbol{v}_h, p_h);\ s_\varkappa \in \mathcal{U}_{ad}^{\varkappa},\ (\boldsymbol{v}_h, p_h) \text{ is a solution of } (\mathcal{P}_h(r_h s_\varkappa))\}.$$

The discretization of $(\mathbb{P})$ then reads as follows:

$$\begin{cases} \text{Find} \quad (s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) \in \mathcal{G}_{\varkappa h} \quad \text{such that} \\ J(s_\varkappa^*, \boldsymbol{v}_h^*, p_h^*) \le J(s_\varkappa, \boldsymbol{v}_h, p_h) \quad \forall (s_\varkappa, \boldsymbol{v}_h, p_h) \in \mathcal{G}_{\varkappa h}. \end{cases} \tag{$\mathbb{P}_{\varkappa h}$}$$

The approximate optimal shape is given by $\Omega(s_\varkappa^*)$.

Next we will analyze the existence of solutions to $(\mathbb{P}_{\varkappa h})$ and their relation to solutions of $(\widehat{\mathbb{P}})$ as $h, \varkappa \to 0+$.

4.3 Existence of Solutions

In order to establish the existence results, we have to impose additional assumptions on the family of triangulations $\{\mathcal{T}_h(s_\varkappa)\}$, $h, \varkappa \to 0+$, which are listed below.

We will suppose that, for any $h, \varkappa > 0$ fixed, the system $\{\mathcal{T}_h(s_\varkappa)\}$, $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ consists of *topologically equivalent triangulations*, meaning that

(T1) the triangulation $\mathcal{T}_h(s_\varkappa)$ has the same number of nodes and the nodes still have the same neighbors for any $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$;

(T2) the positions of the nodes of $\mathcal{T}_h(s_\varkappa)$ depend solely and continuously on variations of the design nodes $\{A_{i-1/2}\}_{i=1}^n$.

For $h, \varkappa \to 0+$ we suppose that

(T3) the family $\{\mathcal{T}_h(s_\varkappa)\}$ is *uniformly regular* with respect to $h, \varkappa$ and $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$: there is $\theta_0 > 0$ such that $\theta(h, s_\varkappa) \ge \theta_0$, $\forall h, \varkappa > 0$, $\forall s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, where $\theta(h, s_\varkappa)$ is the minimal interior angle of all triangles from $\mathcal{T}_h(s_\varkappa)$.

Finally, due to the mixed boundary conditions, we suppose that

(T4) the family $\{\mathcal{T}_h(s_\varkappa)\}$ is *consistent* with the decomposition of $\partial\Omega_h(s_\varkappa)$ into Γ_{out} and $\partial\Omega_h(s_\varkappa) \setminus \Gamma_{out}$.

Let us note that (T3)–(T4) imply the assumptions (A1)–(A2) from the previous section.

One can easily show that ($\mathbb{P}_{\varkappa h}$) leads to the following nonlinear programming problem:

$$\begin{cases} \min_{(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))\in \boldsymbol{U}\times\mathbb{R}^m} \mathcal{J}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) \\ \text{subject to} \\ \quad \boldsymbol{R}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})) = \boldsymbol{0}, \end{cases} \tag{$\mathbb{P}_n$}$$

where $\mathcal{J}$, $\boldsymbol{R}$, $\boldsymbol{q}(\boldsymbol{\alpha})$ is the algebraic representation of J, $(\mathcal{P}_h(s_\varkappa))$, and $(\boldsymbol{v}_h, p_h)$, respectively.

Remark 2 From (T1) it follows that $m := m_1 + m_2$, where $m_1 := \dim W_{0h}$ and $m_2 := \dim L_h$, does not depend on $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ or equivalently on $\boldsymbol{\alpha} \in \boldsymbol{U}$. The components of the residual vector $\boldsymbol{R}$ are given by

$$\begin{aligned} R_k(\boldsymbol{\alpha},\boldsymbol{q}) &:= 2\mu_0\big(\mathbb{D}(\boldsymbol{v}_h),\mathbb{D}(\boldsymbol{\varphi}_h^k)\big)_{\Omega_h(s_\varkappa)} + \rho\left(v_{hj}\frac{\partial v_{hi}}{\partial x_j},\varphi_{hi}^k\right)_{\Omega_h(s_\varkappa)} \\ &\quad + \frac{\rho}{2}((\operatorname{div}\boldsymbol{v}_h)(\boldsymbol{v}_h-\boldsymbol{v}_0),\boldsymbol{\varphi}_h^k)_{\Omega_h(s_\varkappa)} + (|\operatorname{div}\boldsymbol{v}_h|\operatorname{div}\boldsymbol{v}_h,\operatorname{div}\boldsymbol{\varphi}_h^k)_{\Omega_h(s_\varkappa)} \\ &\quad + \langle A_{r_h s_\varkappa}(\boldsymbol{v}_h),\boldsymbol{\varphi}_h^k\rangle_{\Omega_h(s_\varkappa)} + \sigma\int_{\Gamma_{out}}|v_{h2}|v_{h2}\varphi_{h2}^k - (p_h,\operatorname{div}\boldsymbol{\varphi}_h^k)_{\Omega_h(s_\varkappa)}, \\ &\quad k = 1,\dots,m_1, \\ R_{m_1+k}(\boldsymbol{\alpha},\boldsymbol{q}) &:= (\psi_h^k,\operatorname{div}\boldsymbol{v}_h)_{\Omega_h(s_\varkappa)}, \quad k=1,\dots,m_2, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{v}_h &:= \boldsymbol{v}_0 + \sum_{k=1}^{m_1} q_k\boldsymbol{\varphi}_h^k, \quad \boldsymbol{\varphi}_h^k := \boldsymbol{\varphi}_h^k(\boldsymbol{\alpha}), \\ p_h &:= \sum_{k=1}^{m_2} q_{m_1+k}\psi_h^k, \quad \psi_h^k := \psi_h^k(\boldsymbol{\alpha}) \end{aligned} \tag{31}$$

and $\{\boldsymbol{\varphi}_h^k(\boldsymbol{\alpha})\}$, $\{\psi_h^k(\boldsymbol{\alpha})\}$ is a basis of $W_{0h}(s_\varkappa)$ and $L_h(s_\varkappa)$, respectively. The cost function $\mathcal{J}:\mathbb{R}^m\to\mathbb{R}$ does not depend explicitly on $\boldsymbol{\alpha}\in\boldsymbol{U}$ since J does not.

We recall the a priori estimates:

$$\begin{aligned} &\|\nabla\boldsymbol{v}_h\|_{2,\Omega_h(s_\varkappa)}^2 + \|M_{r_h s_\varkappa}|\mathbb{D}(\boldsymbol{v}_h)|\|_{3,\Omega_h(s_\varkappa)}^3 + \|\operatorname{div}\boldsymbol{v}_h\|_{3,\Omega_h(s_\varkappa)}^3 \\ &\quad + \|v_{h2}\|_{3,\Gamma_{out}}^3 + \|p_h\|_{\frac{3}{2},\Omega_h(s_\varkappa)}^{\frac{3}{2}} \le C_E, \end{aligned}$$

where $C_E > 0$ is independent of $h > 0$ and $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$.

The following continuity property of the mapping $\boldsymbol{\alpha}\mapsto\boldsymbol{q}(\boldsymbol{\alpha})$, $\boldsymbol{\alpha}\in\boldsymbol{U}$ is a direct consequence of (T1) and (T2) (for the proof see [24]).

Lemma 9 *Let* $\boldsymbol{\alpha}_N \to \boldsymbol{\alpha}$, $N \to \infty$, *where* $\boldsymbol{\alpha}_N, \boldsymbol{\alpha} \in \boldsymbol{U}$, *and let* $\boldsymbol{q}(\boldsymbol{\alpha}_N)$ *satisfy* $\boldsymbol{R}(\boldsymbol{\alpha}_N, \boldsymbol{q}(\boldsymbol{\alpha}_N)) = \mathbf{0}$. *Then there is a* $\boldsymbol{q}(\boldsymbol{\alpha}) \in \mathbb{R}^m$ *and a subsequence (denoted by the same symbol) such that*

$$\boldsymbol{q}(\boldsymbol{\alpha}_N) \to \boldsymbol{q}(\boldsymbol{\alpha}), \quad N \to \infty \tag{32}$$

and $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) = \mathbf{0}$.

Since $\boldsymbol{U}$ is compact, we immediately obtain the existence of a discrete optimal shape.

Theorem 10 *Problem* $(\mathbb{P}_n)$ *(and equivalently* $(\mathbb{P}_{\varkappa h})$*) has a solution.*

4.4 Convergence Analysis

The key role in our analysis plays the following counterpart of Theorem 3 (recall that the symbol $\tilde{}$ stands for the zero extension of functions).

Lemma 11 *Let* $(s_\varkappa, \boldsymbol{v}_h(s_\varkappa), p_h(s_\varkappa)) \in \mathcal{G}_{\varkappa h}$, $h, \varkappa \to 0+$, $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, *and* $\alpha \in \mathcal{U}_{ad}$ *satisfy*

$$s_\varkappa \rightrightarrows \alpha \quad \text{in } [0, L],\ \varkappa \to 0+ .$$

Then there exists $\widehat{\boldsymbol{v}} \in (W^{1,2}(\widehat{\Omega}))^2$, $\widehat{p} \in L^{\frac{3}{2}}(\widehat{\Omega})$ *and appropriate subsequences such that*

$$\begin{aligned} \tilde{\boldsymbol{v}}_h(s_\varkappa) &\rightharpoonup \widehat{\boldsymbol{v}} && \text{in } \left(W^{1,2}(\widehat{\Omega})\right)^2, \\ \tilde{M}_{r_h s_\varkappa}\mathbb{D}(\tilde{\boldsymbol{v}}_h(s_\varkappa)) &\rightharpoonup \tilde{M}_\alpha \mathbb{D}(\widehat{\boldsymbol{v}}) && \text{in } \left(L^3(\widehat{\Omega})\right)^{2\times 2}, \\ \tilde{p}_h(s_\varkappa) &\rightharpoonup \widehat{p} && \text{in } L^{\frac{3}{2}}(\widehat{\Omega}),\ h, \varkappa \to 0+ . \end{aligned} \tag{33}$$

In addition, denoting $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)}$ *and* $p(\alpha) := \widehat{p}_{|\Omega(\alpha)}$, *then* $\boldsymbol{v}(\alpha) \in \widehat{W}_{\boldsymbol{v}_0}(\alpha)$ *and* $(\boldsymbol{v}(\alpha), p(\alpha))$ *solves* $(\widehat{\mathcal{P}}(\alpha))$.

Remark 3 Since $M_\alpha = 0$ on $\partial\Omega(\alpha) \setminus \Gamma_D$, $\tilde{M}_\alpha$ is continuous in $\overline{\widehat{\Omega}}$. The same holds for the function $l_{m,\alpha}$.

Proof of Lemma 11 We will proceed in the same way as in the proof of Theorem 15 in [3], with several minor changes.

From (14) we know that the sequence $\{\|\boldsymbol{v}_h\|_{r_h s_\varkappa}, \|p_h\|_{\frac{3}{2},\Omega_h(s_\varkappa)}\}$ is bounded and that (33) holds for its appropriate subsequence. Since $r_h s_\varkappa \rightrightarrows \alpha$ in $[0, L]$ as h, $\varkappa \to 0+$, we easily get that $\boldsymbol{v}(\alpha) := \widehat{\boldsymbol{v}}_{|\Omega(\alpha)} \in \widehat{W}_{\boldsymbol{v}_0}(\alpha)$. In addition, $\widehat{p}$ and $\widehat{\boldsymbol{v}}$ vanish in $\widehat{\Omega} \setminus \overline{\Omega}(\alpha)$. From the density property of the system $\{L_h\}$ it follows that $\operatorname{div} \widehat{\boldsymbol{v}} = 0$ a.e. in $\widehat{\Omega}$.

We will focus on the limit passage in $(\mathcal{P}_h(r_h s_\varkappa))$. Let $\boldsymbol{\varphi} \in \mathcal{V}_0(\alpha)$ be given and $\boldsymbol{\varphi}_h$ be the piecewise linear Lagrange interpolant of $\tilde{\boldsymbol{\varphi}}_{|\Omega_h(s_\varkappa)}$ on the triangulation $\mathcal{T}_h(s_\varkappa)$ of $\overline{\Omega}_h(s_\varkappa)$. Since $\operatorname{dist}(\operatorname{supp} \tilde{\boldsymbol{\varphi}}, \Gamma(r_h s_\varkappa)) > 0$ for $h, \varkappa > 0$ small enough, the graph of

$r_h s_\varkappa$ has an empty intersection with supp $\tilde{\boldsymbol{\varphi}}$, which means that $\boldsymbol{\varphi}_h \in W_{0h}(r_h s_\varkappa)$ and it can be used as a test function in $(\mathcal{P}_h(r_h s_\varkappa))$. In addition,

$$\tilde{\boldsymbol{\varphi}}_h \to \tilde{\boldsymbol{\varphi}} \quad \text{in } W^{1,\infty}(\widehat{\Omega})^2,\ h \to 0+, \tag{34}$$

as follows from the well-known approximation results and the uniform regularity assumption $(T3)$ on $\{\mathcal{T}_h(s_\varkappa)\}$. Now we can pass to the limit in the standard terms in (13):

$$\begin{gathered}
(\mathbb{D}(\tilde{\boldsymbol{v}}_h), \mathbb{D}(\tilde{\boldsymbol{\varphi}}_h))_{\widehat{\Omega}} \to (\mathbb{D}(\widehat{\boldsymbol{v}}), \mathbb{D}(\tilde{\boldsymbol{\varphi}}))_{\widehat{\Omega}}, \\
\int_{\Gamma_{out}} |\tilde{v}_{h2}|\tilde{v}_{h2}\tilde{\varphi}_{h2} \to \int_{\Gamma_{out}} |\widehat{v}_2|\widehat{v}_2\tilde{\varphi}_2, \\
\left(\tilde{v}_{hj}\frac{\partial \tilde{v}_{hi}}{\partial x_j}, \tilde{\varphi}_{hi}\right)_{\widehat{\Omega}} \to \left(\widehat{v}_j\frac{\partial \widehat{v}_i}{\partial x_j}, \tilde{\varphi}_i\right)_{\widehat{\Omega}}, \\
(\tilde{p}_h, \operatorname{div}\tilde{\boldsymbol{\varphi}}_h)_{\widehat{\Omega}} \to (\widehat{p}, \operatorname{div}\tilde{\boldsymbol{\varphi}})_{\widehat{\Omega}}, \quad h, \varkappa \to 0+,
\end{gathered} \tag{35}$$

as follows from (33) and (34).

Finally, in order to show that

$$(|\operatorname{div}\tilde{\boldsymbol{v}}_h|\operatorname{div}\tilde{\boldsymbol{v}}_h, \operatorname{div}\tilde{\boldsymbol{\varphi}}_h) + \langle \tilde{A}_{r_h s_\varkappa}(\tilde{\boldsymbol{v}}_h), \tilde{\boldsymbol{\varphi}}_h\rangle \to \langle \tilde{A}_\alpha(\tilde{\boldsymbol{v}}(\alpha)), \tilde{\boldsymbol{\varphi}}\rangle, \tag{36}$$

we use the Vitali theorem. To prove pointwise convergence of $\mathbb{D}(\tilde{\boldsymbol{v}}_h)$ to $\mathbb{D}(\widehat{\boldsymbol{v}})$ we proceed as in the proof of Theorem 15 in [3]. Let $\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega(\alpha);\ \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}$, $\varepsilon > 0$, and $\xi := \xi_\varepsilon \in C_0^\infty(\overline{\Omega}(\alpha))$ such that $\xi \geq 0$ in $\Omega(\alpha)$ and $\xi \equiv 1$ in Ω_ε. We construct a test function $\boldsymbol{\varphi} := \xi(\tilde{\boldsymbol{v}}_{h_1} - \tilde{\boldsymbol{v}}_{h_2} - \boldsymbol{\psi})$, where $h_1, h_2 > 0$ and $\boldsymbol{\psi} \in W_0^{1,3}(\widehat{\Omega})^2$ satisfies

$$\operatorname{div}\boldsymbol{\psi} = \operatorname{div}(\tilde{\boldsymbol{v}}_{h_1} - \tilde{\boldsymbol{v}}_{h_2}) \quad \text{a.e. in } \widehat{\Omega}, \tag{37a}$$

$$\|\boldsymbol{\psi}\|_{1,3,\widehat{\Omega}} \leq C_{div}\|\tilde{\boldsymbol{v}}_{h_1} - \tilde{\boldsymbol{v}}_{h_2}\|_{3,\widehat{\Omega}}, \tag{37b}$$

where $C_{div} > 0$ is independent of h_1 and h_2 (see e.g. [7] for solvability of the divergence equation). Given $\delta > 0$, (37) yields:

$$\|\tilde{\boldsymbol{v}}_{h_1} - \tilde{\boldsymbol{v}}_{h_2}\|_{3,\widehat{\Omega}} \leq \delta, \qquad \|\boldsymbol{\psi}\|_{1,3,\widehat{\Omega}} \leq \delta, \qquad \|\boldsymbol{\varphi}\|_{3,\operatorname{supp}\xi} \leq \delta \tag{38}$$

provided that h_1 and h_2 are sufficiently small. Instead of inserting $\boldsymbol{\varphi}$ directly into $(\mathcal{P}_{h_1}(r_h s_\varkappa))$ and $(\mathcal{P}_{h_2}(r_h s_\varkappa))$, we use the Lagrange interpolants $\boldsymbol{\varphi}_{h_1}$, $\boldsymbol{\varphi}_{h_2}$, respectively. We realize that if h_i, $i = 1, 2$, is small enough, then

$$\|\tilde{\boldsymbol{\varphi}}_{h_i} - \boldsymbol{\varphi}\|_{1,3,\widehat{\Omega}} \leq \delta. \tag{39}$$

We use (13), (38) and (39) to deduce that

$$2\mu_0(\mathbb{D}(\boldsymbol{v}_{h_i}), \mathbb{D}(\boldsymbol{\varphi}_{h_i}))_{\Omega_{h_i}(s_{\varkappa_i})} + \langle A_{r_{h_i} s_{\varkappa_i}}(\boldsymbol{v}_{h_i}), \boldsymbol{\varphi}_{h_i}\rangle_{\Omega_{h_i}(s_{\varkappa_i})} = O(1), \quad i = 1, 2, \tag{40}$$

where $O(1)$ denotes an expression which vanishes as $\delta \to 0$. From the definition of $\boldsymbol{\varphi}$, (38) and (39) we obtain:

$$(\mathbb{D}(\boldsymbol{v}_{h_i}), \mathbb{D}(\boldsymbol{\varphi}_{h_i}))_{\Omega_{h_i}(s_{\varkappa_i})} = (\mathbb{D}(\boldsymbol{v}_{h_i}), \xi \mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}))_{\operatorname{supp} \xi} + O(1), \tag{41a}$$

$$\begin{aligned} &\langle A_{s_{\varkappa_i}}(\boldsymbol{v}_{h_i}), \mathbb{D}(\boldsymbol{\varphi}_{h_i})\rangle_{\Omega_{h_i}(s_{\varkappa_i})} \\ &\quad = (M^3_{r_{h_1} s_{\varkappa_1}} |\mathbb{D}(\boldsymbol{v}_{h_i})| \mathbb{D}(\boldsymbol{v}_{h_i})), \xi \mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}))_{\operatorname{supp} \xi} + O(1), \end{aligned} \tag{41b}$$

$i = 1, 2$. Altogether, (39)–(41) yield:

$$\begin{aligned} 2\mu_0 \|\mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2})\|^2_{2,\Omega_\varepsilon} &\le 2\mu_0 (\mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}), \xi \mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}))_{\operatorname{supp} \xi} \\ &\quad + 2\rho (\xi M^3_{r_{h_1} s_{\varkappa_1}} (|\mathbb{D}(\boldsymbol{v}_{h_1})| \mathbb{D}(\boldsymbol{v}_{h_1}) - |\mathbb{D}(\boldsymbol{v}_{h_2})| \mathbb{D}(\boldsymbol{v}_{h_2})), \mathbb{D}(\boldsymbol{v}_{h_1} - \boldsymbol{v}_{h_2}))_{\operatorname{supp} \xi} \\ &= 2\mu_0 (\mathbb{D}(\boldsymbol{v}_{h_1}), \mathbb{D}(\boldsymbol{\varphi}_{h_1}))_{\operatorname{supp} \xi} + \langle A_{r_{h_1} s_{\varkappa_1}}(\boldsymbol{v}_{h_1}), \boldsymbol{\varphi}_{h_1}\rangle_{\Omega_{h_1}(s_{\varkappa_1})} \\ &\quad - 2\mu_0 (\mathbb{D}(\boldsymbol{v}_{h_2}), \mathbb{D}(\boldsymbol{\varphi}_{h_2}))_{\operatorname{supp} \xi} - \langle A_{r_{h_2} s_{\varkappa_2}}(\boldsymbol{v}_{h_2}), \boldsymbol{\varphi}_{h_2}\rangle_{\Omega_{h_2}(s_{\varkappa_2})} = O(1). \end{aligned}$$

Consequently

$$\mathbb{D}(\tilde{\boldsymbol{v}}_h) \to \mathbb{D}(\widehat{\boldsymbol{v}}), \quad h, \varkappa \to 0+, \text{ a.e. in } \widehat{\Omega}$$

for an appropriate subsequence. From this, (14) and the Vitali theorem we arrive at (36) (note that $\operatorname{div} \boldsymbol{v}_h = \operatorname{tr} \mathbb{D}(\boldsymbol{v}_h)$). Thus $(\boldsymbol{v}(\alpha), p(\alpha))$ solves $(\widehat{\mathcal{P}}(\alpha))$. □

Remark 4 As in the continuous case, due to the lack of a density result for $W_0(\alpha)$, we are not able to prove that the limit $\boldsymbol{v}(\alpha)$ belongs to $W_{\boldsymbol{v}_0}(\alpha)$. Therefore the augmented state problem $(\widehat{\mathcal{P}}(\alpha))$ and shape optimization problem $(\widehat{\mathbb{P}})$ is considered instead of $(\mathcal{P}(\alpha))$ and $(\mathbb{P})$, respectively.

On the basis of the previous lemma we obtain the following convergence result.

Theorem 12 *Let $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ be small enough so that the solutions of $(\mathcal{P}(\alpha))$ and $(\widehat{\mathcal{P}}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, are unique. Let $\{(s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ be a sequence of optimal pairs of $(\mathbb{P}_{\varkappa h})$, $h, \varkappa \to 0+$. Then there is a subsequence of $\{s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ such that*

$$s^*_\varkappa \rightrightarrows \alpha^* \qquad \text{in } [0, L], \tag{42a}$$

$$\tilde{\boldsymbol{v}}^*_h \rightharpoonup \boldsymbol{v}^* \qquad \text{in } \left(W^{1,2}(\widehat{\Omega})\right)^2, \tag{42b}$$

$$\tilde{M}_{r_h s_\varkappa} \mathbb{D}(\tilde{\boldsymbol{v}}^*_h) \rightharpoonup \tilde{M}_\alpha \mathbb{D}(\boldsymbol{v}^*) \qquad \text{in } \left(L^3(\widehat{\Omega})\right)^{2\times 2}, \tag{42c}$$

$$\tilde{p}^*_h \rightharpoonup p^* \qquad \text{in } L^{\frac{3}{2}}(\widehat{\Omega}), \ h, \varkappa \to 0+, \tag{42d}$$

where $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)})$ is an optimal triple for $(\widehat{\mathbb{P}})$. In addition, any accumulation point of $\{s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h)\}$ in the sense of (42) possesses this property.

Proof Let $\overline{\alpha} \in \mathcal{U}_{ad}$ be arbitrary. Then there exists a sequence $\{\overline{s}_\varkappa\}$, $\overline{s}_\varkappa \in \mathcal{U}^\varkappa_{ad}$, such that $\overline{s}_\varkappa \rightrightarrows \overline{\alpha}$ in $[0, L]$, $\varkappa \to 0+$, as follows from the well-known properties of Bézier functions. From Lemma 11 it follows that

$$\tilde{\boldsymbol{v}}_h(\overline{s}_\varkappa) \rightharpoonup \overline{\boldsymbol{v}} \quad \text{in } \big(W^{1,2}(\widehat{\Omega})\big)^2, \tag{43}$$

$$\tilde{M}_{r_h\overline{s}_\varkappa}\mathbb{D}(\tilde{\boldsymbol{v}}_h(\overline{s}_\varkappa)) \rightharpoonup \tilde{M}_\alpha \mathbb{D}(\overline{\boldsymbol{v}}) \quad \text{in } \big(L^3(\widehat{\Omega})\big)^{2\times 2}, \tag{44}$$

$$\tilde{p}_h(\overline{s}_\varkappa) \rightharpoonup \overline{p} \quad \text{in } L^{\frac{3}{2}}(\widehat{\Omega}),\ h, \varkappa \to 0+, \tag{45}$$

where $(\boldsymbol{v}_h(\overline{s}_\varkappa), p_h(\overline{s}_\varkappa))$ are the solutions of $(\mathcal{P}_h(r_h\overline{s}_\varkappa))$ and $(\boldsymbol{v}(\overline{\alpha}), p(\overline{\alpha})) := (\overline{\boldsymbol{v}}_{|\Omega(\overline{\alpha})}, \overline{p}_{|\Omega(\overline{\alpha})})$ is the unique solution of $(\widehat{\mathcal{P}}(\overline{\alpha}))$. Since J is continuous with respect to convergence in (43) and

$$J(s^*_\varkappa, \boldsymbol{v}^*_h, p^*_h) \le J(\overline{s}_\varkappa, \boldsymbol{v}_h(\overline{s}_\varkappa), p_h(\overline{s}_\varkappa)),$$

we have that

$$J(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)}) \le J(\overline{\alpha}, \boldsymbol{v}(\overline{\alpha}), p(\overline{\alpha})).$$

Here $\overline{\alpha} \in \mathcal{U}_{ad}$ is arbitrary, hence $(\alpha^*, \boldsymbol{v}^*_{|\Omega(\alpha^*)}, p^*_{|\Omega(\alpha^*)})$ is a solution of $(\widehat{\mathbb{P}})$. □

Remark 5 Let us mention that the state solutions must be unique for the complete convergence result. Otherwise the limit solutions are optimal only in a subclass of $\widehat{\mathcal{G}}$ formed by all accumulation points of solutions to $(\mathcal{P}_h(r_h s_\varkappa))$, $h, \varkappa \to 0+$.

4.5 Differentiability of the Discrete Cost Function

To establish existence of discrete optimal solutions and their convergence, we have exploited continuity of the cost function with respect to shape variations. In numerical realization, however, the optimization problems are usually solved using gradient-based methods that search for a local minimum. We will therefore examine smoothness of the discrete cost function so that the subsequent numerical procedure is properly justified.

Lemma 13 *Assume that, in addition to* (T2), *the nodal coordinates of $\mathcal{T}_h(s_\varkappa)$ are continuously differentiable with respect to $\boldsymbol{\alpha}$, and that the finite element spaces W_{0h}, L_h are formed by the isoparametric technique. Then $\boldsymbol{R}$ and $\mathcal{J}$ are continuously differentiable w.r.t. $\boldsymbol{\alpha} \in \boldsymbol{U}$ and $\boldsymbol{q} \in \mathbb{R}^m$.*

Proof Due to Remark 2 we observe that $\boldsymbol{R}$ is formed by a sum of integrals over triangles and edges whose differentiability can be analyzed separately. Consider for instance the integral

$$I_T := \int_T M^3_{r_h s_\varkappa} |\mathbb{D}(\boldsymbol{v}_h)| \mathbb{D}(\boldsymbol{v}_h) : \mathbb{D}(\boldsymbol{\varphi}^k_h),$$

$k \in \{1, \dots, m_1\}$, $T \in \mathcal{T}_h(s_\varkappa)$. Since $\boldsymbol{v}_h$ depends linearly on $\boldsymbol{q}$ (as follows from (31)) and $x \mapsto |x|x$ is continuously differentiable, it holds that $\frac{\partial I_T}{\partial \boldsymbol{q}}$ is continuous.

Let $\widehat{T}$ be a reference triangle and j_T the determinant of the corresponding one-to-one mapping $\boldsymbol{\xi} : \widehat{T} \to T$. Then we have

$$I_T = \int_{\widehat{T}} \widehat{M}^3_{r_h s_\varkappa} |\mathbb{D}(\widehat{\boldsymbol{v}}_h)| \mathbb{D}(\widehat{\boldsymbol{v}}_h) : \mathbb{D}(\widehat{\boldsymbol{\varphi}}^k_h)\, j_T,$$

where $\widehat{\boldsymbol{v}}_h(x) := \boldsymbol{v}_h(\boldsymbol{\xi}(x))$ etc. Since the domain of integration is fixed, we obtain:

$$\frac{\partial I_T}{\partial \boldsymbol{\alpha}} = \int_{\widehat{T}} \frac{\partial}{\partial \boldsymbol{\alpha}} \Big(\widehat{M}^3_{r_h s_\varkappa} |\mathbb{D}(\widehat{\boldsymbol{v}}_h)| \mathbb{D}(\widehat{\boldsymbol{v}}_h) : \mathbb{D}(\widehat{\boldsymbol{\varphi}}^k_h)\, j_T \Big).$$

For the isoparametric FEM it holds that $\boldsymbol{v}_h, \nabla \boldsymbol{v}_h, \boldsymbol{\varphi}^k_h, \nabla \boldsymbol{\varphi}^k_h, j_T$ are continuously differentiable w.r.t. $\boldsymbol{\alpha}$ (see [14, Theorem 3.3 on p. 122] for precise formulas). In [24, Lemma 1.2 on p. 8] we have shown that

$$M^3_\alpha(\boldsymbol{x}) = l^2_m(\alpha(x_1), x_2), \quad \boldsymbol{x} \in \Omega(\alpha),\ \alpha \in \mathcal{U}_{ad}$$

where

$$l_m(\boldsymbol{y}) := \frac{y_1}{2} \Big(0.14 - 0.08 d^2(\boldsymbol{y}) - 0.06 d^4(\boldsymbol{y}) \Big),$$
$$d(\boldsymbol{y}) := \left(1 - \frac{2 \min\{y_2, y_1 - y_2\}}{y_1} \right).$$

One easily verifies that d^2 is continuously differentiable. Hence the same applies also to l_m and $\boldsymbol{\alpha} \mapsto M^3_{r_h s_\varkappa}$. Consequently $\frac{\partial I_T}{\partial \boldsymbol{\alpha}}$ is continuous.

The remaining terms appearing in $\boldsymbol{R}$ and $\mathcal{J}$, respectively, can be treated analogously. □

In the following lemma we establish a sufficient condition for invertibility of the matrix $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$.

Lemma 14 *There exists a constant $C_{reg} > 0$ independent of $\varkappa$ and h such that $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q})$ is nonsingular for all $\boldsymbol{\alpha} \in \boldsymbol{U}$ and $\boldsymbol{q} \in \mathbb{R}^m$ provided that $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}} < C_{reg}$.*

Proof It can be shown that

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}) = \begin{pmatrix} \mathbb{A}(\boldsymbol{\alpha}, \boldsymbol{q}) & \mathbb{B}^\top(\boldsymbol{\alpha}) \\ \mathbb{B}(\boldsymbol{\alpha}) & \mathbb{O} \end{pmatrix},$$

where the components a_{kl}, b_{kl} of the matrices $\mathbb{A} := \mathbb{A}(\boldsymbol{\alpha}, \boldsymbol{q})$ and $\mathbb{B} := \mathbb{B}(\boldsymbol{\alpha})$, respectively, are given by the formulae:

$$a_{kl} = 2\mu_0 (\mathbb{D}(\boldsymbol{\varphi}^l_h), \mathbb{D}(\boldsymbol{\varphi}^k_h))_{\Omega_h(s_\varkappa)} + \rho \left(\varphi^l_{hj} \frac{\partial v_{hi}}{\partial x_j} + v_{hj} \frac{\partial \varphi^l_{hi}}{\partial x_j}, \varphi^k_{hi} \right)_{\Omega_h(s_\varkappa)}$$
$$+ \frac{\rho}{2} ((\operatorname{div} \boldsymbol{\varphi}^l_h)(\boldsymbol{v}_h - \boldsymbol{v}_0) + (\operatorname{div} \boldsymbol{v}_h) \boldsymbol{\varphi}^l_h, \boldsymbol{\varphi}^k_h)_{\Omega_h(s_\varkappa)}$$

$$+2\rho\left(M^3_{r_h s_\varkappa}\frac{\mathbb{D}(\boldsymbol{v}_h)}{|\mathbb{D}(\boldsymbol{v}_h)|}\mathbb{D}(\boldsymbol{v}_h):\mathbb{D}(\boldsymbol{\varphi}^l_h)+M^3_{r_h s_\varkappa}|\mathbb{D}(\boldsymbol{v}_h)|\mathbb{D}(\boldsymbol{\varphi}^l_h),\mathbb{D}(\boldsymbol{\varphi}^k_h)\right)_{\Omega_h(s_\varkappa)}$$

$$+\frac{1}{2}(|\operatorname{div}\boldsymbol{v}_h|\operatorname{div}\boldsymbol{\varphi}^l_h,\operatorname{div}\boldsymbol{\varphi}^k_h)_{\Omega_h(s_\varkappa)}$$

$$+2\sigma\int_{\Gamma_{out}}|v_{h2}|\varphi^l_{h2}\varphi^k_{h2},\quad k,l=1,\dots,m_1,$$

$$b_{kl}=-(\psi^l_h,\operatorname{div}\boldsymbol{\varphi}^k_h)_{\Omega_h(s_\varkappa)},\quad k=1,\dots,m_1,\ l=1,\dots,m_2,$$

$$\boldsymbol{v}_h=\boldsymbol{v}_0+\sum_{i=1}^{m_1}q_i\boldsymbol{\varphi}^i_h.$$

For every $\tilde{\boldsymbol{q}}\in\mathbb{R}^{m_1}$ we have that

$$\mathbb{A}\tilde{\boldsymbol{q}}\cdot\tilde{\boldsymbol{q}}\ge 2\mu_0\|\mathbb{D}(\boldsymbol{w}_h)\|^2_{2,\Omega_h(s_\varkappa)}+\rho\left(w_{hj}\frac{\partial v_{hi}}{\partial x_j}+v_{hj}\frac{\partial w_{hi}}{\partial x_j},w_i\right)_{\Omega_h(s_\varkappa)}$$
$$+\frac{\rho}{2}((\operatorname{div}\boldsymbol{w}_h)(\boldsymbol{v}_h-\boldsymbol{v}_0)+(\operatorname{div}\boldsymbol{v}_h)\boldsymbol{w}_h,\boldsymbol{w})_{\Omega_h(s_\varkappa)},$$

where $\boldsymbol{w}_h:=\sum_{i=1}^{m_1}\tilde{q}_i\boldsymbol{\varphi}^i_h$. Using Hölder's, Friedrichs', Korn's inequality and imbedding, we can estimate

$$\rho\left(w_j\frac{\partial v_{hi}}{\partial x_j}+v_{hj}\frac{\partial w_i}{\partial x_j},w_i\right)_{\Omega_h(s_\varkappa)}+\frac{\rho}{2}((\operatorname{div}\boldsymbol{w})(\boldsymbol{v}_h-\boldsymbol{v}_0)+(\operatorname{div}\boldsymbol{v}_h)\boldsymbol{w},\boldsymbol{w})_{\Omega_h(s_\varkappa)}$$
$$\le C(\|\nabla\boldsymbol{v}_h\|_{2,\Omega_h(s_\varkappa)}+\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}})\|\mathbb{D}(\boldsymbol{w})\|^2_{2,\Omega_h(s_\varkappa)},$$

where $C>0$ is independent of $\varkappa$ and h. From (14) and the fact that $C_E\searrow 0$ as $\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}\searrow 0$ (see Remark 2 in [3]) we infer that $\mathbb{A}$ is positive definite if $\|\nabla\boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ is small enough.

Due to (10), the equation $\mathbb{B}^\top\boldsymbol{y}=\boldsymbol{0}$ has only the trivial solution, hence $\mathbb{B}$ has full rank and $\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}$ is nonsingular. □

We are now going to express the gradient of the cost function. Although in our particular situation $\mathcal{J}$ does not explicitly depend on $\boldsymbol{\alpha}$, we will consider the general case and define

$$\mathfrak{J}(\boldsymbol{\alpha}):=\mathcal{J}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha})).$$

The control-to-state mapping is in general multi-valued, thus differentiation of $\mathfrak{J}$ has sense only if restricted to a particular branch of solutions. Since $\boldsymbol{R}$ is smooth, differentiability of the mapping $\boldsymbol{\alpha}\mapsto\boldsymbol{q}(\boldsymbol{\alpha})$ of solutions to the equation

$$\boldsymbol{R}(\boldsymbol{\alpha},\boldsymbol{q}(\boldsymbol{\alpha}))=\boldsymbol{0}\tag{46}$$

follows from the implicit function theorem, provided that the matrix $\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{q}}$ is nonsingular.

In order to avoid computation of $\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\boldsymbol{\alpha}}$, the adjoint state $\boldsymbol{p} := \boldsymbol{p}(\boldsymbol{\alpha})$ is introduced through the equation

$$\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))\right)^{\top} \boldsymbol{p} = \frac{\partial \mathcal{J}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})). \tag{47}$$

Then the gradient of $\mathfrak{J}$ reads:

$$\frac{\partial \mathfrak{J}}{\partial \alpha_k}(\boldsymbol{\alpha}) = \frac{\partial \mathcal{J}}{\partial \alpha_k}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) - \boldsymbol{p} \cdot \left(\frac{\partial \boldsymbol{R}}{\partial \alpha_k}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))\right), \quad k = 1, \ldots, n. \tag{48}$$

We summarize the above ideas in the following statement.

Theorem 15 *Suppose that the hypothesis of Lemma 13 holds. Let* $(\bar{\boldsymbol{\alpha}}, \boldsymbol{q}(\bar{\boldsymbol{\alpha}})) \in \boldsymbol{U} \times \mathbb{R}^m$ *be a solution to* (46) *and the matrix* $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$ *be nonsingular at* $(\bar{\boldsymbol{\alpha}}, \boldsymbol{q}(\bar{\boldsymbol{\alpha}}))$. *Then there is a neighbourhood of this point in which the mapping*

$$\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha})$$

defines a continuously differentiable branch of solutions to the system (46). *Moreover, the cost function* $\mathfrak{J}$ *is continuously differentiable on this branch and its gradient is expressed through* (48) *and* (47).

If $\|\nabla \boldsymbol{v}_0\|_{3,\widehat{\Omega}}$ is small enough so that $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$ is invertible and solutions to the state problem are unique, then the control-to-state mapping is single-valued and formulas (48) and (47) determine the gradient of $\mathfrak{J}$ in the usual sense. Note that the condition guaranteeing uniqueness of the solutions to the discrete state problems is independent of $\boldsymbol{\alpha}$ and h.

Although invertibility of $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$ is guaranteed only for "small data", the result of this subsection is not restricted only to that case. Roughly speaking, convergence of the Newton method for numerical realization of the state problem goes hand in hand with the differentiability of the state problem.

5 Numerical Realization

In this section we present a method of numerical realization of the shape optimization problem. We would like to emphasize that our implementation is not restricted to this particular problem but it can be applied to a wide range of shape optimization and optimal control problems that can be formulated like $(\mathbb{P}_n)$.

5.1 State Problem

We will start with the numerical solution of the discrete state problem $(\mathcal{P}_h(\alpha))$ (see Sect. 3 for definition of $(\mathcal{P}_h(\alpha))$, $\alpha \in \widetilde{\mathcal{U}}_{ad}$) whose algebraic form is (46).

We assume that $\boldsymbol{\alpha}$ is given and consider (46) as a system of m nonlinear algebraic equations for the vector of unknowns $\boldsymbol{q} := \boldsymbol{q}(\boldsymbol{\alpha}) \in \mathbb{R}^m$ which will be solved by the Newton-Raphson method:

$$\text{Given } \boldsymbol{q}_k \in \mathbb{R}^m, \text{ define } \boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\right)^{-1} \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k). \tag{49}$$

Let us recall that the sequence $\{\boldsymbol{q}_k\}$, $k = 0, 1, \ldots$, converges provided that the initial guess $\boldsymbol{q}_0$ is close enough to the solution of (46). Thus we have to supply a good approximation of $\boldsymbol{q}$ at the beginning. This is usually done by using some other algorithm (e.g. the fixed point iterations) before the Newton-Raphson method is used. The main advantage of this method is that if $\boldsymbol{R}$ is twice continuously differentiable and the inverse of $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$ exists, then convergence of (49) is at least quadratic. Instead of computing the inverse matrix $(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k))^{-1}$, we solve for every k the linear system

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}_k)\Delta \boldsymbol{q}_k = \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k) \tag{50}$$

for the unknown $\Delta \boldsymbol{q}_k \in \mathbb{R}^m$ and put $\boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \Delta \boldsymbol{q}_k$. For the solution of (50) we used the package SuperLU, which performs an LU decomposition with partial pivoting (see [6] for detailed description).

In our program we do not implement the analytical form of $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$. Instead, we only specify how to assemble the residual vector $\boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)$. The matrix of the linearized system (50) is obtained automatically by using tools of the automatic differentiation. The residual vector is decomposed into the sum of area and boundary integrals, which are further calculated element by element or edge by edge using suitable numerical quadratures.

The algorithm for the numerical solution of the state problem now reads as follows:

Algorithm 1 Solution of the discrete state problem

1. Choose the tolerance $r_{max} > 0$ for the residuum and the max. number of the Newton iterations $k_{max} \in \mathbb{N}$.
2. Choose $\boldsymbol{q}_0 \in \mathbb{R}^m$.
3. For $k = 0, \ldots, k_{max} - 1$:
   - Compute $\boldsymbol{b}_k := \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)$ and set $\mathbb{C}_k := \frac{\partial \boldsymbol{R}(\boldsymbol{\alpha}, \boldsymbol{q}_k)}{\partial \boldsymbol{q}}$.
   - Solve $\mathbb{C}_k \Delta \boldsymbol{q}_k = \boldsymbol{b}_k$ and set $\boldsymbol{q}_{k+1} := \boldsymbol{q}_k - \Delta \boldsymbol{q}_k$, $r_k := |\boldsymbol{b}_k|$.
   - If $r_k < r_{max}$ then go to 4.
4. If $r_k < r_{max}$ then set $\boldsymbol{q} := \boldsymbol{q}_k$, otherwise report error.

5.2 Shape Optimization Problem

We solve numerically the mathematical programming problem $(\mathbb{P}_n)$. Since the function to be minimized is smooth, we will use a gradient based minimization algorithm supplied by the gradient information derived in Sect. 4.5. Most of the tedious work will be done by means of the automatic differentiation (see [10, 14]).

For the numerical minimization itself we used the following packages:

- KNITRO—a robust tool for many types of smooth optimization problems (see [4, 5, 27]).
- NAG C library—in particular the function e04wdc which is intended for smooth optimization and uses the sequential quadratic programming [8].

Both packages provide a Fortran/C interface that allows to supply arbitrary routines for the cost function and gradient evaluation. A comparison of both packages and the obtained results can be found in Sect. 5.3 where results of several model examples are presented.

The evaluation of the cost function $\mathfrak{J}$ is done by the following chain:

$$\boldsymbol{\alpha} \mapsto \boldsymbol{q}(\boldsymbol{\alpha}) \mapsto \mathfrak{J}(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})).$$

Since the first mapping is in general multi-valued, we restrict ourselves to a single branch corresponding to the initial state $(\boldsymbol{\alpha}_0, \boldsymbol{q}(\boldsymbol{\alpha}_0))$, so that $\mathfrak{J}(\boldsymbol{\alpha})$ and $\nabla\mathfrak{J}(\boldsymbol{\alpha})$ are well-defined, at least locally. We assume also that the solutions $\boldsymbol{q}(\boldsymbol{\alpha})$ obtained by the iterative process lie on the same branch. Instead of the global minimum of $\mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$ we search for a local minimizer of $\mathfrak{J}(\boldsymbol{\alpha})$.

$$\frac{\partial \mathfrak{J}}{\partial \alpha_k}(\boldsymbol{\alpha}) = \frac{\partial \mathcal{J}}{\partial \alpha_k}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) - \boldsymbol{p} \cdot \left(\frac{\partial \boldsymbol{R}}{\partial \alpha_k}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) \right), \quad k = 1, \dots, n, \tag{51}$$

$$\left(\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})) \right)^{\mathrm{T}} \boldsymbol{p} = \frac{\partial \mathcal{J}}{\partial \boldsymbol{q}}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha})). \tag{52}$$

The gradient $\nabla\mathfrak{J}(\boldsymbol{\alpha})$ is computed by the method described in Sect. 4.5, see (47)–(48). In particular, we have to solve only one additional linear problem for the adjoint state $\boldsymbol{p}$. Let us also notice that for our particular cost function used in the computations it holds that

$$\frac{\partial \mathcal{J}}{\partial \alpha_k}(\boldsymbol{\alpha}, \boldsymbol{q}) = 0, \quad k = 1, \dots, n.$$

The implementation of (47)–(48) is simple, provided that the partial derivatives $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{\alpha}}$, $\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{q}}$, $\frac{\partial \mathcal{J}}{\partial \boldsymbol{q}}$ are computed in a smart way. Their hand-coding is in most cases elaborate and error-prone, requiring an additional algebraic sensitivity analysis. For this reason we compute them with the aid of the automatic differentiation and the operator overloading feature of C++ (see [25] for details on the implementation).

For the computations of the state problem for different $\boldsymbol{\alpha} \in \boldsymbol{U}$ (or $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, equivalently) we need to construct triangulations of $\Omega_h(s_\varkappa)$ that satisfy (T1)–(T4). We use an approach which exploits the shape of $\Omega_h(s_\varkappa)$: We choose a suitable $\bar{s}_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ and create a triangulation $\mathcal{T}_h(\bar{s}_\varkappa)$. Then, given $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$, we define the triangulation of $\Omega_h(s_\varkappa)$ from $\mathcal{T}_h(\bar{s}_\varkappa)$ in such a way, that every node $(x(\bar{s}_\varkappa), y(\bar{s}_\varkappa))$ of $T_h(\bar{s}_\varkappa)$ is shifted in the vertical direction:

$$
\begin{aligned}
x(s_\varkappa) &:= x(\bar{s}_\varkappa),\\
y(s_\varkappa) &:= y(\bar{s}_\varkappa)\frac{s_\varkappa(x(s_\varkappa))}{\bar{s}_\varkappa(x(s_\varkappa))}.
\end{aligned}
\tag{53}
$$

The map $s_\varkappa \mapsto (x(s_\varkappa), y(s_\varkappa))$ is continuously differentiable and due to the definition of $\mathcal{U}_{ad}^{\varkappa}$, the assumptions (T1)–(T4) are satisfied.

The evaluation of $\mathfrak{J}$ and $\nabla\mathfrak{J}$ can be summarized as follows:

Algorithm 2 Evaluation of the discrete cost function and its gradient

1. Given $\boldsymbol{\alpha} \in \boldsymbol{U}$, solve the state problem and obtain $\boldsymbol{q}(\boldsymbol{\alpha})$;
2. Evaluate $\mathfrak{J}(\boldsymbol{\alpha}) := \mathcal{J}(\boldsymbol{\alpha}, \boldsymbol{q}(\boldsymbol{\alpha}))$;
3. Solve the adjoint equation (47) to obtain $\boldsymbol{p}(\boldsymbol{\alpha})$;
4. Evaluate $\nabla\mathfrak{J}$ using (48).

5.3 Model Examples

We end up with several numerical examples. Let us note that the parameters used in the following computations do not correspond to any real industrial application.

5.3.1 State Problem

Traditionally the paper machine header has been designed with a linearly tapered header. We use this header design to test the state problem solver. The computational domain is 9.5 m long and 1 m wide (see Fig. 4). This domain is partitioned by using a uniform triangular mesh into 8000 triangles. The velocity is approximated by continuous piecewise quadratic functions, while the pressure by continuous linear functions. For this type of approximation (known as the Taylor-Hood element) it is known that the Babuška-Brezzi condition (11) holds true [2, 26]. The resulting number of degrees of freedom of $\boldsymbol{q}$ is 28663.

The pulp is modelled as an incompressible fluid with the laminar viscosity $\mu_0 = 10^{-3}$ Pa s and the density $\rho = 10^3$ kg/m^3. The inlet and outlet velocity profiles are chosen as follows:

$$
\begin{aligned}
\boldsymbol{v}_{D|\{0\}\times(0,H_1)} &= \left(4\left(1 - \left(\frac{2x_2}{H_1} - 1\right)^8\right), 0\right) \text{ m/s},\\
\boldsymbol{v}_{D|\{L\}\times(0,H_2)} &= \left(1 - \left(\frac{2x_2}{H_2} - 1\right)^8, 0\right) \text{ m/s}.
\end{aligned}
$$

If we define the kinematic viscosity $\nu := \mu_0/\rho$, then ν^{-1} gives the Reynolds number 10^6 in case of the standard Navier-Stokes equations. This usually requires the use of a stabilized numerical scheme. However our turbulence model produces enough

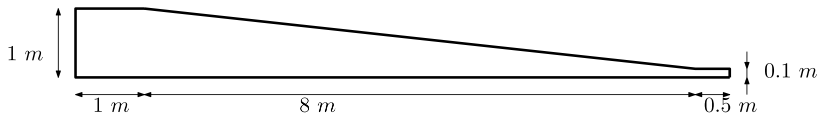
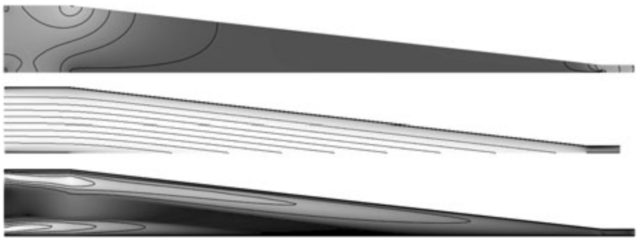


Fig. 4 Dimensions of the computational header

Fig. 5 Solution of the state problem (for $\sigma = 10^3$ and linearly tapered header): pressure p, velocity magnitude $|\boldsymbol{v}|$ and streamlines, dynamic viscosity μ



turbulent viscosity so that the state problem can be solved without any additional stabilization. As the initial approximation we chose a solution of a similar problem with a higher viscosity. The stopping criterion for the residuum is $r_{max} = 10^{-9}$. The nonlinear loop needed from 2 to 10 iterations, each of which took about 7.1 s on AMD Opteron 246 with 2 GB RAM. The direct solver SuperLU was efficient enough for this problem size, requiring only 20% of one Newton iteration time, while the rest was spent on the residual assembly. Solution of the state problem in the linearly tapered header is depicted in Fig. 5.

5.3.2 Shape Optimization Problem

The traditional linearly tapered header serves as a starting point for the shape optimization. The number of design parameters is set to $n = 20$. Due to the well-known properties of the Bézier functions the derivative of $s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}$ can be estimated as follows:

$$|s'_\varkappa| \leq \frac{\alpha_{max} - \alpha_{min}}{\varkappa} = \frac{\alpha_{max} - \alpha_{min}}{L_2} n \quad \forall s_\varkappa \in \mathcal{U}_{ad}^{\varkappa}.$$

Therefore for reasonably small n the constraint γ on the derivative will be removed from the definition of $\mathcal{U}_{ad}^{\varkappa}$. We then obtain a nonlinear optimization problem with simple bounds only. We set $\alpha_{max} = H_1$ and $\alpha_{min} = H_2$. The boundary segment $\tilde{\Gamma} \subset \Gamma_{out}$ used in the definition of the cost function is $\tilde{\Gamma} = (1.5, 8.5)$. The outflow suction coefficient $\sigma = 10^3$ in what follows.

We ran the computation repeatedly with two different target velocity profiles v_{opt}. In the first case we used a constant target velocity $v_{opt} = -0.443$ m/s. We tested two optimization packages in order to compare the obtained results and the performance. All parameters were left default, only in case of KNITRO solver we tried several values of the initial trust region parameter δ. Both packages, KNITRO and NAG, converged apparently to the same shape. However NAG turned out to be superior in terms of the required cost function and gradient evaluations. KNITRO solver ended in all cases after approximately 100 iterations, achieving the KKT optimality conditions with the error smaller than 10^{-3}. On the other hand, NAG C library needed 73 itera-

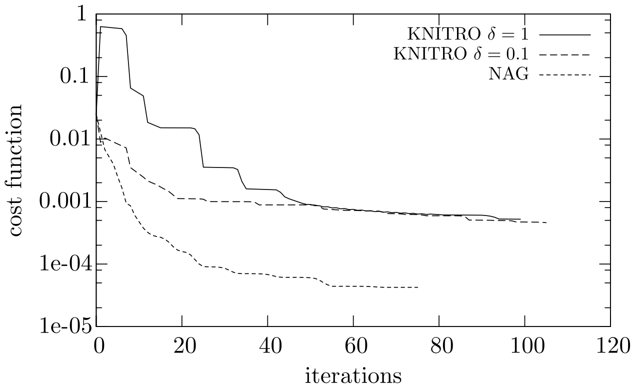


Fig. 6 Convergence history of the used optimization algorithms

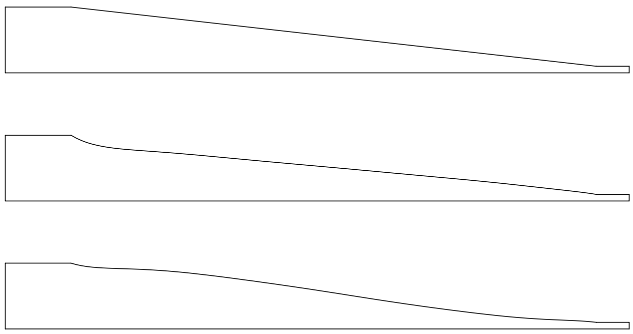


Fig. 7 Shapes of the header (from the top): Initial, optimal for the constant and non-constant target velocity

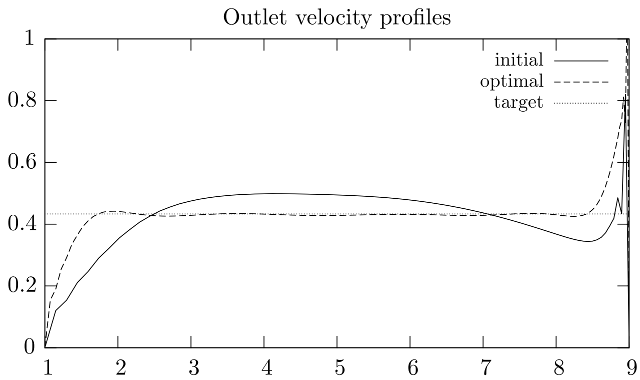


Fig. 8 Initial and optimal outlet velocity (constant target velocity)

tions to get the optimality error smaller than 2×10^{-6}. The value of the cost function decreased from 2.5×10^{-2} to 4.2×10^{-5} in case of NAG. In Fig. 6 the convergence history of all algorithms is shown.

In the second case a function

$$v_{opt} = -0.65 \sin\left(\frac{x - L_1}{L_2}\pi\right) \text{ m/s}$$

was chosen as the target outlet velocity. Here the computation ended after 44 iterations using NAG and the cost function value decreased from 8.7×10^{-2} to 1.1×10^{-3}.

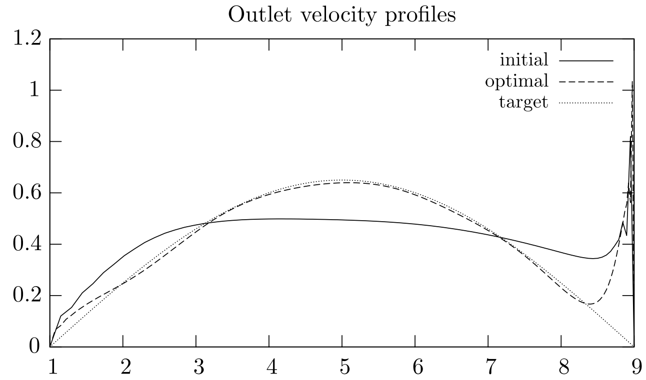


Fig. 9 Initial and optimal outlet velocity (non-constant target velocity)

Fig. 10 Solution of the state problem: pressure p, velocity magnitude $|\boldsymbol{v}|$ and streamlines, dynamic viscosity μ, optimal shape for the constant target velocity

Fig. 11 Solution of the state problem: pressure p, velocity magnitude $|\boldsymbol{v}|$ and streamlines, dynamic viscosity μ, optimal shape for the non-constant target velocity

Computed optimal shapes are depicted in Fig. 7. The optimal velocity profiles for the constant and the non-constant target are shown in Fig. 8, and in Fig. 9, respectively, and the corresponding solutions of the state problem in Figs. 10 and 11.

There is no reason to expect that the cost function is convex, therefore the found minima are possibly only local ones. However, all the used algorithms converged to very similar shapes that are close to the one obtained in [14], where a different method was applied. Thus, there is a chance that the final design is close to the global minimum. In any case, for practical purposes it is usually sufficient to find a local minimum which improves the initial state.

One can see from Fig. 7 that the difference between the initial and optimized shapes is not too big. This indicates that the cost function is very sensitive with respect to shape variations. In spite of this fact, the proposed examples reveal that it is possible to control the outflow velocity and consequently the quality of produced paper by appropriate change of the header geometry.

6 Conclusion

The paper consists of 5 sections. After explaining the physical motivation in Sect. 1, we formulate the problem and recall the existence results for the continuous case in Sect. 2. Due to an algebraic turbulence model the weak formulation of the state problem involves the weighted Sobolev spaces.

The main part of the paper is devoted to approximation and numerical realization of the problem formulated beforehand: In Sect. 3 a finite element discretization of the flow problem is studied. The existence of discrete solutions and their convergence to a solution of the continuous problem is proved. Section 4 describes an approximation of shapes, existence of discrete optimal shapes, and their convergence to a solution of the original shape optimization problem. The results of these two sections are obtained using the technique developed in [3], sharing many similarities with [17–19] and [21].

Finally, an algorithm for numerical realization is described in Sect. 5. The proposed method takes the advantage of the automatic differentiation which significantly simplifies and speeds up the computer program. The model examples show that very good results can be obtained and that the mathematical modelling together with numerical analysis can bring a significant contribution to the paper making engineering.

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4.3 Shape optimization and slip boundary conditions

The choice of boundary conditions at the solid-fluid interface is not always straightforward. Under certain circumstances, fluids may move along the surface. Apart of inviscid fluids, where the impermeability condition

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \tag{4.3}$$

is satisfactory at solid boundaries, the slip behaviour is relevant also for viscous fluids e.g. in presence of a hydrophobic [49, 14], nonwetting [5, 7], chemically patterned surfaces [47, 53] or in general a surface with micro/nanosize structure [38, 52, 10]. Then, one has to provide an appropriate constitutive relation between the tangential part of the velocity and of the tangent shear stress. The Navier condition, stating that $\boldsymbol{v}_\tau$ is a linear function of $(\mathbb{S}\boldsymbol{n})_\tau$, where $\mathbb{T} = -p\mathbb{I} + \mathbb{S}$, is a natural choice in many cases. However, in some applications the resistance of the fluid to the tangential force is observed, which implies that the fluid is at rest until the stress reaches certain threshold value. Mathematically it can be expressed by the relations:

$$|(\mathbb{S}\boldsymbol{n})_\tau| \leq g, \quad g\boldsymbol{v}_\tau = -|\boldsymbol{v}_\tau|(\mathbb{S}\boldsymbol{n})_\tau. \tag{4.4}$$

In the following reprints we present results for three types of threshold-slip boundary conditions (see Figure 4.2), for simplicity used with the Stokes system. The first paper [26] deals with the case of given slip bound g and

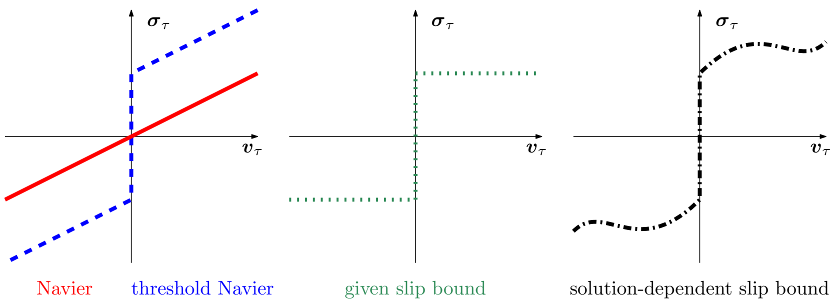

Figure 4.2: Examples of slip laws (4.4).

the shape optimization problem approximated by a problem where the impermeability condition is penalized. The second paper [25] considers slip bound depending on the tangential velocity ($g = g(|\boldsymbol{v}_\tau|)$) and suggests an augmented formulation, where the tangential and the normal stress appear as new unknowns. In the last paper [27] we present a numerical analysis and solution for the case of slip bound linearly depending on the tangent velocity ($g = \sigma_0 + \sigma_1 |\boldsymbol{v}_\tau|$). The shape optimization problem is solved by penalizing the impermeability (4.3) and regularizing the non-smooth boundary condition. The main results include:

- Existence analysis of the Stokes problem with threshold slip boundary conditions. Due to the boundary condition, the problem becomes non-smooth and leads to a variational inequality combined with the incompressibility constraint. We also introduce new augmented formulations which split the tangential and normal stress allowing its separate treatment and approximation.

- Domain dependence of solutions and existence of optimal shapes. In this respect, the main contribution lies in the construction of a suitable extension operator which takes into account the slip boundary condition.

- Treatment of the impermeability condition (4.3) and the non-smooth slip law in the approximate schemes. On curved boundaries, the numerical realization of (4.3) is troublesome. We use either penalization or regularization, which introduces an additional unknown. In all cases it is shown that for vanishing penalty and regularization parameters the solutions converge to the solution of the original problem.

Reprints

- J. Haslinger, J. Stebel, T. Sassi. Shape optimization for Stokes problem with threshold slip. *Applications of Mathematics*, **59**(6):631-652, 2014.

- J. Haslinger, J. Stebel. Stokes problem with a solution dependent slip bound: Stability of solutions with respect to domains. *Zeitschrift für Angewandte Mathematik und Mechanik*, **96**(9):1049-1060, 2016.

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SHAPE OPTIMIZATION FOR STOKES PROBLEM WITH THRESHOLD SLIP

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Abstract. We study the Stokes problems in a bounded planar domain Ω with a friction type boundary condition that switches between a slip and no-slip stage. Our main goal is to determine under which conditions concerning the smoothness of Ω solutions to the Stokes system with the slip boundary conditions depend continuously on variations of Ω. Having this result at our disposal, we easily prove the existence of a solution to optimal shape design problems for a large class of cost functionals. In order to release the impermeability condition, whose numerical treatment could be troublesome, we use a penalty approach. We introduce a family of shape optimization problems with the penalized state relations. Finally we establish convergence properties between solutions to the original and modified shape optimization problems when the penalty parameter tends to zero.

Keywords: Stokes problem; friction boundary condition; shape optimization

MSC 2010: 49Q10, 76D07

1. Introduction

An important part of mathematical modeling of fluid flow is the proper choice of boundary conditions. Solid impermeable walls are traditionally described by the no-slip condition, i.e.,

$$\boldsymbol{u} = \boldsymbol{0},$$

where $\boldsymbol{u}$ denotes the velocity field. In some applications, however, one can observe a tangential velocity along the surface. In this case it is more realistic to use some

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kind of the slip condition. Navier [14] proposed the condition

$$\boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \quad \lambda \geqslant 0,$$

saying that the tangential velocity $\boldsymbol{u}_\tau$ should be proportional to the shear stress $\boldsymbol{\sigma}_\tau$. Relations of this type are often used especially in non-Newtonian fluid mechanics, see e.g. [13], [4].

In this paper we introduce a system with a friction-type condition, which switches between a slip and no-slip stage depending on the magnitude of the shear stress. Due to its non-smoothness, the weak formulation of the considered problem leads to a variational inequality. To demonstrate the difficulties arising from this fact and still to keep ideas clear, we consider the Stokes problem in a planar domain Ω.

Problems involving friction-type boundary conditions have been analysed e.g. in [6], [7], [15]. The main goal of this paper is to study under which conditions concerning the smoothness of Ω solutions to the Stokes problem with threshold slip depend continuously on variations of Ω. This is the basic property enabling us to prove the existence of optimal shapes for a large class of optimal shape design problems.

It should be stressed that domain dependence of solutions subject to slip boundary conditions is more delicate than in the case of no-slip. In particular, the control-to-state mapping for problems with slip boundary conditions can be discontinuous for some sequences of equi-Lipschitz domains [1], which cannot happen when no slip is considered. It is also known that uniform $C^{1,1}$ regularity of boundary perturbations is sufficient for continuous dependence of solutions subject to Navier's slip condition [17]. We refer to [3] for more details on this subject.

The slip conditions bring another difficulty also for the numerical treatment. On polygonal computational domains the impermeability condition cannot be applied directly due to insufficient approximation of the normal vector. One possible remedy is to use a penalty approach [12]. We introduce a family of shape optimization problems with the penalized states and establish mutual relations between solutions to the original and modified optimization problems when the penalty parameter tends to zero.

The paper is organized as follows: In the next section we present the fluid flow model and define a class of shape optimization problems. The domain dependence of solutions to the state problem is analysed in Section 3. In Section 4 we define a family of shape optimization problems governed by the Stokes system with threshold slip but with a penalized form of the impermeability condition. Discretizations of these problems together with the convergence analysis are presented in Section 5.

Throughout the paper, the following notation will be used: $H^k(Q)$, $k \geqslant 0$ integer, stands for the classical Sobolev space of functions which are together with their generalized derivatives up to order k square integrable in Q ($H^0(Q) := L^2(Q)$) with the norm denoted by $\|\cdot\|_{k,Q}$. For the norm in $L^\infty(Q)$ we use the notation $\|\cdot\|_{\infty,Q}$. Finally, c denotes a generic, positive constant. To emphasize that c depends on a particular parameter p, we shall write $c := c(p)$.

2. Formulation of the problem

Let us consider the Stokes problem in a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. The slip boundary conditions are prescribed on a part of the boundary S and the no-slip condition on $\Gamma = \partial\Omega \setminus \overline{S}$:

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \tag{2.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \tag{2.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma, \tag{2.1c}$$

$$u_\nu = 0 \text{ on } S, \tag{2.1d}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant g \text{ on } S, \tag{2.1e}$$

$$\boldsymbol{u}_\tau \neq \boldsymbol{0} \Rightarrow \|\boldsymbol{\sigma}_\tau\| = g \ \&\ \exists \lambda \geqslant 0\colon\ \boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \text{ on } S. \tag{2.1f}$$

Here $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure, and $\boldsymbol{f}$ is the external force. Further, $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ denote the unit outward normal and tangential vector to $\partial\Omega$, respectively. If $\boldsymbol{a} \in \mathbb{R}^2$ is a vector, then $a_\nu := \boldsymbol{a} \cdot \boldsymbol{\nu}$, $\boldsymbol{a}_\tau := \boldsymbol{a} - a_\nu \boldsymbol{\nu}$ are its normal component and the tangential part on $\partial\Omega$, respectively. The Euclidean norm of $\boldsymbol{a}$ is denoted by $\|\boldsymbol{a}\|$. Finally, $\boldsymbol{\sigma}_\tau := (\partial \boldsymbol{u}/\partial \boldsymbol{\nu})_\tau$ stands for the shear stress and $g > 0$ a.e. on S is a given slip bound. By the classical solution of (2.1) we mean any couple of sufficiently smooth functions $(\boldsymbol{u}, p)$ satisfying the differential equations and the boundary conditions in (2.1).

To give the weak formulation of (2.1) we shall need the following function spaces:

$$V(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2;\ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma,\ v_\nu = 0 \ \text{ on } S\}, \tag{2.2}$$

$$V_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in V(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ a.e. in } \ \Omega\}, \tag{2.3}$$

$$L^2_0(\Omega) = \left\{ q \in L^2(\Omega);\ \int_\Omega q = 0 \right\}. \tag{2.4}$$

The weak formulation of (2.1) reads as follows:

$$(\mathcal{P}) \qquad \begin{aligned} &\text{Find } (\boldsymbol{u}, p) \in V(\Omega) \times L^2_0(\Omega) \text{ such that} \\ &\forall\, \boldsymbol{v} \in V(\Omega)\colon\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j(\boldsymbol{v}_\tau) - j(\boldsymbol{u}_\tau) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega}, \\ &\forall\, q \in L^2_0(\Omega)\colon\ b(\boldsymbol{u}, q) = 0, \end{aligned}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u}\colon \nabla\boldsymbol{v} := \int_\Omega \nabla u_i \cdot \nabla v_i, \tag{2.5a}$$

$$b(\boldsymbol{v},q) = \int_\Omega q \operatorname{div}\boldsymbol{v}, \tag{2.5b}$$

$$j(\boldsymbol{\varphi}) = \int_S g\|\boldsymbol{\varphi}\|. \tag{2.5c}$$

Remark 1. Since we consider a two-dimensional case, we have that $\|\boldsymbol{v}_\tau\| = |\boldsymbol{v}\cdot\boldsymbol{\tau}|$ on S.

The following existence and uniqueness result is known [6].

Theorem 1. *Let $\boldsymbol{f} \in (L^2(\Omega))^2$, $g \in L^\infty(S)$, $g > 0$ a.e. on S. Then $(\mathcal{P})$ has a unique solution $(\boldsymbol{u},p)$ and*

$$\|\nabla\boldsymbol{u}\|_{0,\Omega} + \|p\|_{0,\Omega} \leqslant c(\|\boldsymbol{f}\|_{0,\Omega} + \|g\|_{\infty,S}), \tag{2.6}$$

where c is a positive constant which does not depend on $\boldsymbol{f}$ and g.

Up to now, the domain Ω was given. From now on, we shall consider a specific family of domains, namely

$$\mathcal{O} = \{\Omega(\alpha);\ \alpha \in \mathcal{U}_{\mathrm{ad}}\},$$

where (see Figure 1)

$$\Omega(\alpha) = \{(x_1,x_2);\ x_1 \in (0,1),\ x_2 \in (\alpha(x_1),\gamma)\}, \tag{2.7}$$

$$\mathcal{U}_{\mathrm{ad}} = \{\alpha \in C^{1,1}([0,1]);\ \alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max} \text{ in } [0,1],\ |\alpha^{(j)}| \leqslant C_j,\ j = 1,2 \text{ a.e. in } (0,1)\}. \tag{2.8}$$

Here γ, $\alpha_{\min}$, $\alpha_{\max}$, C_1, C_2 are given positive constants chosen in such a way that $\mathcal{U}_{\mathrm{ad}} \neq \emptyset$.

The boundary $\partial\Omega(\alpha)$ is split into $S(\alpha)$ and $\Gamma(\alpha) = \partial\Omega(\alpha) \setminus \overline{S}(\alpha)$, where

$$S(\alpha) = \{(x_1,x_2);\ x_1 \in (0,1),\ x_2 = \alpha(x_1)\},\ \alpha \in \mathcal{U}_{\mathrm{ad}},$$

i.e., $S(\alpha)$ is the graph of α. On any $\Omega(\alpha)$ we shall solve the Stokes system with the slip boundary conditions on $S(\alpha)$ and the no-slip condition on $\Gamma(\alpha)$. To emphasize the fact that the state problem is parametrized by $\alpha \in \mathcal{U}_{\mathrm{ad}}$ we shall use the following notation: $V(\alpha) := V(\Omega(\alpha))$, $V_{\mathrm{div}}(\alpha) := V_{\mathrm{div}}(\Omega(\alpha))$, $L^2_0(\alpha) := L^2_0(\Omega(\alpha))$. Similarly,

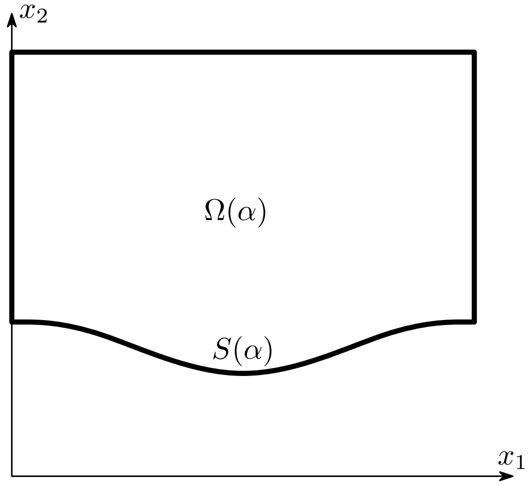

Figure 1. Geometry of the domain $\Omega(\alpha)$.

the bilinear forms a_α, b_α and the non-differentiable term j_α denote the ones from (2.5) with Ω, S replaced by $\Omega(\alpha)$ and $S(\alpha)$, respectively. The weak form of the state problem on $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\rm ad}$ reads as follows:

$$(\mathcal{P}(\alpha)) \qquad \begin{aligned} &\text{Find } (\boldsymbol{u}(\alpha), p(\alpha)) \in V(\alpha) \times L^2_0(\alpha) \text{ such that} \\ &\forall\, \boldsymbol{v} \in V(\alpha)\colon\ a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)) - b_\alpha(\boldsymbol{v} - \boldsymbol{u}(\alpha), p(\alpha)) \\ &\qquad + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}, \\ &\forall\, q \in L^2_0(\alpha)\colon\ b_\alpha(\boldsymbol{u}(\alpha), q) = 0. \end{aligned}$$

In what follows we shall suppose that $\boldsymbol{f} \in (L^2_{\rm loc}(\mathbb{R}^2))^2$ and, for simplicity of our analysis, that g is a positive constant.

Finally, let $J\colon \Delta \to \mathbb{R}$ be a cost functional, $\Delta = \{(\alpha, \boldsymbol{y}, q);\ \alpha \in \mathcal{U}_{\rm ad},\ \boldsymbol{y} \in V(\alpha),\ q \in L^2_0(\alpha)\}$ and $\mathfrak{J}(\alpha) = J(\alpha, \boldsymbol{u}(\alpha), p(\alpha))$, where $(\boldsymbol{u}(\alpha), p(\alpha))$ is the unique solution of $(\mathcal{P}(\alpha))$. Next we shall study the following optimal shape design problem:

$$(\mathbb{P}) \qquad \text{Find } \alpha^* \in \mathcal{U}_{\rm ad} \text{ such that } \forall\, \alpha \in \mathcal{U}_{\rm ad}\colon\ \mathfrak{J}(\alpha^*) \leqslant \mathfrak{J}(\alpha).$$

To prove that $(\mathbb{P})$ has a solution we shall need the following lower-semicontinuity property of J:

$$(2.9) \qquad \left.\begin{aligned} &\alpha_n \to \alpha \ \text{ in } C^1([0,1]),\ \alpha_n, \alpha \in \mathcal{U}_{\rm ad} \\ &\boldsymbol{y}_n \rightharpoonup \boldsymbol{y} \ \text{ in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_n, \boldsymbol{y} \in (H^1_0(\widehat{\Omega}))^2 \\ &q_n \rightharpoonup q \ \text{ in } L^2(\widehat{\Omega}),\ q_n, q \in L^2_0(\widehat{\Omega}) \end{aligned}\right\}$$
$$\Rightarrow \liminf_{n\to\infty} J(\alpha_n, \boldsymbol{y}_n|_{\Omega(\alpha_n)}, q_n|_{\Omega(\alpha_n)}) \geqslant J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}),$$

where $\widehat{\Omega}$ is a domain which contains all $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{\mathrm{ad}}$. Here and in what follows, $\widehat{\Omega} = (0,1) \times (0,\gamma)$ with γ from the definition of $\Omega(\alpha)$. Our first goal will be to prove the following result.

Theorem 2. *Let* (2.9) *be satisfied. Then* ($\mathbb{P}$) *has a solution.*

3. Stability of solutions with respect to shape variations

In this section we shall prove that the solutions of $(\mathcal{P}(\alpha))$ depend on $\alpha \in \mathcal{U}_{\mathrm{ad}}$ in a continuous way, which is the basic property used to prove the existence of a solution to ($\mathbb{P}$). To this end we have to introduce convergence of domains belonging to $\mathcal{O}$ and convergence of functions with variable domains of their definition.

Definition 1. Let $\Omega(\alpha_n) \in \mathcal{O}$, $n = 1, 2, \dots$ be given. We say that the sequence $\{\Omega(\alpha_n)\}$ tends to $\Omega(\alpha) \in \mathcal{O}$ (and write $\Omega(\alpha_n) \to \Omega(\alpha)$) if

$$\alpha_n \to \alpha \text{ in } C^1([0,1]).$$

Definition 2. Let $\boldsymbol{y}_n \in V(\alpha_n)$, $\alpha_n \in \mathcal{U}_{\mathrm{ad}}$, $n = 1, 2, \dots$ be given. We say that the sequence $\{\boldsymbol{y}_n\}$ tends weakly to $\boldsymbol{y} \in V(\alpha)$, $\alpha \in \mathcal{U}_{\mathrm{ad}}$ (and write $\boldsymbol{y}_n \rightharpoonup \boldsymbol{y}$) if

$$\pi_{\alpha_n} \boldsymbol{y}_n \rightharpoonup \pi_\alpha \boldsymbol{y} \text{ (weakly) in } (H^1(\widehat{\Omega}))^2, \tag{3.1}$$

where for any $\beta \in \mathcal{U}_{\mathrm{ad}}$, $\pi_\beta \in \mathcal{L}(V(\beta), H_0^1(\widehat{\Omega}))$ denotes an extension mapping from $\Omega(\beta)$ on $\widehat{\Omega}$, whose norm can be estimated independently of $\beta \in \mathcal{U}_{\mathrm{ad}}$. If weak convergence in (3.1) can be replaced by the strong one, we say that $\{\boldsymbol{y}_n\}$ tends strongly to $\boldsymbol{y}$ (and write $\boldsymbol{y}_n \to \boldsymbol{y}$).

For functions belonging to $H_0^1(\alpha_n) := H_0^1(\Omega(\alpha_n))$ or $L_0^2(\alpha_n)$ the situation is much simpler since one can use the zero extension outside of $\Omega(\alpha_n)$.

Definition 3. Let $z_n \in H_0^1(\alpha_n)$, $\alpha_n \in \mathcal{U}_{\mathrm{ad}}$, $n = 1, 2, \dots$ We say that the sequence $\{z_n\}$ tends to $z \in H_0^1(\alpha)$ weakly, strongly (and write $z_n \rightharpoonup z$, $z_n \to z$, respectively) if

$$\begin{aligned} z_n^0 &\rightharpoonup z^0 \quad \text{in } H_0^1(\widehat{\Omega}), \\ z_n^0 &\to z^0 \quad \text{in } H_0^1(\widehat{\Omega}), \end{aligned}$$

respectively. Here the symbol "^{0}" stands for the zero extension of functions from their domain of definition on $\widehat{\Omega}$ (analogously we define convergence of a sequence $\{q_n\}$, $q_n \in L_0^2(\alpha_n)$).

R e m a r k 2. Since all domains belonging to $\mathcal{O}$ satisfy the so-called uniform cone property, such an extension mapping from Definition 2 can be easily constructed. Indeed, first we use the uniform extension mapping from $V(\beta)$ to $H^1(\mathbb{R}^2)$, whose existence is guarenteed, as follows from [5]. Then extended functions are multiplied by a suitable cut-off function in order to get zero traces on the boundary of $\widehat{\Omega}$.

The following auxiliary result is a direct consequence of the Arzelà-Ascoli and Lebesgue theorem (see e.g. [16], [9] for further details on convergence of domains).

Lemma 1. *It holds:*

(i) *the system $\mathcal{O}$ is compact with respect to convergence from Definition 1;*

(ii) *if $\Omega(\alpha_n) \to \Omega(\alpha)$, $\alpha_n, \alpha \in \mathcal{U}_{\rm ad}$, then*

$$\chi_n \to \chi \quad \textit{in } L^q(\widehat{\Omega})\ \forall\, q \in [1,\infty),$$

where χ_n, χ are the characteristic functions of $\Omega(\alpha_n)$ and $\Omega(\alpha)$, respectively.

First we show that the constant c in (2.6) can be chosen to be independent of $\alpha \in \mathcal{U}_{\rm ad}$.

Lemma 2. *There exists a constant $c > 0$ such that*

$$\|\pi_\alpha \boldsymbol{u}(\alpha)\|_{1,\widehat{\Omega}} + \|p^0(\alpha)\|_{0,\widehat{\Omega}} \leqslant c \tag{3.2}$$

holds for any $\alpha \in \mathcal{U}_{\rm ad}$.

P r o o f. Using test functions $\boldsymbol{v} \in V_{\rm div}(\alpha)$, $\alpha \in \mathcal{U}_{\rm ad}$, problem $(\mathcal{P}(\alpha))$ takes the form:

$$a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v}-\boldsymbol{u}(\alpha)) + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) \geqslant (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}, \quad \boldsymbol{v} \in V_{\rm div}(\alpha). \tag{3.3}$$

Inserting $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}(\alpha)$ into (3.3) we obtain:

$$\begin{aligned} |\boldsymbol{u}(\alpha)|^2_{1,\Omega(\alpha)} := \|\nabla \boldsymbol{u}(\alpha)\|^2_{0,\Omega(\alpha)} &\leqslant a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) + j_\alpha(\boldsymbol{u}_\tau(\alpha)) = (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)} \\ &\leqslant \|\boldsymbol{f}\|_{0,\widehat{\Omega}} \|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}, \end{aligned} \tag{3.4}$$

where for simplicity of notation $\pi_\alpha \boldsymbol{u} := \pi_\alpha \boldsymbol{u}(\alpha)$. The seminorm on the left of (3.4) can be estimated from below by the Friedrichs inequality with a constant $c > 0$ which does not depend on $\alpha \in \mathcal{U}_{\rm ad}$ [9]. Thus

$$c\|\boldsymbol{u}(\alpha)\|^2_{1,\Omega(\alpha)} \leqslant |\boldsymbol{u}(\alpha)|^2_{1,\Omega(\alpha)} \leqslant \|\boldsymbol{f}\|_{0,\widehat{\Omega}} \|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}.$$

From this and the fact that also the norm of π_α can be estimated uniformly with respect to $\alpha \in \mathcal{U}_{\rm ad}$, the boundedness of $\|\pi_\alpha \boldsymbol{u}(\alpha)\|_{1,\widehat{\Omega}}$ follows. To prove the boundedness of the pressure we proceed as follows: Using the fact that

$$a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) - b_\alpha(\boldsymbol{u}(\alpha), p(\alpha)) + j_\alpha(\boldsymbol{u}_\tau(\alpha)) = (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha))},$$

we obtain from the inequality in $(\mathcal{P}(\alpha))$:

$$b_\alpha(\boldsymbol{v}, p(\alpha)) \leqslant a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v}) + j_\alpha(\boldsymbol{v}_\tau) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha)} \leqslant c\|\boldsymbol{v}\|_{1,\Omega(\alpha)}, \ \boldsymbol{v} \in V(\alpha), \tag{3.5}$$

where $c > 0$ does not depend on $\alpha \in \mathcal{U}_{\rm ad}$, making use of the boundedness of $\|\pi_\alpha \boldsymbol{u}\|_{1,\widehat{\Omega}}$ and the uniform boundedness of the trace mapping $\mathrm{Tr}_\alpha \in \mathcal{L}(H^1(\Omega(\alpha)), L^2(\Omega(\alpha)))$ with respect to $\alpha \in \mathcal{U}_{\rm ad}$ [9]. From (3.5) it follows that

$$\sup_{\boldsymbol{v} \in V(\alpha)} \frac{b_\alpha(\boldsymbol{v}, p(\alpha))}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \leqslant c. \tag{3.6}$$

From [8] we know that there is a mapping $\mathcal{B}_\alpha \in \mathcal{L}(L_0^2(\alpha), (H_0^1(\alpha))^2)$ such that $\operatorname{div} \mathcal{B}_\alpha q = q$ a.e. in $\Omega(\alpha)$, whose norm is bounded independently of $\alpha \in \mathcal{U}_{\rm ad}$ (see also [3], Section 4)[1]. The choice $\boldsymbol{v} := \mathcal{B}_\alpha p(\alpha)$ in (3.6) yields:

$$\sup_{\boldsymbol{v} \in V(\alpha)} \frac{b_\alpha(\boldsymbol{v}, p(\alpha))}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \geqslant \frac{b_\alpha(\mathcal{B}_\alpha p(\alpha), p(\alpha))}{\|\mathcal{B}_\alpha p(\alpha)\|_{1,\Omega(\alpha)}} = \frac{\|p(\alpha)\|^2_{0,\Omega(\alpha)}}{\|\mathcal{B}_\alpha p(\alpha)\|_{1,\Omega(\alpha)}} \geqslant \bar{c}\|p(\alpha)\|_{0,\Omega(\alpha)},$$

where the constant $\bar{c} > 0$ is independent of $\alpha \in \mathcal{U}_{\rm ad}$. This concludes the proof. □

We shall also need the following auxiliary result.

Lemma 3. *Let $\alpha_n, \alpha \in \mathcal{U}_{\rm ad}$ be such that $\alpha_n \to \alpha$ in $C^1([0,1])$ and let $\boldsymbol{v} \in V(\alpha)$ be given. Then there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ and a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$ such that $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \ k \to \infty. \tag{3.7}$$

In addition, for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k}). \tag{3.8}$$

P r o o f. Let $\boldsymbol{\nu}^\alpha := \boldsymbol{\nu}^\alpha(x_1)$, $\boldsymbol{\nu}^{\alpha_n} := \boldsymbol{\nu}^{\alpha_n}(x_1)$ denote the unit outward normal vector to $S(\alpha)$ and $S(\alpha_n)$, respectively. By the same symbols we shall denote their

[1] In fact, the norm of $\mathcal{B}_\alpha$ depends only on $\|\alpha\|_{1,\infty,[0,1]}$, i.e., it is uniformly bounded for $\alpha \in \{\beta \in C^{0,1}([0,1]);\ 0 \leqslant \beta \leqslant \alpha_{\max},\ |\beta'| \leqslant C_1 \text{ in } [0,1]\}$.

natural extensions defined in $\widehat{\Omega}$, i.e., $\boldsymbol{\nu}^\alpha(x) := \boldsymbol{\nu}^\alpha(x_1)$ and $\boldsymbol{\nu}^{\alpha_n}(x) := \boldsymbol{\nu}^{\alpha_n}(x_1)$, $x = (x_1, x_2) \in \widehat{\Omega}$. We set

$$\varphi(x) := \boldsymbol{v}(x) \cdot \boldsymbol{\nu}^\alpha(x), \quad \boldsymbol{\psi}(x) := \boldsymbol{v}_{\tau^\alpha}(x), \quad x \in \Omega(\alpha).$$

Then $\varphi \in H_0^1(\Omega(\alpha))$, $\boldsymbol{\psi} \in (H^1(\Omega(\alpha)))^2$ and $\boldsymbol{\psi} = \mathbf{0}$ on $\Gamma(\alpha)$. Using the density arguments, one can find sequences $\{\varphi_k\}$, $\varphi_k \in C_0^\infty(\Omega(\alpha))$ and $\{\boldsymbol{\psi}_k\}$, $\boldsymbol{\psi}_k \in (C^\infty(\overline{\Omega}(\alpha)))^2$, $\operatorname{dist}(\operatorname{supp} \boldsymbol{\psi}_k, \Gamma(\alpha)) > 0$ for all $k \in \mathbb{N}$ such that

$$\begin{gathered} \varphi_k \to \varphi \quad \text{in } H_0^1(\Omega(\alpha)), \\ \boldsymbol{\psi}_k \to \boldsymbol{\psi},\ k \to \infty, \quad \text{in } (H^1(\Omega(\alpha))^2 \end{gathered}$$

and also

$$\begin{gathered} \varphi_k^0 \to \varphi^0 \quad \text{in } H_0^1(\widehat{\Omega}), \\ \pi_\alpha \boldsymbol{\psi}_k \to \pi_\alpha \boldsymbol{\psi}, \quad \text{in } (H^1(\widehat{\Omega}))^2. \end{gathered}$$

Moreover, we may assume that $\operatorname{dist}(\operatorname{supp} \pi_\alpha \boldsymbol{\psi}_k, \widehat{\Gamma}) > 0$ for all $k \in \mathbb{N}$ where $\widehat{\Gamma} := \partial\widehat{\Omega} \setminus [0,1] \times \{0\}$. The sequence $\{\boldsymbol{v}_k\}$ satisfying (3.7)–(3.8) will be constructed as follows. Suppose for the moment that there exists a filter of indices $\{n_k\}$, $k \to \infty$, such that for any $k \in \mathbb{N}$ it holds that $S(\alpha_{n_k}) \cap \operatorname{supp} \varphi_k^0 = \emptyset$ and in addition there exist functions $\boldsymbol{N}_{n_k} \in (C^{0,1}(\widehat{\Omega}))^2$ such that $\boldsymbol{N}_{n_k}|_{\partial\Omega(\alpha_{n_k})} = \boldsymbol{\nu}^{\alpha_{n_k}}$ and

$$\boldsymbol{N}_{n_k} \to \boldsymbol{\nu}^\alpha \quad \text{in } (H^1(\widehat{\Omega}))^2,\ k \to \infty. \tag{3.9}$$

Define $\boldsymbol{v}_k$ by:

$$\boldsymbol{v}_k = \varphi_k^0 \boldsymbol{N}_{n_k} + (\pi_\alpha \boldsymbol{\psi}_k)_{\tau_{n_k}} = \varphi_k^0 \boldsymbol{N}_{n_k} + \pi_\alpha \boldsymbol{\psi}_k - (\pi_\alpha \boldsymbol{\psi}_k \cdot \boldsymbol{N}_{n_k}) \boldsymbol{N}_{n_k}. \tag{3.10}$$

From this and the definition of n_k it immediately follows that $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$, $\boldsymbol{v}_k = \mathbf{0}$ on $\Gamma(\alpha_{n_k})$ and $\boldsymbol{v}_k \cdot \boldsymbol{\nu}^{\alpha_{n_k}}|_{S(\alpha_{n_k})} = \varphi_k^0|_{S(\alpha_{n_k})} = 0$. Hence, $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k})$. Passing to the limit with $k \to \infty$ in (3.10), we obtain:

$$\boldsymbol{v}_k \to \varphi^0 \boldsymbol{\nu}^\alpha + \pi_\alpha \boldsymbol{\psi} - (\pi_\alpha \boldsymbol{\psi} \cdot \boldsymbol{\nu}^\alpha) \boldsymbol{\nu}^\alpha =: \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2.$$

It is easy to see that $\overline{\boldsymbol{v}}$ satisfies $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$.

It remains to prove (3.9). Since $\alpha_n \to \alpha$ in $C^1([0,1])$, we have

$$\boldsymbol{\nu}^{\alpha_n} \to \boldsymbol{\nu}^\alpha \quad \text{in } C(\overline{\widehat{\Omega}}) \tag{3.11}$$

and from the definition of $\mathcal{O}$ it follows that

$$\|\nabla\boldsymbol{\nu}^{\beta}\|_{\infty,\widehat{\Omega}} \leqslant C_2 \quad \text{for every } \beta \in \mathcal{U}_{\mathrm{ad}}. \tag{3.12}$$

Let $\xi_k \in C^{\infty}([0,\infty))$ be functions satisfying $0 \leqslant \xi_k \leqslant 1$ in $[0,\infty)$, $\xi_k|_{[0,1/(2k)]} = 1$, and $\xi_k|_{[1/k,\infty)} = 0$ for every $k \in \mathbb{N}$. For $k, n \in \mathbb{N}$ we set

$$\boldsymbol{N}_{n,k}(x) := \xi_k(|x_2 - \alpha(x_1)|)(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}) + \boldsymbol{\nu}^{\alpha}.$$

It is readily seen that $\boldsymbol{N}_{n,k} \in (C^{0,1}(\overline{\widehat{\Omega}}))^2$ for all $k, n \in \mathbb{N}$ and

$$\|\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \leqslant \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \quad \text{as } n \to \infty \tag{3.13}$$

uniformly with respect to $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$ be fixed. Then from the definition of ξ_k it follows that there exists an index $n_0 := n_0(k) \in \mathbb{N}$ such that $\boldsymbol{N}_{n,k}|_{\partial\Omega_n} = \boldsymbol{\nu}^{\alpha_n}$ for any $n \geqslant n_0$. Furthermore:

$$\begin{aligned} \|\nabla(\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha})\|_{0,\widehat{\Omega}} &\leqslant \max_{(x_1,x_2)\in\widehat{\Omega}} |\nabla(\xi_k(|x_2 - \alpha(x_1)|))| \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} \\ &\quad + \|\nabla(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha})\|_{0,\{|x_2-\alpha(x_1)|<1/k\}} \\ &\leqslant \sqrt{1 + C_1^2}\, \|\xi_k'\|_{\infty,[0,\infty)} \|\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha}\|_{0,\widehat{\Omega}} + 2C_2/k. \end{aligned} \tag{3.14}$$

From this we see (still keeping $k \in \mathbb{N}$ fixed) that there exists an index $n_1 := n_1(k) \in \mathbb{N}$ such that $\|\nabla(\boldsymbol{N}_{n,k} - \boldsymbol{\nu}^{\alpha})\|_{0,\widehat{\Omega}} = O(1/k)$ for any $n \geqslant n_1$. Setting $\boldsymbol{N}_{n_k} := \boldsymbol{N}_{n_k,k}$, where $n_k = \max\{n_0, n_1\}$, we obtain (3.9), making use of (3.13). $\square$

The main result of this section is the following stability result.

Theorem 3. *Let α_n, $\alpha \in \mathcal{U}_{\mathrm{ad}}$ be such that $\alpha_n \to \alpha$ in $C^1([0,1])$ and denote by $(\boldsymbol{u}_n, p_n) := (\boldsymbol{u}(\alpha_n), p(\alpha_n)) \in V(\alpha_n) \times L_0^2(\alpha_n)$ the unique solution of $(\mathcal{P}(\alpha_n))$. Suppose that there exists an element $(\bar{\boldsymbol{u}}, \bar{p}) \in (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ such that*

$$\pi_{\alpha_n}\boldsymbol{u}_n \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \tag{3.15a}$$

$$p_n^0 \rightharpoonup \bar{p} \quad \text{in } L_0^2(\widehat{\Omega}). \tag{3.15b}$$

Then $(\boldsymbol{u}(\alpha), p(\alpha)) := (\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ solves $(\mathcal{P}(\alpha))$.

Proof. First we show that $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)}) \in V_{\mathrm{div}}(\alpha) \times L_0^2(\alpha)$. The fact that $\boldsymbol{u}(\alpha) := \bar{\boldsymbol{u}}|_{\Omega(\alpha)} = \mathbf{0}$ on $\Gamma(\alpha)$ and $p(\alpha) := \bar{p}|_{\Omega(\alpha)} \in L_0^2(\Omega(\alpha))$ is readily seen. It remains to prove that $\operatorname{div} \boldsymbol{u}(\alpha) = 0$ in $\Omega(\alpha)$ and $\boldsymbol{u}(\alpha) \cdot \boldsymbol{\nu}^{\alpha} = 0$ on $S(\alpha)$. This is equivalent to verifying that

$$\int_{\Omega(\alpha)} \boldsymbol{u}(\alpha) \cdot \nabla\varphi = 0 \quad \forall\, \varphi \in H^1(\Omega(\alpha)),\ \varphi = 0 \text{ on } \Gamma(\alpha). \tag{3.16}$$

Let φ from (3.16) be given and denote by $\tilde{\varphi} \in H^1(\widehat{\Omega})$ its extension such that $\tilde{\varphi} = 0$ on $\partial\widehat{\Omega} \setminus [0,1] \times \{0\}$. Since $\boldsymbol{u}_n \in V(\alpha_n)$ for all $n \in \mathbb{N}$, we get

$$\int_{\Omega(\alpha_n)} \boldsymbol{u}_n \cdot \nabla\tilde{\varphi} = 0 \ \Leftrightarrow\ \int_{\widehat{\Omega}} \chi_n \pi_{\alpha_n} \boldsymbol{u}_n \cdot \nabla\tilde{\varphi} = 0, \tag{3.17}$$

where χ_n is the characteristic function of $\Omega(\alpha_n)$. Letting $n \to \infty$ in (3.17), we obtain

$$\int_{\widehat{\Omega}} \chi_n \pi_{\alpha_n} \boldsymbol{u}_n \cdot \nabla\tilde{\varphi} \to \int_{\widehat{\Omega}} \chi \bar{\boldsymbol{u}} \cdot \nabla\tilde{\varphi} = \int_{\Omega(\alpha)} \boldsymbol{u}(\alpha) \cdot \nabla\varphi = 0,$$

where χ is the characteristic function of $\Omega(\alpha)$, making use of Lemma 1 (ii) and (3.15a). Hence, $\boldsymbol{u}(\alpha) \in V_{\mathrm{div}}(\alpha)$. Now we show that the pair $(\boldsymbol{u}(\alpha), p(\alpha))$ satisfies the inequality in $(\mathcal{P}(\alpha))$.

Let $\boldsymbol{v} \in V(\alpha)$ be given and construct the sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ satisfying (3.7) and (3.8). Since $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})} \in V(\alpha_{n_k})$ for an appropriate $n_k \in \mathbb{N}$, it can be used as a test function in $(\mathcal{P}(\alpha_{n_k}))$ (to simplify notation we shall write $a_{n_k} := a_{\alpha_{n_k}}$, $b_{n_k} := b_{\alpha_{n_k}}$, $j_{n_k} := j_{\alpha_{n_k}}$):

$$\begin{aligned} a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) - b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) + j_{n_k}(\boldsymbol{v}_{k\tau}) - j_{n_k}(\boldsymbol{u}_{n_k\tau}) \\ \geqslant (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})}. \end{aligned} \tag{3.18}$$

Letting $k \to \infty$ in (3.18) and using Lemma 1 (ii), (3.7), (3.15) we obtain (for details we refer to [9]):

$$\limsup_{k\to\infty} a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) \leqslant a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)), \tag{3.19a}$$

$$\lim_{k\to\infty} b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) = b_\alpha(\boldsymbol{v} - \boldsymbol{u}(\alpha), p(\alpha)), \tag{3.19b}$$

$$\lim_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})} = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}. \tag{3.19c}$$

The frictional term can be written as

$$\begin{aligned} j_{n_k}(\boldsymbol{v}_{k\tau}) &= g \int_0^1 |\boldsymbol{v}_{k\tau} \circ \alpha_{n_k}| \sqrt{1 + |\alpha'_{n_k}|^2}\, \mathrm{d}x_1 \\ &= g \int_0^1 |\boldsymbol{v}_k \circ \alpha_{n_k} - (\boldsymbol{v}_k \circ \alpha_{n_k} \cdot \boldsymbol{\nu}^{\alpha_{n_k}}) \boldsymbol{\nu}^{\alpha_{n_k}}|^2 \sqrt{1 + |\alpha'_{n_k}|^2}\, \mathrm{d}x_1. \end{aligned}$$

From [9] we know that

$$\boldsymbol{v}_k \circ \alpha_{n_k} \to \boldsymbol{v} \circ \alpha \quad \text{in } (L^2((0,1)))^2,\ k \to \infty.$$

Therefore,

$$j_{n_k}(\boldsymbol{v}_{k\tau}) \to j_\alpha(\boldsymbol{v}_\tau), \quad k \to \infty,$$

using the fact that $\boldsymbol{\nu}^{\alpha_{n_k}} \rightrightarrows \boldsymbol{\nu}^\alpha$, $\alpha'_{n_k} \rightrightarrows \alpha'$ (uniformly) in $[0,1]$ (similarly for $j_{n_k}(\boldsymbol{u}_{n_k\tau})$). From this and (3.19) we see that $(\boldsymbol{u}(\alpha), p(\alpha))$ satisfies the inequality in $(\mathcal{P}(\alpha))$, i.e., $(\boldsymbol{u}(\alpha), p(\alpha))$ solves $(\mathcal{P}(\alpha))$. □

R e m a r k 3. It is easy to show that (3.15a) implies that

$$\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n \to \chi \nabla \bar{\boldsymbol{u}} \quad \text{in } (L^2(\widehat{\Omega}))^2, \tag{3.20}$$

where χ_n, χ are the characteristic functions of $\Omega(\alpha_n)$ and $\Omega(\alpha)$, respectively. To prove (3.20) it is sufficient to show that

$$\|\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n\|_{0,\widehat{\Omega}} \to \|\chi \nabla \bar{\boldsymbol{u}}\|_{0,\widehat{\Omega}}, \quad n \to \infty.$$

Indeed,

$$\begin{aligned}\|\chi_n \nabla \pi_{\alpha_n} \boldsymbol{u}_n\|^2_{0,\widehat{\Omega}} = a_{\alpha_n}(\boldsymbol{u}_n, \boldsymbol{u}_n) &= b_{\alpha_n}(\boldsymbol{u}_n, p_n^0) - j_{\alpha_n}(\boldsymbol{u}_{n\tau}) + (\boldsymbol{f}, \boldsymbol{u}_n)_{0,\Omega(\alpha_n)} \\ &\to b_\alpha(\boldsymbol{u}(\alpha), p(\alpha)) - j_\alpha(\boldsymbol{u}_\tau(\alpha)) + (\boldsymbol{f}, \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)} \\ &= a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{u}(\alpha)) = \|\chi \nabla \bar{\boldsymbol{u}}\|^2_{0,\widehat{\Omega}}.\end{aligned}$$

From (3.20) it easily follows that

$$\boldsymbol{u}_n \to \boldsymbol{u}(\alpha) \quad \text{in } (H^1_{\text{loc}}(\Omega(\alpha)))^2$$

(see [9]).

P r o o f of Theorem 2. Let $\{(\boldsymbol{u}_n, p_n)\}$, where $(\boldsymbol{u}_n, p_n)$ solves $(\mathcal{P}(\alpha_n))$, be a minimizing sequence in $(\mathbb{P})$. Since $\{(\pi_{\alpha_n} \boldsymbol{u}_n, p_n^0)\}$ is bounded in $(H^1(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ as follows from Lemma 2, one can find its subsequence (denoted by the same symbol) such that (3.15) holds true. The existence of a solution to $(\mathbb{P})$ is then an easy consequence of (2.9) and Theorem 3. □

4. Shape optimization with the penalized state problem

The aim of this section is to analyse a new shape optimization problem for the Stokes system with threshold slip but with a penalization of the impermeability condition (2.1d). In addition to the notation introduced in the previous sections we denote

$$\widetilde{V}(\alpha) = \{\boldsymbol{v} \in (H^1(\Omega(\alpha)))^2\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma(\alpha)\},$$
$$\widetilde{V}_{\mathrm{div}}(\alpha) = \{\boldsymbol{v} \in \widetilde{V}(\alpha);\ b_\alpha(\boldsymbol{v}, q) = 0\ \forall\, q \in L^2_0(\alpha)\}, \quad \alpha \in \mathcal{U}_{\mathrm{ad}},$$

and define the penalty term

$$c_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_0^1 (\boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^\alpha)(\boldsymbol{v} \circ \alpha \cdot \boldsymbol{\nu}^\alpha)\,\mathrm{d}x_1,$$

where $\boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^\alpha := \boldsymbol{u}(x_1, \alpha(x_1)) \cdot \boldsymbol{\nu}^\alpha(x_1)$, $x_1 \in (0,1)$. This bilinear form will be used to approximate the boundary condition $\boldsymbol{u} \cdot \boldsymbol{\nu}^\alpha = 0$ on $S(\alpha)$.

Let $\alpha \in \mathcal{U}_{\mathrm{ad}}$ be fixed and $\varepsilon > 0$ be a penalty parameter. The penalized form of $(\mathcal{P}(\alpha))$ reads as follows

$(\mathcal{P}(\alpha)_\varepsilon)$ Find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \widetilde{V}(\alpha) \times L^2_0(\alpha)$ such that

$$\begin{aligned}
&\forall\, \boldsymbol{v} \in \widetilde{V}(\alpha)\colon\ a_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) - b_\alpha(\boldsymbol{v} - \boldsymbol{u}_\varepsilon, p_\varepsilon)\\
&\qquad + j_\alpha(\boldsymbol{v}_\tau) - j_\alpha(\boldsymbol{u}_{\varepsilon\tau}) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon) \geqslant (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_{0,\Omega(\alpha)},\\
&\forall\, q \in L^2_0(\alpha)\colon\ b_\alpha(\boldsymbol{u}_\varepsilon, q) = 0.
\end{aligned}$$

Using the same technique as in [6] one can show that $(\mathcal{P}(\alpha)_\varepsilon)$ has a unique solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ for any $\varepsilon > 0$. Moreover,

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } (H^1(\Omega(\alpha)))^2, \tag{4.1a}$$
$$p_\varepsilon \rightharpoonup p \quad \text{in } L^2_0(\alpha),\ \varepsilon \to 0+ \tag{4.1b}$$

and $(\boldsymbol{u}, p)$ is the unique solution of $(\mathcal{P}(\alpha))$.

Now we introduce the following family of shape optimization problems with the state problem $(\mathcal{P}(\alpha)_\varepsilon)$. For any $\varepsilon > 0$ fixed, we define

$(\mathbb{P}_\varepsilon)$ Find $\alpha^*_\varepsilon \in \mathcal{U}_{\mathrm{ad}}$ such that $\forall\, \alpha \in \mathcal{U}_{\mathrm{ad}}\colon\ \mathfrak{J}_\varepsilon(\alpha^*_\varepsilon) \leqslant \mathfrak{J}_\varepsilon(\alpha)$,

where $\mathfrak{J}_\varepsilon(\alpha) := J(\alpha, \boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ with $(\boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ being the solution of $(\mathcal{P}(\alpha)_\varepsilon)$. Using a similar approach as in Section 3 (see also [9]) one can prove the following result.

Theorem 4. *Let* (2.9) *be satisfied. Then* $(\mathbb{P}_\varepsilon)$ *has a solution for any* $\varepsilon > 0$.

In the subsequent part of this section we shall analyse the mutual relation between solutions of $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$. We start with the following result.

Lemma 4. *There exists a constant* $c := c(\|\boldsymbol{f}\|_{0,\widehat{\Omega}}) > 0$ *independent of* $\alpha \in \mathcal{U}_{\mathrm{ad}}$ *and* $\varepsilon > 0$ *such that the solution* $(\boldsymbol{u}_\varepsilon(\alpha), p_\varepsilon(\alpha))$ *of* $(\mathcal{P}(\alpha)_\varepsilon)$ *is bounded:*

$$\|\pi_\alpha \boldsymbol{u}_\varepsilon(\alpha)\|_{1,\widehat{\Omega}} + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon(\alpha), \boldsymbol{u}_\varepsilon(\alpha)) + \|p^0_\varepsilon(\alpha)\|_{0,\widehat{\Omega}} \leqslant c. \tag{4.2}$$

P r o o f. The boundedness of the first two terms in (4.2) follows easily from the fact that $\boldsymbol{u}_\varepsilon(\alpha) \in \widetilde{V}_{\mathrm{div}}(\alpha)$ and satisfies

$$\begin{aligned} &a_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + j_\alpha(\boldsymbol{u}_{\varepsilon\tau}) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) \\ &\qquad \leqslant a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) + j_\alpha(\boldsymbol{v}_\tau) + \frac{1}{\varepsilon} c_\alpha(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_{0,\Omega(\alpha)} \quad \forall\, \boldsymbol{v} \in \widetilde{V}_{\mathrm{div}}(\alpha), \end{aligned} \tag{4.3}$$

making use of the definitions of $(\mathcal{P}(\alpha)_\varepsilon)$ and $\widetilde{V}_{\mathrm{div}}(\alpha)$. Inserting $\boldsymbol{v} \equiv \boldsymbol{0}$ into the right-hand side of (4.3) we obtain the claim. To show the boundedness of $\{p_\varepsilon(\alpha)\}$ we proceed as follows: From the inequality in $(\mathcal{P}(\alpha)_\varepsilon)$ we see that

$$b_\alpha(\boldsymbol{v}, p_\varepsilon(\alpha)) \leqslant a_\alpha(\boldsymbol{u}_\varepsilon(\alpha), \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in (H^1_0(\Omega(\alpha)))^2.$$

Thus (see also Lemma 2)

$$\bar{c}\|p_\varepsilon\|_{0,\Omega(\alpha)} \leqslant \sup_{\substack{\boldsymbol{v} \in (H^1_0(\Omega(\alpha)))^2 \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{b_\alpha(\boldsymbol{v}, p_\varepsilon)}{\|\boldsymbol{v}\|_{1,\Omega(\alpha)}} \leqslant c,$$

making use of the boundedness of $\{\|\boldsymbol{u}_\varepsilon(\alpha)\|_{1,\Omega(\alpha)}\}$. Since also $\bar{c}$ does not depend on $\alpha \in \mathcal{U}_{\mathrm{ad}}$ and $\varepsilon > 0$, we arrive at (4.2). □

The key role in our analysis plays the following stability type result.

Lemma 5. *Let* $\alpha_n \to \alpha$ *in* $C^1([0,1])$, $\alpha_n, \alpha \in \mathcal{U}_{\mathrm{ad}}$ *and* $\{(\boldsymbol{u}_n, p_n)\}$ *be the sequence of solutions to* $(\mathcal{P}(\alpha_n)_{\varepsilon_n})$, $\varepsilon_n \to 0+$. *Then there exist a subsequence of* $\{(\boldsymbol{u}_n, p_n)\}$ *(denoted by the same symbol) and a pair* $(\bar{u}, \bar{p}) \in (H^1_0(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ *such that*

$$\pi_{\alpha_n} \boldsymbol{u}_n \rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2, \tag{4.4a}$$

$$p^0_n \rightharpoonup \bar{p} \quad \text{in } L^2_0(\widehat{\Omega}),\ n \to \infty. \tag{4.4b}$$

In addition, the pair $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ *is a solution of* $(\mathcal{P}(\alpha))$.

P r o o f. The existence of a subsequence satisfying (4.4) follows from Lemma 4. Clearly, $\bar{\boldsymbol{u}}|_{\Omega(\alpha)} \in \widetilde{V}_{\rm div}(\alpha)$. Next we show that $\boldsymbol{u} := \bar{\boldsymbol{u}}|_{\Omega(\alpha)}$ satisfies (2.1d) on $S(\alpha)$. From (4.2) we see that

$$0 \leqslant c_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \leqslant \varepsilon_n c \to 0 \quad \text{as } n \to \infty, \tag{4.5}$$

where for brevity $c_n := c_{\alpha_n}$. On the other hand,

$$c_n(\boldsymbol{u}_n, \boldsymbol{u}_n) \to c_\alpha(\boldsymbol{u}, \boldsymbol{u}) \quad \text{as } n \to \infty. \tag{4.6}$$

Indeed,

$$\begin{aligned} &\|\boldsymbol{u}_n \circ \alpha_n \cdot \boldsymbol{\nu}^{\alpha_n} - \boldsymbol{u} \circ \alpha \cdot \boldsymbol{\nu}^{\alpha}\|_{0,(0,1)} \\ &\leqslant \|(\boldsymbol{u}_n \circ \alpha_n - \boldsymbol{u} \circ \alpha) \cdot \boldsymbol{\nu}^{\alpha_n}\|_{0,(0,1)} + \|\boldsymbol{u} \circ \alpha(\boldsymbol{\nu}^{\alpha_n} - \boldsymbol{\nu}^{\alpha})\|_{0,(0,1)} \to 0, \quad n \to \infty. \end{aligned} \tag{4.7}$$

Convergence of the first term on the right of (4.7) is shown in [9], Lemma 2.21. From (4.5) and (4.6) it follows that $\boldsymbol{u} \cdot \boldsymbol{\nu}^\alpha = 0$ on $S(\alpha)$, hence $\boldsymbol{u} \in V_{\rm div}(\alpha)$.

It remains to show that $\boldsymbol{u}$ solves $(\mathcal{P}(\alpha))$. Let $\overline{\boldsymbol{v}} \in V(\alpha)$ be given. Then accordingly to Lemma 3 there exists a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ satisfying (3.7) and (3.8). Since $\boldsymbol{v}_k|_{\Omega(\alpha_{n_k})}$ can be used as a test function in $(\mathcal{P}(\alpha_{n_k})_{\varepsilon_{n_k}})$, we obtain:

$$a_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) - b_{n_k}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) + j_{n_k}(\boldsymbol{v}_{k\tau}) - j_{n_k}(\boldsymbol{u}_{n_k}) \geqslant (\boldsymbol{f}, \boldsymbol{v}_k)_{0,\Omega(\alpha_{n_k})}.$$

Here we used the fact that

$$\frac{1}{\varepsilon_{n_k}} c_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) = -\frac{1}{\varepsilon_{n_k}} c_{n_k}(\boldsymbol{u}_{n_k}, \boldsymbol{u}_{n_k}) \leqslant 0.$$

The rest of the proof is identical with the one of Theorem 3. $\square$

To establish a relation between solutions of $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$ we shall also need the continuity of J in the following sense

$$\left.\begin{aligned} &\alpha_n \to \alpha \quad \text{in } C^1([0,1]),\ \alpha_n, \alpha \in \mathcal{U}_{\rm ad} \\ &\boldsymbol{y}_n \to \boldsymbol{y} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_n, \boldsymbol{y} \in (H^1_0(\widehat{\Omega}))^2 \\ &q_n \rightharpoonup q \quad \text{in } L^2(\widehat{\Omega}),\ q_n, q \in L^2_0(\widehat{\Omega}) \end{aligned}\right\} \tag{4.8}$$
$$\Rightarrow \lim_{n\to\infty} J(\alpha_n, \boldsymbol{y}_n|_{\Omega(\alpha_n)}, q_n|_{\Omega(\alpha_n)}) = J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}).$$

Theorem 5. *Let (2.9) and (4.8) be satisfied. Then from any sequence $\{\alpha_\varepsilon^*\}$ of solutions to $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$, one can choose a subsequence (denoted by the same symbol) and find a triplet $(\alpha^*, \boldsymbol{u}^*, p^*) \in \mathcal{U}_{\mathrm{ad}} \times (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega})$ such that*

$$\alpha_\varepsilon^* \to \alpha^* \quad \text{in } C^1([0,1]), \tag{4.9a}$$

$$\pi_{\alpha_\varepsilon^*} \boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*) \rightharpoonup \boldsymbol{u}^* \quad \text{in } (H^1(\widehat{\Omega}))^2, \tag{4.9b}$$

$$p_\varepsilon^0(\alpha_\varepsilon^*) \rightharpoonup p^* \quad \text{in } L_0^2(\widehat{\Omega}),\ \varepsilon \to 0+. \tag{4.9c}$$

Moreover, α^* is a solution of $(\mathbb{P})$ and $(\boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)})$ solves $(\mathcal{P}(\alpha^*))$. Besides that, any accumulation point of $\{(\alpha_\varepsilon^*, \boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*), p_\varepsilon(\alpha_\varepsilon^*))\}$ in the sense of (4.9) has this property.

P r o o f. The existence of a subsequence $\{\alpha_\varepsilon^*\}$ satisfying (4.9a) follows from the Arzelà-Ascoli theorem. Furthermore, (4.9b), (4.9c), and the fact that $(\boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)})$ solves $(\mathcal{P}(\alpha^*))$ are proven in Lemma 5. Let $\overline{\alpha} \in \mathcal{U}_{\mathrm{ad}}$ be given and $(\boldsymbol{u}(\overline{\alpha}), p(\overline{\alpha}))$ be the unique solution of $(\mathcal{P}(\overline{\alpha}))$. From (4.1) we know that

$$\begin{aligned}
\boldsymbol{u}_\varepsilon(\overline{\alpha}) &\to \boldsymbol{u}(\overline{\alpha}) \quad \text{in } (H^1(\Omega(\overline{\alpha})))^2,\\
p_\varepsilon(\overline{\alpha}) &\rightharpoonup p(\overline{\alpha}) \quad \text{in } L_0^2(\Omega(\overline{\alpha})),\ \varepsilon \to 0+
\end{aligned}$$

and also

$$\begin{aligned}
\pi_{\overline{\alpha}} \boldsymbol{u}_\varepsilon(\overline{\alpha}) &\to \pi_{\overline{\alpha}} \boldsymbol{u}(\overline{\alpha}) \quad \text{in } (H^1(\widehat{\Omega}))^2,\\
p_\varepsilon^0(\overline{\alpha}) &\rightharpoonup p^0(\overline{\alpha}) \quad \text{in } L_0^2(\widehat{\Omega}),\ \varepsilon \to 0+.
\end{aligned} \tag{4.10}$$

The definition of $(\mathbb{P}_\varepsilon)$ yields

$$J(\alpha_\varepsilon^*, \boldsymbol{u}_\varepsilon(\alpha_\varepsilon^*), p_\varepsilon(\alpha_\varepsilon^*)) \leqslant J(\overline{\alpha}, \boldsymbol{u}_\varepsilon(\overline{\alpha}), p_\varepsilon(\overline{\alpha})).$$

Letting ε tend to zero on the filter of indices for which (4.9) holds, we obtain

$$J(\alpha^*, \boldsymbol{u}^*|_{\Omega(\alpha^*)}, p^*|_{\Omega(\alpha^*)}) \leqslant J(\overline{\alpha}, \boldsymbol{u}(\overline{\alpha}), p(\overline{\alpha}))\ \forall\, \overline{\alpha} \in \mathcal{U}_{\mathrm{ad}},$$

making use of (2.9), (4.8), and (4.10). $\square$

5. Approximation of $(\mathbb{P}_\varepsilon)$

In this section, a finite-dimensional approximation of $(\mathbb{P}_\varepsilon)$ will be proposed and analysed. Next we shall assume that $\varepsilon > 0$ is fixed. We introduce a finite element discretization of $(\mathcal{P}(\alpha)_\varepsilon)$ and a discretization of the set $\mathcal{U}_{\text{ad}}$. We will show that the discrete shape optimization problem has a solution. Finally, we will study convergence properties of such solutions if the discretization parameter $h \to 0+$.

5.1. Formulation of the discrete problem. We start with the approximation of the admissible set $\mathcal{U}_{\text{ad}}$. Since for finite element methods it is convenient to use polygonal domains, we will consider piecewise linear approximations of $\mathcal{U}_{\text{ad}}$. On the other hand, as $\mathcal{U}_{\text{ad}}$ contains $C^{1,1}$-functions, this approximation of $\mathcal{U}_{\text{ad}}$ becomes external and some technical difficulties arise, especially in the convergence analysis.

Let $d \in \mathbb{N}$ be given and set $h := 1/d$. By δ_h we denote the equidistant partition of $[0,1]$:

$$\delta_h\colon\ 0 = a_0 < a_1 < \ldots < a_d = 1,$$

where

$$a_j = jh,\ j = 0,1,\ldots,d.$$

The set of discrete admissible shapes $\mathcal{U}^h_{\text{ad}}$ consists of continuous, piecewise linear functions on δ_h which satisfy constraints analogous to those imposed in (2.8):

$$\begin{aligned}
\mathcal{U}^h_{\text{ad}} := \{\alpha_h \in C([0,1]);\ & \alpha_{h|[a_{i-1},a_i]} \in P_1([a_{i-1},a_i])\ \forall\, i = 1,\ldots,d;\\
& \alpha_{\min} \leqslant \alpha_h(a_i) \leqslant \alpha_{\max}\ \forall\, i = 0,\ldots,d;\\
& |\alpha_h(a_i) - \alpha_h(a_{i-1})| \leqslant C_1 h\ \forall\, i = 1,\ldots,d;\\
& |\alpha_h(a_{i+1}) - 2\alpha_h(a_i) + \alpha_h(a_{i-1})| \leqslant C_2 h^2\ \forall\, i = 1,\ldots,d-1\}.
\end{aligned}$$

The positive constants $\alpha_{\min}$, $\alpha_{\max}$, C_1 and C_2 are the same as in (2.8). We denote the set of discrete admissible shapes by

$$\mathcal{O}_h := \{\Omega(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\text{ad}}\}.$$

The symbol $\mathcal{T}_h(\alpha_h)$ will denote a triangulation of $\overline{\Omega}(\alpha_h)$ with the norm h. We will consider the system $\{\mathcal{T}_h(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\text{ad}}\}$ which consists of *topologically equivalent triangulations*, i.e.:

(T1) the number of nodes as well as the neighbours of each triangle in $\mathcal{T}_h(\alpha_h)$ is the same for all $\alpha_h \in \mathcal{U}^h_{\text{ad}}$;

(T2) the position of the nodes in $\mathcal{T}_h(\alpha_h)$ depends continuously on α_h;

(T3) the triangulations $\mathcal{T}_h(\alpha_h)$ are compatible with the decomposition of $\partial\Omega(\alpha_h)$ into $S(\alpha_h)$ and $\Gamma(\alpha_h)$ for any $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$.

In order to establish convergence results we will also need:

(T4) the system $\{\mathcal{T}_h(\alpha_h);\ \alpha_h \in \mathcal{U}^h_{\mathrm{ad}}\}$ is *uniformly regular* with respect to $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$, i.e., there exists a constant $\theta_0 > 0$ such that

$$\theta_h(\alpha_h) \geqslant \theta_0 \ \forall\, h > 0 \ \forall\, \alpha_h \in \mathcal{U}^h_{\mathrm{ad}},$$

where $\theta_h(\alpha_h)$ denotes the minimal interior angle of all triangles from $\mathcal{T}_h(\alpha_h)$.

In order to give a finite element discretization of the state problem, we define the spaces of piecewise polynomial functions

$$\widetilde{V}_h(\alpha_h) := \{\boldsymbol{v}_h \in (C(\overline{\Omega}(\alpha_h)))^2;\ \boldsymbol{v}_{h|T} \in (P_2(T))^2 \ \forall T \in \mathcal{T}_h(\alpha_h),\ \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma(\alpha_h)\},$$

$$L_h(\alpha_h) := \left\{q_h \in C(\overline{\Omega}(\alpha_h));\ q_{h|T} \in P_1(T) \ \forall T \in \mathcal{T}_h(\alpha_h),\ \int_{\Omega(\alpha_h)} q_h = 0\right\}.$$

Let $\varepsilon > 0$, $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$ be given. The discrete penalized state problem reads as follows:

$$(\mathcal{P}_{h\varepsilon}(\alpha_h)) \qquad \begin{aligned} &\text{Find } (\boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon}) := (\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)) \in \widetilde{V}_h(\alpha_h) \times L_h(\alpha_h) \text{ s.t.}\\ &\forall\, \boldsymbol{v}_h \in \widetilde{V}_h(\alpha_h):\ a_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon}) - b_{\alpha_h}(\boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon})\\ &\qquad\qquad + j_{\alpha_h}(\boldsymbol{v}_{h\tau}) - j_{\alpha_h}(\boldsymbol{u}_{h\varepsilon\tau}) + \frac{1}{\varepsilon} c_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon})\\ &\qquad\qquad \geqslant (\boldsymbol{f}, \boldsymbol{v}_h - \boldsymbol{u}_{h\varepsilon})_{0,\Omega(\alpha_h)},\\ &\forall\, q_h \in L_h(\alpha_h):\ b_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, q_h) = 0. \end{aligned}$$

Since the pair $\widetilde{V}_h(\alpha_h)$ and $L_h(\alpha_h)$ satisfies the Babuška-Brezzi condition (see (5.2) below), problem $\mathcal{P}_{h\varepsilon}(\alpha_h)$ has a unique solution.

Lemma 6. *There exists a constant $c := c(\|\boldsymbol{f}\|_{0,\widehat{\Omega}}) > 0$ independent of $\varepsilon > 0$, $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$ such that the solution $(\boldsymbol{u}_{h\varepsilon}, p_{h\varepsilon})$ of $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ is bounded:*

$$\|\pi_{\alpha_h}\boldsymbol{u}_{h\varepsilon}\|_{1,\widehat{\Omega}} + \frac{1}{\varepsilon} c_{\alpha_h}(\boldsymbol{u}_{h\varepsilon}, \boldsymbol{u}_{h\varepsilon}) + \|p^0_{h\varepsilon}\|_{0,\widehat{\Omega}} \leqslant c. \tag{5.1}$$

P r o o f. The boundedness of the first two terms in (5.1) can be shown exactly as in the proof of Lemma 4. The pressure estimate will be proven provided that the discrete inf-sup condition

$$\inf_{q \in L_h(\alpha_h)\setminus\{0\}} \ \sup_{\boldsymbol{v} \in \widetilde{V}_h(\alpha_h)\setminus\{\boldsymbol{0}\}} \frac{b_{\alpha_h}(q, \boldsymbol{v})}{\|q\|_{0,\Omega(\alpha_h)}\|\boldsymbol{v}\|_{1,\Omega(\alpha_h)}} \geqslant c \tag{5.2}$$

holds with a constant $c > 0$ independent of $h > 0$ and $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$. Indeed, in [2], Chapter VI.6, it is shown that (5.2) holds with a constant $c := c(\bar{c})$, where $\bar{c}$ is the constant in the inf-sup condition for the spaces $L_0^2(\alpha_h)$ and $\widetilde{V}(\alpha_h)$. As we have pointed out before, $\bar{c}$ does not depend on α_h, and so neither does c. □

Analogously to the continuous setting, the discrete shape optimization problem is defined as the minimization of $\mathfrak{J}_{h\varepsilon}$ on $\mathcal{U}_{\mathrm{ad}}^h$, where

$$\mathfrak{J}_{h\varepsilon}(\alpha_h) := J(\alpha_h, \boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)),$$

with $(\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h))$ being the solution of $(\mathcal{P}_{h\varepsilon}(\alpha_h))$. Thus, for each $\varepsilon > 0$ and $h > 0$, the discrete shape optimization problem reads:

$$(\mathbb{P}_{h\varepsilon}) \qquad \text{Find } \alpha_{h\varepsilon}^* \in \mathcal{U}_{\mathrm{ad}}^h \text{ such that } \forall\, \alpha_h \in \mathcal{U}_{\mathrm{ad}}^h\colon\ \mathfrak{J}_{h\varepsilon}(\alpha_{h\varepsilon}^*) \leqslant \mathfrak{J}_{h\varepsilon}(\alpha_h).$$

Adapting the approach from the previous section to the discrete case, one can easily show that the graph

$$\begin{aligned}\mathcal{G}_{h\varepsilon} := \{&(\alpha_h, \boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h));\ \alpha_h \in \mathcal{U}_{\mathrm{ad}}^h,\\ &(\boldsymbol{u}_{h\varepsilon}(\alpha_h), p_{h\varepsilon}(\alpha_h)) \text{ is the solution of } \mathcal{P}_{h\varepsilon}(\alpha_h)\}\end{aligned}$$

is compact for any $\varepsilon > 0$ and $h > 0$, so the following result is straightforward.

Theorem 6. *Let $h, \varepsilon > 0$ be fixed and $\mathfrak{J}_{h\varepsilon}$ be lower semicontinuous on $\mathcal{U}_{\mathrm{ad}}^h$. Then $(\mathbb{P}_{h\varepsilon})$ has a solution.*

5.2. Convergence analysis. In this section we will analyse the mutual relation between solutions to $(\mathbb{P}_{h\varepsilon})$ and $(\mathbb{P}_{\varepsilon})$ as $h \to 0+$ keeping $\varepsilon > 0$ fixed, aiming to show that the discrete optimal shapes converge in some sense to an optimal shape of the continuous setting.

We start by recalling some auxiliary results concerning the relationship between $\mathcal{U}_{\mathrm{ad}}^h$, $h \to 0+$, and $\mathcal{U}_{\mathrm{ad}}$, which can be proven using the same arguments as in [10], [11].

Lemma 7. *For any $\alpha \in \mathcal{U}_{\mathrm{ad}}$ there exists a sequence $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$ such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$.*

Lemma 8. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}_{\mathrm{ad}}^h$ be such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$. Then $\alpha \in \mathcal{U}_{\mathrm{ad}}$ and there exists a subsequence $\{\alpha_{h_m}\} \subset \{\alpha_h\}$ satisfying:*

$$\alpha_{h_m}' \to \alpha' \quad \text{in } L^\infty(0,1), \quad h_m \to 0+. \tag{5.3}$$

In order to pass to the limit in the variational inequality we also need the following result.

Lemma 9. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$ be such that $\alpha_h \to \alpha$ in $C([0,1])$, $h \to 0+$ and let $\boldsymbol{v} \in \widetilde{V}(\alpha)$ be given. Then there exist a sequence $\{\boldsymbol{v}_h\}$, $\boldsymbol{v}_h \in (H^1(\widehat{\Omega}))^2$, and a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$ such that $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$, $\overline{\boldsymbol{v}}|_{\Omega(\alpha)} = \boldsymbol{v}$ and*

$$\boldsymbol{v}_h \to \overline{\boldsymbol{v}} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ h \to 0+. \tag{5.4}$$

P r o o f. Let $\eta > 0$ be arbitrary and set $\overline{\boldsymbol{v}} := \pi_\alpha \boldsymbol{v} \in (H^1_0(\widehat{\Omega}))^2$. By the density argument one can find $\boldsymbol{\varphi} \in (C^\infty_0(\widehat{\Omega}))^2$ such that

$$\|\boldsymbol{\varphi} - \overline{\boldsymbol{v}}\|_{1,\widehat{\Omega}} < \frac{\eta}{2}. \tag{5.5}$$

Let $\Theta(\alpha_h) = \widehat{\Omega} \setminus \overline{\Omega}(\alpha_h)$ and $\widehat{\mathcal{T}}_h(\alpha_h)$ be a triangulation of $\overline{\Theta}(\alpha_h)$ such that the nodes of $\mathcal{T}_h(\alpha_h)$ and $\widehat{\mathcal{T}}_h(\alpha_h)$ on $S(\alpha_h)$ coincide and, moreover, the family $\{\widehat{\mathcal{T}}_h(\alpha_h)\}$, $h \to 0$, satisfies (T1), (T2) and (T4). By r_h we denote the piecewise quadratic Lagrange interpolation operator in $\overline{\widehat{\Omega}}$ with the triangulation $\mathcal{T}_h(\alpha_h) \cup \widehat{\mathcal{T}}_h(\alpha_h)$. From (T4) it follows that there exists a constant $c > 0$ independent of $h > 0$ and $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$ such that

$$\|r_h\boldsymbol{\varphi} - \boldsymbol{\varphi}\|_{1,\widehat{\Omega}} \leqslant ch\|\boldsymbol{\varphi}\|_{2,\widehat{\Omega}} \quad \forall \boldsymbol{\varphi} \in (H^2(\widehat{\Omega}))^2. \tag{5.6}$$

We set $\boldsymbol{v}_h := r_h\boldsymbol{\varphi}$. Then clearly $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$ for every $h > 0$. Moreover, from (5.6) we see that there exists $h_0 := h_0(\eta) > 0$ such that for any $h \leqslant h_0$ it holds that

$$\|\boldsymbol{v}_h - \boldsymbol{\varphi}\|_{1,\widehat{\Omega}} < \frac{\eta}{2},$$

which together with (5.5) completes the proof. □

The following lemma establishes convergence properties of solutions to $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ as $h \to 0+$.

Lemma 10. *Let $\{\alpha_h\}$, $\alpha_h \in \mathcal{U}^h_{\mathrm{ad}}$, $h \to 0+$, be an arbitrary sequence. Then there exist its subsequence (denoted by the same symbol), a function $\alpha \in \mathcal{U}_{\mathrm{ad}}$, and a pair $(\bar{\boldsymbol{u}}, \bar{p}) \in (H^1_0(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ such that*

$$\begin{aligned} \alpha_h &\to \alpha \quad \text{in } C([0,1]),\\ \pi_{\alpha_h}\boldsymbol{u}_{h\varepsilon}(\alpha_h) &\rightharpoonup \bar{\boldsymbol{u}} \quad \text{in } (H^1(\widehat{\Omega}))^2,\\ p_{h\varepsilon}(\alpha_h)^0 &\rightharpoonup \bar{p} \quad \text{in } L^2(\widehat{\Omega}),\ h \to 0+. \end{aligned}$$

Moreover, $(\bar{\boldsymbol{u}}|_{\Omega(\alpha)}, \bar{p}|_{\Omega(\alpha)})$ is the solution to $(\mathcal{P}(\alpha)_\varepsilon)$.

P r o o f. The existence of convergent subsequences follows from Lemma 6, the Arzelà-Ascoli theorem and Lemma 8. From Lemma 9 we know that for any $\boldsymbol{v} \in \widetilde{V}(\alpha)$ one can find a sequence $\{\boldsymbol{v}_h\}$, $\boldsymbol{v}_h|_{\Omega(\alpha_h)} \in \widetilde{V}_h(\alpha_h)$ satisfying (5.4). The limit passage for $h \to 0+$ in $(\mathcal{P}_{h\varepsilon}(\alpha_h))$ can be done as in the proof of Theorem 3, making use of (5.3). $\square$

To establish the convergence of solutions to $(\mathbb{P}_{h\varepsilon})$ as $h \to 0+$ we shall need the continuity of J in the following sense:

$$(5.7)\qquad \left.\begin{aligned} \alpha_h &\to \alpha \quad \text{in } C([0,1]),\ \alpha_h \in \mathcal{U}^h_{\rm ad}, \alpha \in \mathcal{U}_{\rm ad} \\ \pi_{\alpha_h}\boldsymbol{y}_h &\rightharpoonup \boldsymbol{y} \quad \text{in } (H^1(\widehat{\Omega}))^2,\ \boldsymbol{y}_h \in \widetilde{V}_h(\alpha_h), \boldsymbol{y} \in (H^1_0(\widehat{\Omega}))^2 \\ q^0_h &\rightharpoonup q \quad \text{in } L^2(\widehat{\Omega}),\ q_h \in L_h(\alpha_h), q \in L^2_0(\widehat{\Omega}) \end{aligned}\right\} \Rightarrow \lim_{h\to 0+} J(\alpha_h, \boldsymbol{y}_h, q_h) = J(\alpha, \boldsymbol{y}|_{\Omega(\alpha)}, q|_{\Omega(\alpha)}).$$

We have the following convergence result.

Theorem 7. *Let $\{\alpha^*_{h\varepsilon}\}$, $h \to 0+$, be a sequence of solutions to $(\mathbb{P}_{h\varepsilon})$, $h \to 0+$, and let (5.7) be satisfied. Then there exist: a subsequence of $\{\alpha^*_{h\varepsilon}\}$ (denoted by the same symbol) and a triplet $(\alpha^*_\varepsilon, \boldsymbol{u}^*_\varepsilon, p^*_\varepsilon) \in \mathcal{U}_{\rm ad} \times (H^1_0(\widehat{\Omega}))^2 \times L^2_0(\widehat{\Omega})$ such that*

$$\begin{aligned} \alpha^*_{h\varepsilon} &\to \alpha^*_\varepsilon \quad \text{in } C([0,1]), \\ \pi_{\alpha^*_{h\varepsilon}}\boldsymbol{u}_{h\varepsilon}(\alpha^*_{h\varepsilon}) &\rightharpoonup \boldsymbol{u}^*_\varepsilon \quad \text{in } (H^1(\widehat{\Omega}))^2, \\ p^0_{h\varepsilon}(\alpha^*_{h\varepsilon}) &\rightharpoonup p^*_\varepsilon \quad \text{in } L^2(\widehat{\Omega}),\ h \to 0+. \end{aligned}$$

Moreover, α^*_ε is a solution of $(\mathbb{P}_\varepsilon)$ and $(\boldsymbol{u}^*_\varepsilon|_{\Omega(\alpha^*_\varepsilon)}, p^*_\varepsilon|_{\Omega(\alpha^*_\varepsilon)})$ solves $(\mathcal{P}_\varepsilon(\alpha^*_\varepsilon))$.

The p r o o f is analogous to the one of Theorem 5.

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Stokes problem with a solution dependent slip bound: Stability of solutions with respect to domains

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We study the Stokes problem in a bounded planar domain Ω with a friction type boundary condition that switches between a slip and no-slip stage. Unlike our previous work [8], in the present paper the threshold value may depend on the velocity field. Besides the usual velocity-pressure formulation, we introduce an alternative formulation with three Lagrange multipliers which allows a more flexible treatment of the impermeability condition as well as optimum design problems with cost functions depending on the shear and/or normal stress. Our main goal is to determine under which conditions concerning smoothness of the boundary of Ω, solutions to the Stokes system depend continuously on variations of Ω. Having this result at our disposal, we easily prove the existence of a solution to optimal shape design problems for a large class of cost functionals.

1 Introduction

This paper analyses one property of the Stokes system defined in $\Omega \subset \mathbb{R}^2$ with a slip type boundary condition, namely the continuous dependence of its solutions on the shape of Ω. This property plays the crucial role in the existence analysis of optimal shape design problems. The no-slip boundary condition, i.e. the vanishing velocity on the boundary, is widely used in practice. It characterizes the adhesion of the fluid on the solid wall. This condition is acceptable for small velocities and on a macroscopic level. On the other hand, there are many situations (flow of the fluid on hydrophobic surfaces, polymer melts flow, problems with multiple interfaces, micro/nanofluidics etc.) where the slip of the fluid occurs. To get a more realistic model, the slip has to be taken into account. For the physical justification of different types of slip conditions we refer to [14] and [9]. The mathematical analysis of the Stokes and Navier-Stokes system with the slip and leak boundary conditions has been done in [3] and extended to non-stationary problems in [4]. The regularity of solutions to the Stokes system with slip and leak boundary conditions has been established in [15]. In [1] the stick–slip condition is considered as an implicit constitutive equation on the boundary, having a monotone 2-graph property, and the existence of weak solutions to Bingham and Navier-Stokes fluids is proven.

Shape optimization involving fluid models with slip boundary conditions as the state problem is of a great practical importance. Slip boundary conditions affect the velocity profile and hence the velocity gradient of the fluid in the vicinity of the wall. The velocity gradient is an important factor in the transformation of the mechanical energy to heat, the process representing the energy loss. Shape optimization of the interior of hydraulic elements may reduce the velocity gradient resulting in energy savings. In [8] a class of shape optimization problems for the Stokes system with the threshold boundary conditions involving a priori given slip bound has been studied. The existence result for the continuous setting of the problem and convergence analysis for appropriate discretizations of the continuous model have been established.

Nevertheless it is known from experiments that the slip bound may depend on the solution itself, e.g. on values of the tangential component of the velocity. The aim of this paper is to extend the existing stability results to this type of the slip boundary condition. Besides the standard velocity-pressure formulation used in [8] we present a new weak formulation adding another two Lagrange multipliers: one releasing the impermeability condition and the other regularizing the non-smooth slip functional. This new formulation turns out to be useful in numerical solution of this problem. Moreover, it

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enables us to approximate directly the normal and shear stress and to use these quantities as arguments of appropriate objective functionals to control the stress distribution along the slip part of the boundary.

The paper is organized as follows: in Sect. 2 we present the velocity-pressure formulation of the Stokes system with a solution dependent slip bound. Using fixed point arguments we prove that such problem has at least one solution for any slip bound represented by a continuous, positive function g having a polynomial growth. If in addition, g is one-sided Lipschitz continuous with sufficiently small modulus, then the solution is unique. Section 3 deals with a four-field formulation of the problem whose solution is represented by the velocity u, pressure p, normal, tangential shear stress σ^ν, and σ^τ, respectively. In Sect. 4 we prove that the graph of the respective generally multi-valued solution mappings considered as a function of the shape of the slip part of the boundary, is closed in an appropriate topology. On the basis of these results the existence of solutions to a class of optimal shape design problems will be proven in Sect. 5.

2 The velocity-pressure formulation of the problem

Unlike [8], where the slip bound was given, the present paper deals with a more general case, namely the slip bound will be a function of the tangential velocity.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$. The slip boundary conditions are prescribed on an open, non-empty part S of the boundary and the no-slip condition on $\Gamma = \partial\Omega \setminus \overline{S}$, $\Gamma \neq \emptyset$:

$$-\operatorname{div}(2\mu\mathbb{D}\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.1c}$$

$$u_\nu = 0 \quad \text{on } S, \tag{2.1d}$$

$$|\sigma^\tau| \leq g(|u_\tau|) \quad \text{on } S, \tag{2.1e}$$

$$g(|u_\tau|)u_\tau = -|u_\tau|\sigma^\tau \quad \text{on } S. \tag{2.1f}$$

Here $\mu > 0$ is the (constant) viscosity, $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure, $\mathbb{D}(\boldsymbol{u})$ is the symmetric part of the gradient of $\boldsymbol{u}$ and $\boldsymbol{f}$ is the external force. Further, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\boldsymbol{\tau} = (\nu_2, -\nu_1)$ denote the unit outward normal, and tangential vector to $\partial\Omega$, respectively. If $\boldsymbol{a} \in \mathbb{R}^2$ is a vector then $a_\nu := \boldsymbol{a} \cdot \boldsymbol{\nu}$, $a_\tau := \boldsymbol{a} \cdot \boldsymbol{\tau}$ is its normal, and the tangential component on $\partial\Omega$, respectively. Finally, $\sigma^\tau := (2\mu(\mathbb{D}\boldsymbol{u})\boldsymbol{\nu})_\tau$ stands for the shear stress and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given slip bound function. By a classical solution of (2.1) we mean any couple of sufficiently smooth functions $(\boldsymbol{u}, p)$ satisfying the differential equations and the boundary conditions in (2.1).

Remark 2.1.

(i) Note that if μ is constant then $\operatorname{div}(2\mu\mathbb{D}\boldsymbol{v}) = \mu\Delta\boldsymbol{v}$ for any sufficiently smooth $\boldsymbol{v}$ satisfying $\operatorname{div}\boldsymbol{v} = 0$ in Ω. Throughout the paper we shall assume that $2\mu = 1$.

(ii) The condition (2.1f) says that a slip may occur only if the equality holds in (2.1e).

To give the weak formulation of (2.1) we shall need the following function sets:

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2 |\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\ v_\nu = 0 \text{ on } S\}, \tag{2.2}$$

$$\boldsymbol{V}_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}(\Omega) |\ \operatorname{div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega\}, \tag{2.3}$$

$$L^2_0(\Omega) = \{q \in L^2(\Omega) |\ \int_\Omega q = 0\}, \tag{2.4}$$

$$L^2_+(S) = \{\varphi \in L^2(S) |\ \varphi \geq 0 \text{ a.e. on } S\}, \tag{2.5}$$

$$H^{1/2}(S) = \{\varphi \in L^2(S) |\ \exists v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma : v = \varphi \text{ on } S\}, \tag{2.6}$$

$$H^{1/2}_+(S) = \{\varphi \in H^{1/2}(S) |\ \varphi \geq 0 \text{ a.e. on } S\}. \tag{2.7}$$

Remark 2.2. If $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ and $S \in C^{1,1}$ then it is readily seen that $v_{\tau|S} \in H^{1/2}(S)$.

From now on we shall suppose that $S \in C^{1,1}$ (for the definition see [12]*).*

The trace space $H^{1/2}(S)$ is equipped with the norm

$$\|\varphi\|_{1/2,S} = \inf_{\substack{\boldsymbol{v}\in \boldsymbol{V}(\Omega)\\ v_\tau=\varphi}} |\boldsymbol{v}|_{1,\Omega} = |\boldsymbol{w}(\varphi)|_{1,\Omega},$$

where $\boldsymbol{w}(\varphi) \in \boldsymbol{V}(\Omega)$ is the solution to

$$\left.\begin{aligned} \Delta \boldsymbol{w}(\varphi) &= \boldsymbol{0} \text{ in } \Omega,\\ \boldsymbol{w}(\varphi) &= \boldsymbol{0} \text{ on } \Gamma,\\ w_\nu(\varphi) &= 0 \text{ on } S,\\ w_\tau(\varphi) &= \varphi \text{ on } S. \end{aligned}\right\}$$

Further we introduce the following forms:

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbb{D}\boldsymbol{u} : \mathbb{D}\boldsymbol{v}, \quad b(\boldsymbol{v},q) = \int_\Omega q \operatorname{div} \boldsymbol{v}, \quad j(\varphi, v_\tau) = \int_S g(\varphi)|v_\tau|,$$
$$\boldsymbol{u},\boldsymbol{v} \in (H^1(\Omega))^2, q \in L^2(\Omega), \varphi \in H^{1/2}_+(S). \quad (2.8)$$

We shall assume that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is *continuous* and there exists a positive constant c_g and $r \in (1,\infty)$ such that

$$g(x) \le c_g(1 + x^{r-1}) \ \forall x \in \mathbb{R}_+, \quad (2.9)$$

so that the mapping $\varphi \mapsto g(\varphi)$ is bounded and continuous from $L^r_+(S)$ to $L^{r'}_+(S)$, $r' := r/(r-1)$. We also note that $H^{1/2}(S)$ is compactly embedded into $L^q(S)$ for any $q \in [1,\infty)$ (see [12]).

The weak formulation of (2.1) reads as follows:

$$\left.\begin{aligned} &\textit{Find } (\boldsymbol{u}, p) \in \boldsymbol{V}(\Omega)\times L^2_0(\Omega) \textit{ such that}\\ &\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega):\ a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p)\\ &\qquad\qquad + j(|u_\tau|, v_\tau) - j(|u_\tau|, u_\tau) \ge (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_{0,\Omega},\\ &\forall q \in L^2_0(\Omega) : b(\boldsymbol{u}, q) = 0. \end{aligned}\right\} \quad (\mathcal{P})$$

We will show that under the above mentioned assumptions on g, problem $(\mathcal{P})$ has at least one solution for any $\boldsymbol{f} \in (L^2(\Omega))^2$. To this end we use the weak variant of Schauder's fixed point theorem [10].

For a given function $\varphi \in H^{1/2}_+(S)$ we consider the auxiliary problem:

$$\left.\begin{aligned} &\textit{Find } (\boldsymbol{u}^\varphi, p^\varphi) \in \boldsymbol{V}(\Omega)\times L^2_0(\Omega) \textit{ such that}\\ &\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega):\ a(\boldsymbol{u}^\varphi,\boldsymbol{v}-\boldsymbol{u}^\varphi) - b(\boldsymbol{v}-\boldsymbol{u}^\varphi,p^\varphi)\\ &\qquad\qquad + j(\varphi, v_\tau) - j(\varphi, u^\varphi_\tau) \ge (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}^\varphi)_{0,\Omega},\\ &\forall q \in L^2_0(\Omega) : b(\boldsymbol{u}^\varphi, q) = 0. \end{aligned}\right\} \quad (\mathcal{P}^\varphi)$$

We know that for every $\varphi \in H^{1/2}_+(S)$ there exist: a unique solution $(\boldsymbol{u}^\varphi, p^\varphi)$ of $(\mathcal{P}^\varphi)$ and positive constants $\overline{c}$, $\overline{\overline{c}}$ such that

$$\|\nabla \boldsymbol{u}^\varphi\|_{0,\Omega} \le \overline{c}\,\|\boldsymbol{f}\|_{0,\Omega}, \quad (2.10a)$$

$$\|p^\varphi\|_{0,\Omega} \le \overline{\overline{c}}(\|\boldsymbol{f}\|_{0,\Omega} + \|g(\varphi)\|_{r',S}), \quad (2.10b)$$

holds for every $\boldsymbol{f} \in (L^2(\Omega))^2$ and $\varphi \in H^{1/2}(S)$. To prove (2.10) we proceed as in [3] making use of the Korn inequality, the inf-sup condition for the pressure and the growth condition (2.9).

Let us define the mapping $\Psi : H^{1/2}_+(S) \to H^{1/2}_+(S)$ by

$$\Psi(\varphi) = |u^\varphi_\tau| \text{ on } S.$$

Then $(\mathcal{P})$ is equivalent to the problem of finding a fixed point of Ψ in $H^{1/2}_+(S)$.

Theorem 2.3. *The mapping Ψ has the following properties:*

(i) $\Psi(B) \subset B$, *where* $B = \{\varphi \in H^{1/2}_+(S) |\, \|\varphi\|_{1/2,S} \le \overline{c}\}$ *and $\overline{c}$ is the constant from (2.10a).*

(ii) Ψ *is weakly continuous in* $H_+^{1/2}(S)$, *i.e.*

$$\varphi_k \rightharpoonup \varphi \text{ in } H^{1/2}(S), \varphi_k, \varphi \in H_+^{1/2}(S) \Rightarrow \Psi(\varphi_k) \rightharpoonup \Psi(\varphi) \text{ in } H^{1/2}(S).$$

P r o o f . The property *(i)* follows immediately from

$$\| \, |u_\tau^\varphi| \, \|_{1/2,S} \leq \|u_\tau^\varphi\|_{1/2,S} \leq \|\nabla \boldsymbol{u}^\varphi\|_{0,\Omega} \leq \overline{c},$$

making use of (2.10a) and the definition of the norm in $H^{1/2}(S)$.

Let $(\boldsymbol{u}^k, p^k)$ denote the solution to $(\mathcal{P}^{\varphi_k})$. Assume that $\varphi_k \rightharpoonup \varphi$ in $H^{1/2}(S)$ and consequently $\varphi_k \to \varphi$ in $L^q(S)$ $\forall q \in [1, \infty)$. Since the sequence $\{(\boldsymbol{u}^k, p^k)\}$ is bounded in $\boldsymbol{V}(\Omega) \times L_0^2(\Omega)$ as follows from (2.10), there exists a subsequence (denoted by the index k') such that

$$\boldsymbol{u}^{k'} \rightharpoonup \overline{\boldsymbol{u}} \text{ in } (H^1(\Omega))^2, \quad p^{k'} \rightharpoonup \overline{p} \text{ in } L_0^2(\Omega), k' \to \infty.$$

It is easy to show that $(\overline{\boldsymbol{u}}, \overline{p})$ is a solution of $(\mathcal{P}^\varphi)$, i.e. $(\overline{\boldsymbol{u}}, \overline{p}) = (\boldsymbol{u}^\varphi, p^\varphi)$. Indeed,

$$\left.\begin{aligned} &\limsup_{k'\to\infty} a(\boldsymbol{u}^{k'}, \boldsymbol{v} - \boldsymbol{u}^{k'}) \leq a(\overline{\boldsymbol{u}}, \boldsymbol{v} - \overline{\boldsymbol{u}}), \\ &\qquad (\boldsymbol{f}, \boldsymbol{v} - u^{k'})_{0,\Omega} \to (\boldsymbol{f}, \boldsymbol{v} - \overline{\boldsymbol{u}})_{0,\Omega}, \\ &\qquad b(\boldsymbol{v} - \boldsymbol{u}^{k'}, p^{k'}) \to b(\boldsymbol{v} - \overline{\boldsymbol{u}}, \overline{p}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega). \end{aligned}\right\} \tag{2.11}$$

To prove

$$\int_S g(\varphi_{k'})(|v_\tau| - |u_\tau^{k'}|) \to \int_S g(\varphi)(|v_\tau| - |\overline{u}_\tau|), k' \to \infty, \tag{2.12}$$

we use that

$$\varphi_{k'} \to \varphi \text{ in } L^r(S) \Rightarrow g(\varphi_{k'}) \to g(\varphi) \text{ in } L^{r'}(S), \tag{2.13}$$

and

$$|u_\tau^{k'}| \to |\overline{u}_\tau| \text{ in } L^r(S). \tag{2.14}$$

Clearly (2.13) and (2.14) imply (2.12). From (2.11) and (2.12) it follows that $(\overline{\boldsymbol{u}}, \overline{p})$ is a solution to $(\mathcal{P}^\varphi)$. Since this solution is unique, then

$$(\boldsymbol{u}^k, p^k) \rightharpoonup (\overline{\boldsymbol{u}}, \overline{p}) \text{ weakly in } (H^1(\Omega))^2 \times L_0^2(\Omega), k \to \infty,$$

i.e. $(\overline{\boldsymbol{u}}, \overline{p}) = (\boldsymbol{u}^\varphi, p^\varphi)$. Finally

$$\boldsymbol{u}^k \rightharpoonup \boldsymbol{u}^\varphi \text{ in } (H^1(\Omega))^2 \Rightarrow |\boldsymbol{u}^k| \rightharpoonup |\boldsymbol{u}^\varphi| \text{ in } (H^1(\Omega))^2 \Rightarrow |u_\tau^k| \rightharpoonup |u_\tau^\varphi| \text{ in } H^{1/2}(S)$$
$$\Leftrightarrow \Psi(\varphi_k) \rightharpoonup \Psi(\varphi) \text{ in } H^{1/2}(S)$$

proving *(ii)*. □

The weak variant of Schauder's fixed-point theorem and Theorem 2.3 ensure the existence of at least one fixed point of Ψ in $H_+^{1/2}(S)$. Thus $(\mathcal{P})$ has at least one solution.

Next we shall study under which conditions, problem $(\mathcal{P})$ has a unique solution.

Theorem 2.4. *In addition to (2.9), let* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *be one-sided Lipschitz continuous in* $\mathbb{R}_+$*:*

$$(g(x_1) - g(x_2))(x_2 - x_1) \leq L(x_1 - x_2)^2 \; \forall x_1, x_2 \in \mathbb{R}_+, \tag{2.15}$$

with the constant $L \geq 0$ *satisfying*

$$L < \frac{1}{c^2}, \tag{2.16}$$

where c *is the norm of the trace mapping* $\operatorname{tr} : \boldsymbol{V}(\Omega) \to L^2(S)$, $\operatorname{tr} \boldsymbol{v} = v_\tau$, *assuming that* $\boldsymbol{V}(\Omega)$ *is equipped with the norm* $\|\mathbb{D}(\cdot)\|_{0,\Omega}$. *Then* Ψ *has a unique fixed point.*

Proof. Let $\varphi_1, \varphi_2 \in H_+^{1/2}(S)$ be two fixed points of Ψ and $(\boldsymbol{u}^i, p^i)$ be solutions to $(\mathcal{P}(\varphi_i))$, $i = 1, 2$. Then $\boldsymbol{u}^i \in \boldsymbol{V}_{\mathrm{div}}(\Omega)$ and

$$a(\boldsymbol{u}^i, \boldsymbol{v} - \boldsymbol{u}^i) + j(|u_\tau^i|, v_\tau) - j(|u_\tau^i|, u_\tau^i) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^i)_{0,\Omega}\ \forall \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{div}}(\Omega).$$

By a standard technique we obtain:

$$\frac{1}{c^2}\left\| |u_\tau^2| - |u_\tau^1| \right\|_{0,S}^2 \leq \left\| \mathbb{D}(\boldsymbol{u}^1 - \boldsymbol{u}^2) \right\|_{0,\Omega}^2 = a(\boldsymbol{u}^1 - \boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2) \leq \int_S (g(|u_\tau^1|) - g(|u_\tau^2|))(|u_\tau^2| - |u_\tau^1|)$$
$$\leq_{(2.15)} L \left\| |u_\tau^2| - |u_\tau^1| \right\|_{0,S}^2. \quad (2.17)$$

From this and (2.16) we see that $\varphi_1 = |u_\tau^1| = |u_\tau^2| = \varphi_2$. □

Remark 2.5. Any non-decreasing function g automatically satisfies (2.15) with the constant $L = 0$ so that $(\mathcal{P})$ has a unique solution. If $L > 0$ then (2.15) permits a "small" decrease of g and the solution to $(\mathcal{P})$ is unique provided that (2.16) is satisfied.

3 Four-field formulation of $(\mathcal{P}^\varphi)$ and $(\mathcal{P})$

The pressure p in the velocity-pressure formulation introduced in the previous section is the Lagrange multiplier associated with the incompressibility condition in Ω. This section presents another formulation involving two additional Lagrange multipliers σ^ν, σ^τ defined on S releasing the impermeability condition $u_\nu = 0$ on S, and regularizing the non-differentiable functional j. To this end we shall need the additional function spaces:

$$W(\Omega) = \{v \in H^1(\Omega) |\ v = 0 \text{ on } \Gamma\}, \quad (3.1)$$

$$\boldsymbol{W}(\Omega) = W(\Omega) \times W(\Omega), \quad (3.2)$$

$$H^{-1/2}(S) = (H^{1/2}(S))' \text{ (dual of } H^{1/2}(S)), \quad (3.3)$$

$$\boldsymbol{H}^{1/2}(S) = H^{1/2}(S) \times H^{1/2}(S), \quad (3.4)$$

$$\boldsymbol{H}^{-1/2}(S) = (\boldsymbol{H}^{1/2}(S))'. \quad (3.5)$$

If $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \boldsymbol{H}^{-1/2}(S)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \boldsymbol{H}^{1/2}(S)$ then

$$\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle := \langle \mu_1, \varphi_1 \rangle + \langle \mu_2, \varphi_2 \rangle.$$

Since $S \in C^{1,1}$, the mapping

$$\mathrm{tr} : \boldsymbol{v} \mapsto (v_\nu, v_\tau), \text{ where } v_\nu = \boldsymbol{v}_{|S} \cdot \boldsymbol{\nu},\ v_\tau = \boldsymbol{v}_{|S} \cdot \boldsymbol{\tau},$$

maps $\boldsymbol{W}(\Omega)$ onto $\boldsymbol{H}^{1/2}(S)$. If $\boldsymbol{\mu} = (\mu^\nu, \mu^\tau) \in \boldsymbol{H}^{-1/2}(S)$ then

$$\langle \boldsymbol{\mu}, \mathrm{tr}\, \boldsymbol{v} \rangle := \langle \mu^\nu, v_\nu \rangle + \langle \mu^\tau, v_\tau \rangle.$$

Analogously to the previous section, the space $\boldsymbol{H}^{1/2}(S)$ is equipped with the norm

$$\|\boldsymbol{\varphi}\|_{1/2,S} = \inf_{\substack{\boldsymbol{v} \in \boldsymbol{W}(\Omega) \\ \mathrm{tr}\, \boldsymbol{v} = \boldsymbol{\varphi}}} |\boldsymbol{v}|_{1,\Omega} = |\boldsymbol{w}(\boldsymbol{\varphi})|_{1,\Omega}, \quad (3.6)$$

where $\boldsymbol{w}(\boldsymbol{\varphi})$ solves:

$$\left.\begin{aligned} \Delta \boldsymbol{w}(\boldsymbol{\varphi}) &= \boldsymbol{0} \text{ in } \Omega, \\ \boldsymbol{w}(\boldsymbol{\varphi}) &= \boldsymbol{0} \text{ on } \Gamma, \\ \mathrm{tr}\, \boldsymbol{w}(\boldsymbol{\varphi}) &= \boldsymbol{\varphi} \text{ on } S. \end{aligned}\right\} \quad (3.7)$$

The standard dual norm in $\boldsymbol{H}^{-1/2}(S)$ is given by

$$[\![\boldsymbol{\mu}]\!]_{-1/2,S} = \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{H}^{1/2}(S) \\ \boldsymbol{\varphi} \neq \boldsymbol{0}}} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\varphi} \rangle}{\|\boldsymbol{\varphi}\|_{1/2,S}} = \sup_{\substack{\boldsymbol{v} \in \boldsymbol{W}(\Omega) \\ \mathrm{tr}\, \boldsymbol{v} \neq \boldsymbol{0}}} \frac{\langle \boldsymbol{\mu}, \mathrm{tr}\, \boldsymbol{v} \rangle}{\|\mathrm{tr}\, \boldsymbol{v}\|_{1/2,S}},$$

where $\| \ \|_{1/2,S}$ is defined by (3.6). One can introduce another norm on $\boldsymbol{H}^{-1/2}(S)$, namely

$$\|\boldsymbol{\mu}\|_{-1/2,S} = \sup_{\substack{\boldsymbol{v}\in\boldsymbol{W}(\Omega)\\ \boldsymbol{v}\neq\boldsymbol{0}}} \frac{\langle\boldsymbol{\mu}, \operatorname{tr}\boldsymbol{v}\rangle}{|\boldsymbol{v}|_{1,\Omega}}.$$

It is known [5] that

$$[\![\boldsymbol{\mu}]\!]_{-1/2,S} = \|\boldsymbol{\mu}\|_{-1/2,S} \quad \forall\boldsymbol{\mu}\in\boldsymbol{H}^{-1/2}(S). \tag{3.8}$$

To regularize the functional $j(\varphi,\cdot)$ we use the closed convex set $K(\varphi)\subset H^{-1/2}(S)$ defined by

$$K(\varphi) = \{\mu^\tau \in L^{r'}(S) |\ |\mu^\tau| \le g(\varphi) \text{ a.e. on } S\}, \quad \varphi\in H^{1/2}_+(S).$$

It is readily seen that

$$j(\varphi, v_\tau) = \int_S g(\varphi)|v_\tau| = \sup_{\mu^\tau\in K(\varphi)} \int_S \mu^\tau v_\tau.$$

Hence

$$j(\varphi, v_\tau) \ge (\mu^\tau, v_\tau)_S := \int_S \mu^\tau v_\tau \quad \forall\mu^\tau\in K(\varphi). \tag{3.9}$$

The *four-field formulation* of $(\mathcal{P}^\varphi)$ reads as follows:

$$\left.\begin{array}{ll} \textit{Find } (\boldsymbol{u}, p, \sigma^\nu, \sigma^\tau) \in \boldsymbol{W}(\Omega)\times L^2_0(\Omega)\times H^{-1/2}(S)\times K(\varphi) \textit{ s.t.} & \\ \forall\boldsymbol{v}\in\boldsymbol{W}(\Omega): \quad a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p) - \langle\sigma^\nu, v_\nu\rangle - (\sigma^\tau, v_\tau)_S = (\boldsymbol{f},\boldsymbol{v})_{0,\Omega}, & \\ \forall q\in L^2_0(\Omega): \quad b(\boldsymbol{u},q) = 0, & \\ \forall\mu^\nu\in H^{-1/2}(S): \langle\mu^\nu, u_\nu\rangle = 0, & \\ \forall\mu^\tau\in K(\varphi): \quad (\mu^\tau+\sigma^\tau, u_\tau)_S \le 0. & \end{array}\right\} \tag{$\mathcal{M}^\varphi$}$$

Suppose that $(\mathcal{M}^\varphi)$ has a solution. In what follows we give its interpretation. From $(\mathcal{M}^\varphi)_{2,3}$ we see that $\boldsymbol{u}\in\boldsymbol{V}_{\text{div}}(\Omega)$, where $\boldsymbol{V}_{\text{div}}(\Omega)$ is defined by (2.3). Using test functions $\boldsymbol{v}\in\boldsymbol{V}(\Omega)$ in $(\mathcal{M}^\varphi)_1$ we get

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p) - (\sigma^\tau, v_\tau - u_\tau)_S = (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_{0,\Omega} \quad \forall\boldsymbol{v}\in\boldsymbol{V}(\Omega). \tag{3.10}$$

From $(\mathcal{M}^\varphi)_4$ it follows that

$$-(\sigma^\tau, u_\tau)_S = \sup_{\mu^\tau\in K(\varphi)} (\mu^\tau, u_\tau)_S = j(\varphi, u_\tau),$$

which together with (3.9) yields

$$-(\sigma^\tau, v_\tau - u_\tau)_S \le j(\varphi, v_\tau) - j(\varphi, u_\tau).$$

From this and (3.10) we obtain:

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - b(\boldsymbol{v}-\boldsymbol{u},p) + j(\varphi,v_\tau) - j(\varphi,u_\tau) \ge (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_{0,\Omega} \quad \forall\boldsymbol{v}\in\boldsymbol{V}(\Omega),$$

i.e. the couple $(\boldsymbol{u},p)\in\boldsymbol{V}(\Omega)\times L^2_0(\Omega)$ solves $(\mathcal{P}^\varphi)$. The formal application of Green's formula to $(\mathcal{M}^\varphi)_1$ gives:

$$\sigma^\nu = -p + ((\mathbb{D}\boldsymbol{u})\boldsymbol{v})_\nu \quad \text{and} \quad \sigma^\tau = ((\mathbb{D}\boldsymbol{u})\boldsymbol{v})_\tau \ \text{ on } S.$$

On the contrary, if $(\boldsymbol{u},p)$ is a solution to $(\mathcal{P}^\varphi)$ there exists a unique couple $(\sigma^\nu,\sigma^\tau)\in H^{-1/2}(S)\times K(\varphi)$ such that $(\boldsymbol{u},p,\sigma^\nu,\sigma^\tau)$ is a solution to $\mathcal{M}^\varphi$. This is a consequence of the next theorem.

Theorem 3.1. *Problem $(\mathcal{M}^\varphi)$ has a unique solution $(\boldsymbol{u},p,\sigma^\nu,\sigma^\tau)$ for any $\varphi\in H^{1/2}_+(S)$. In addition, the couple $(\boldsymbol{u},p)$ solves $(\mathcal{P}^\varphi)$.*

Proof. To prove the existence and uniqueness of a solution to $(\mathcal{M}^\varphi)$ it is sufficient to show that the bilinear form $c(\boldsymbol{v},(q,\boldsymbol{\mu})) := -b(\boldsymbol{v},q) - \langle\boldsymbol{\mu}, \operatorname{tr}\boldsymbol{v}\rangle$ satisfies the LBB-condition. It is well-known (see e.g. [5]) that

$$\exists\gamma>0: \sup_{\substack{\boldsymbol{v}\in(H^1_0(\Omega))^2\\ \boldsymbol{v}\neq\boldsymbol{0}}} \frac{b(\boldsymbol{v},q)}{|\boldsymbol{v}|_{1,\Omega}} \ge \gamma\,\|q\|_{0,\Omega}\,\forall q\in L^2_0(\Omega)$$

and from (3.8) we have

$$\sup_{\substack{\boldsymbol{v}\in \boldsymbol{W}(\Omega)\\ \boldsymbol{v}\neq \boldsymbol{0}}} \frac{\langle \boldsymbol{\mu}, \operatorname{tr} \boldsymbol{v}\rangle}{|\boldsymbol{v}|_{1,\Omega}} = [\![\boldsymbol{\mu}]\!]_{-1/2,S} \ \forall \boldsymbol{\mu} \in \boldsymbol{H}^{-1/2}(S).$$

Then there exists a constant $\tilde{\gamma} > 0$ such that

$$\sup_{\substack{\boldsymbol{v}\in \boldsymbol{W}(\Omega)\\ \boldsymbol{v}\neq \boldsymbol{0}}} \frac{c(\boldsymbol{v}, (q, \boldsymbol{\mu}))}{|\boldsymbol{v}|_{1,\Omega}} \geq \tilde{\gamma}(\|q\|_{0,\Omega} + [\![\boldsymbol{\mu}]\!]_{-1/2,S}) \ \forall (q, \boldsymbol{\mu}) \in L_0^2(\Omega) \times \boldsymbol{H}^{-1/2}(S)$$

as follows from Theorem 3.1 in [11]. □

Any quadruplet $(\boldsymbol{u}, p, \sigma^\nu, \sigma^\tau)$ is said to be a *solution of the problem with the solution dependent slip coefficient* if it solves $(\mathcal{M}^\varphi)$ with $\varphi = |u_\tau|$ on S:

$$\left.\begin{aligned}
&(\boldsymbol{u}, p, \sigma^\nu, \sigma^\tau) \in \boldsymbol{W}(\Omega) \times L_0^2(\Omega) \times H^{-1/2}(S) \times K(|u_\tau|) \ \textit{s.t.}\\
&\forall \boldsymbol{v} \in \boldsymbol{W}(\Omega): \qquad a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) - \langle \sigma^\nu, v_\nu\rangle - (\sigma^\tau, v_\tau)_S = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega},\\
&\forall q \in L_0^2(\Omega): \qquad b(\boldsymbol{u}, q) = 0,\\
&\forall \mu^\nu \in H^{-1/2}(S): \langle \mu^\nu, u_\nu\rangle = 0,\\
&\forall \mu^\tau \in K(|u_\tau|): \quad (\mu^\tau + \sigma^\tau, u_\tau)_S \leq 0.
\end{aligned}\right\} \qquad (\mathcal{M})$$

On the basis of the results of Sect. 2 and Theorem 3.1 we arrive at the following theorem.

Theorem 3.2. *Problem $(\mathcal{M})$ has a solution. In addition, if g satisfies the assumptions of Theorem 2.4, then the solution is unique.*

4 Stability of solutions with respect to boundary variations

The aim of this section is to show that solutions to $(\mathcal{P})$ and $(\mathcal{M})$ depend continuously on the shape of Ω. We shall suppose that only the part S of $\partial\Omega$ where the slip conditions are prescribed, is subject to variations. In addition, for the sake of simplicity of our presentation we shall assume that S is represented by the graph of a function α which belongs to an appropriate class $\mathcal{U}_{ad}$. Here and in what follows $\mathcal{U}_{ad}$ will be defined by

$$\mathcal{U}_{ad} = \{\alpha \in C^{1,1}([0,1]) |\ 0 < \alpha_{min} \leq \alpha \leq \alpha_{max} \text{ in } [0,1], |\alpha^{(j)}| \leq C_j,\ j = 1, 2 \text{ a.e. in } (0,1)\}, \qquad (4.1)$$

where $\alpha_{min}, \alpha_{max}$ and $C_j > 0$, $j = 1, 2$ are given. With any $\alpha \in \mathcal{U}_{ad}$ we associate the domain

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1 \in (0,1), x_2 \in (\alpha(x_1), \omega)\},$$

where $\omega > 0$ is a constant which does not depend on $\alpha \in \mathcal{U}_{ad}$. Further $\partial\Omega(\alpha) = \overline{S}(\alpha) \cup \overline{\Gamma}(\alpha)$, where

$$S(\alpha) = \{(x_1, \alpha(x_1)) \in \mathbb{R}^2 |\ x_1 \in (0,1)\}$$

is the slip part of $\partial\Omega(\alpha)$. The family of admissible domains consists of all $\Omega(\alpha)$ with $\alpha \in \mathcal{U}_{ad}$. We shall also assume that $\boldsymbol{f} \in (L^2_{loc}(\mathbb{R}^2))^2$.

4.1 Stability of $(\mathcal{P})$

Let $\alpha \in \mathcal{U}_{ad}$ be given and denote by $(\boldsymbol{u}(\alpha), p(\alpha)) \in \boldsymbol{V}(\Omega(\alpha)) \times L_0^2(\Omega(\alpha))$ a (not necessarily unique) solution to $(\mathcal{P}(\alpha))$ defined in $\Omega(\alpha)$:

$$\left.\begin{aligned}
\forall \boldsymbol{v} \in \boldsymbol{V}(\Omega(\alpha)): \quad & a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)) - b_\alpha(\boldsymbol{v} - \boldsymbol{u}(\alpha)), p(\alpha))\\
&\qquad + j_\alpha(|u_\tau(\alpha)|, v_\tau) - j_\alpha(|u_\tau(\alpha)|, u_\tau(\alpha))\\
&\qquad \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)},\\
\forall q \in L_0^2(\Omega(\alpha)): \quad & b_\alpha(\boldsymbol{u}(\alpha), q) = 0,
\end{aligned}\right\} \qquad (\mathcal{P}(\alpha))$$

where a_α, b_α and j_α are defined by (2.8) on $\Omega := \Omega(\alpha)$, $S := S(\alpha)$.

Remark 4.1. Let us note that the condition (2.16) ensuring the uniqueness of the solution to $(\mathcal{P}(\alpha))$ can be chosen to be independent of $\alpha \in \mathcal{U}_{ad}$. Indeed, the constant c in (2.16) can be bounded from above uniformly with respect to $\alpha \in \mathcal{U}_{ad}$ making use of Lemma 2.19 in [7] and also the constant in the Korn inequality can be chosen to be independent of $\alpha \in \mathcal{U}_{ad}$ ([6, 13]).

Let $\widehat{\Omega} := (0, 1) \times (0, \omega) \supset \Omega(\alpha)$ $\forall \alpha \in \mathcal{U}_{ad}$ be the hold-all domain and $\pi_\alpha \in \mathcal{L}(\boldsymbol{V}(\Omega(\alpha)), (H_0^1(\widehat{\Omega}))^2)$ an extension mapping from $\Omega(\alpha)$ to $\widehat{\Omega}$. Since all $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ satisfy the uniform cone property, there exists π_α whose norm can be estimated independently of $\alpha \in \mathcal{U}_{ad}$ (see [2]). Finally, the upper index "0" stands for the zero extension of functions from $\Omega(\alpha)$ to $\widehat{\Omega}$.

Theorem 4.2. *There exists a constant* $c := c(\boldsymbol{f}, c_g, r) > 0$ *independent of* $\alpha \in \mathcal{U}_{ad}$ *such that*

$$\|\pi_\alpha \boldsymbol{u}(\alpha)\|_{1,\widehat{\Omega}} + \|p^0(\alpha)\|_{0,\widehat{\Omega}} \le c \tag{4.2}$$

holds for any solution $(\boldsymbol{u}(\alpha), p(\alpha))$ *to* $(\mathcal{P}(\alpha))$.

Proof. The estimate of the first term in (4.2) follows from (2.10a) and Remark 4.1 concerning the Korn inequality. Similarly, the estimate of the pressure term follows from (2.10b) using that the constant in the inf-sup condition for pressure can be chosen independently of $\alpha \in \mathcal{U}_{ad}$ (see [7]) and the growth condition (2.9). □

Let

$$\mathcal{G}_{\mathcal{P}} := \{(\alpha, \boldsymbol{u}(\alpha), p(\alpha)) |\ \alpha \in \mathcal{U}_{ad}, (\boldsymbol{u}(\alpha), p(\alpha)) \text{ solves } (\mathcal{P}(\alpha))\}$$

be the graph of the generally multivalued solution mapping $\Phi : \alpha \mapsto (\boldsymbol{u}(\alpha), p(\alpha))$, $\alpha \in \mathcal{U}_{ad}$.

Theorem 4.3. *The graph* $\mathcal{G}_{\mathcal{P}}$ *is closed in the following sense:*

$$\left.\begin{aligned} &\alpha_n \to \alpha \text{ in } C^1([0,1]), \alpha_n, \alpha \in \mathcal{U}_{ad},\\ &(\pi_{\alpha_n}\boldsymbol{u}_n, p_n^0) \rightharpoonup (\overline{\boldsymbol{u}}, \overline{p}) \text{ in } (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega}),\\ &\text{where } (\alpha_n, \boldsymbol{u}_n, p_n) := (\alpha_n, \boldsymbol{u}(\alpha_n), p(\alpha_n)) \in \mathcal{G}_{\mathcal{P}} \end{aligned}\right\} \Rightarrow (\overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)}) \text{ solves } (\mathcal{P}(\alpha))$$

and hence $(\alpha, \overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)}) \in \mathcal{G}_{\mathcal{P}}$.

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}(\Omega(\alpha))$ be given. From Lemma 3 in [8] we know that there exist: a function $\overline{\boldsymbol{v}} \in (H^1(\widehat{\Omega}))^2$, a sequence $\{\boldsymbol{v}_k\}$, $\boldsymbol{v}_k \in (H^1(\widehat{\Omega}))^2$ and a filter of indices $\{n_k\}$ such that $\overline{\boldsymbol{v}}_{|\Omega(\alpha)} = \boldsymbol{v}$,

$$\left.\begin{aligned} &\boldsymbol{v}_k \to \overline{\boldsymbol{v}} \text{ in } (H^1(\widehat{\Omega}))^2, k \to \infty,\\ &\boldsymbol{v}_{k|\Omega(\alpha_{n_k})} \in \boldsymbol{V}(\Omega(\alpha_{n_k})). \end{aligned}\right\} \tag{4.3}$$

Therefore $\boldsymbol{v}_{k|\Omega(\alpha_{n_k})}$ can be used as a test function in $(\mathcal{P}(\alpha_{n_k}))$:

$$\left.\begin{aligned} &a_{\alpha_{n_k}}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) - b_{\alpha_{n_k}}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k})\\ &\qquad + j_{\alpha_{n_k}}(|u_{n_k\tau}|, v_{k\tau}) - j_{\alpha_{n_k}}(|u_{n_k\tau}|, u_{n_k\tau})\\ &\ge (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})},\\ &\forall q \in L_0^2(\Omega(\alpha_{n_k})) : b_{\alpha_{n_k}}(\boldsymbol{u}_{n_k}, q) = 0. \end{aligned}\right\} \tag{4.4}$$

Denote $(\boldsymbol{u}(\alpha), p(\alpha)) := (\overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)})$. The fact that $\boldsymbol{u}(\alpha) \in \boldsymbol{V}_{\mathrm{div}}(\Omega(\alpha))$ and the following limit passages can be proven exactly as in [8]:

$$\begin{aligned} &\limsup_{k\to\infty} a_{\alpha_{n_k}}(\boldsymbol{u}_{n_k}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k}) \le a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v} - \boldsymbol{u}(\alpha)),\\ &\lim_{k\to\infty} b_{\alpha_{n_k}}(\boldsymbol{v}_k - \boldsymbol{u}_{n_k}, p_{n_k}) = b_\alpha(\boldsymbol{v} - \boldsymbol{u}(\alpha), p(\alpha)),\\ &\lim_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v}_k - \boldsymbol{u}_{n_k})_{0,\Omega(\alpha_{n_k})} = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(\alpha))_{0,\Omega(\alpha)}, \end{aligned}$$

making use of (4.3). It remains to pass to the limit in the slip terms.

From (4.3) and weak convergence $\pi_{\alpha_{n_k}}(\boldsymbol{u}_{n_k}) \rightharpoonup \overline{\boldsymbol{u}}$ in $(H^1(\widehat{\Omega}))^2$ it follows ([7]):

$$\left.\begin{aligned} \boldsymbol{u}_{n_k|S(\alpha_{n_k})} \circ \alpha_{n_k} &\to \overline{\boldsymbol{u}}_{|S(\alpha)} \circ \alpha\\ \boldsymbol{v}_{k|S(\alpha_{n_k})} \circ \alpha_{n_k} &\to \boldsymbol{v}_{|S(\alpha)} \circ \alpha \end{aligned}\right\} \text{ in } (L^r(0,1))^2. \tag{4.5}$$

For the proof of (4.5) with $r = 2$ we refer to Lemma 2.21 in [7]. However the result is true for any $r \in [1, \infty)$ using that $H^1(\widehat{\Omega})$ is compactly embedded in $L^r(0, 1)$. From (4.5) we easily obtain:

$$\left.\begin{aligned} u_{n_k\tau} \circ \alpha_{n_k} &:= (\boldsymbol{u}_{n_k|S(\alpha_{n_k})} \cdot \boldsymbol{\tau}^{\alpha_{n_k}}) \circ \alpha_{n_k} \to (\overline{\boldsymbol{u}}_{|S(\alpha)} \cdot \boldsymbol{\tau}^{\alpha}) \circ \alpha = \overline{u}_\tau \circ \alpha \\ v_{k\tau} \circ \alpha_{n_k} &:= (\boldsymbol{v}_{k|S(\alpha_{n_k})} \cdot \boldsymbol{\tau}^{\alpha_{n_k}}) \circ \alpha_{n_k} \to (\boldsymbol{v}_{|S(\alpha)} \cdot \boldsymbol{\tau}^{\alpha}) \circ \alpha = v_\tau \circ \alpha \end{aligned}\right\} \text{ in } L^r(0,1), \tag{4.6}$$

since $\boldsymbol{\tau}^{\alpha_{n_k}} \circ \alpha_{n_k} \rightrightarrows \boldsymbol{\tau}^{\alpha} \circ \alpha$ (uniformly) in $[0, 1]$, where $\boldsymbol{\tau}^{\beta}$ stands for the unit tangential vector to $S(\beta)$, $\beta \in \mathcal{U}_{ad}$ (see [8]). Consequently,

$$g(|u_{n_k\tau} \circ \alpha_{n_k}|) \to g(|\overline{u}_\tau \circ \alpha|) \text{ in } L^{r'}(0, 1). \tag{4.7}$$

Hence

$$j_{\alpha_{n_k}}(|u_{n_k\tau}|, v_{k\tau}) = \int_0^1 g(|u_{n_k\tau} \circ \alpha_{n_k}|)|v_{k\tau} \circ \alpha_{n_k}|\sqrt{1 + (\alpha'_{n_k})^2}\, dx_1$$

$$\to \int_0^1 g(|\overline{u}_\tau \circ \alpha|)|v_\tau \circ \alpha|\sqrt{1 + (\alpha')^2}\, dx_1 = j_\alpha(|\overline{u}_\tau|, v_\tau), k \to \infty, \tag{4.8}$$

as follows from $(4.6)_2$, (4.7) and convergence $\alpha_n \to \alpha$ in $C^1([0, 1])$. The limit passage for the second slip term in (4.4) can be done in the same way. □

Remark 4.4. Theorem 4.3 automatically guarantees the following property of the sequence $\{\boldsymbol{u}_n\}$:

$$\boldsymbol{u}_{n|Q} \to \boldsymbol{u}(\alpha)_{|Q} \text{ in } (H^1(Q))^2 \tag{4.9}$$

for any domain Q such that $\overline{Q} \subset \Omega(\alpha)$, i.e. $\boldsymbol{u}_n \to \boldsymbol{u}(\alpha)$ in $(H^1_{loc}(\Omega(\alpha)))^2$.

Indeed, let χ, χ_n be the characteristic function of $\Omega(\alpha)$ and $\Omega(\alpha_n)$, respectively. Inserting $\boldsymbol{v} = \boldsymbol{0}, 2\boldsymbol{u}_n$ into $(\mathcal{P}(\alpha_n))$ we obtain:

$$\left\|\chi_n \mathbb{D}(\pi_{\alpha_n}\boldsymbol{u}_n)\right\|^2_{0,\widehat{\Omega}} = -j_{\alpha_n}(|u_{n\tau}|, u_{n\tau}) + (\chi_n \boldsymbol{f}, \pi_{\alpha_n}\boldsymbol{u}_n)_{0,\widehat{\Omega}}.$$

Hence

$$\lim_{n\to\infty} \left\|\chi_n \mathbb{D}(\pi_{\alpha_n}\boldsymbol{u}_n)\right\|^2_{0,\widehat{\Omega}} = -j_\alpha(|\overline{u}_\tau|, \overline{u}_\tau) + (\chi \boldsymbol{f}, \overline{\boldsymbol{u}})_{0,\widehat{\Omega}} = \|\chi \mathbb{D}\overline{\boldsymbol{u}}\|^2_{0,\widehat{\Omega}} \tag{4.10}$$

arguing as in (4.8), using that $\chi_n \to \chi$ in $L^q(\widehat{\Omega})$ for any $q \in [1, \infty)$ and the fact that $\boldsymbol{u}(\alpha) := \overline{\boldsymbol{u}}_{|\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$. Let Q be as above. Since $\alpha_n \to \alpha$ in $C^1([0, 1])$ it holds that $Q \subset \Omega(\alpha_n)$ for any n large enough. From (4.10) it follows:

$$\lim_{n\to\infty} \|\mathbb{D}\boldsymbol{u}_n\|^2_{0,Q} = \left\|\mathbb{D}\boldsymbol{u}(\alpha)\right\|^2_{0,Q},$$

which together with the assumptions of Theorem 4.3 and the Korn inequality prove (4.9).

4.2 Stability of $(\mathcal{M})$

Let $\mathcal{U}_{ad}$ be defined again by (4.1). We keep notation of Subsect. 4.1, i.e. the meaning of $\Omega(\alpha)$, $S(\alpha)$, j_α, a_α, b_α, $\boldsymbol{\nu}^\alpha$, $\boldsymbol{\tau}^\alpha$ remains. In addition, $\langle\ ,\ \rangle_\alpha$ will denote the duality pairing between $H^{-1/2}(S(\alpha))$ and $H^{1/2}(S(\alpha))$. If $\mu^\nu \in H^{-1/2}(S(\alpha))$ and $\boldsymbol{v} \in (H^1(\Omega(\alpha)))^2$ then

$$\langle \mu^\nu, v_\nu \rangle_\alpha := \langle \mu^\nu, \boldsymbol{v}_{|S(\alpha)} \cdot \boldsymbol{\nu}^\alpha \rangle_\alpha \quad \forall \alpha \in \mathcal{U}_{ad}.$$

Recall that

$$(\mu^\tau, v_\tau)_{S(\alpha)} = \int_{S(\alpha)} \mu^\tau v_\tau \quad \forall \mu^\tau \in L^{r'}(S(\alpha)), v_\tau \in L^r(S(\alpha)).$$

Problem $(\mathcal{M})$ formulated on $\Omega := \Omega(\alpha)$ will be denoted by $(\mathcal{M}(\alpha))$.

Let

$$\mathcal{G}_\mathcal{M} = \{(\alpha, \boldsymbol{u}(\alpha), p(\alpha), \sigma^\nu(\alpha), \sigma^\tau(\alpha))|\ \alpha \in \mathcal{U}_{ad}, (\boldsymbol{u}(\alpha), p(\alpha), \sigma^\nu(\alpha), \sigma^\tau(\alpha)) \text{ solves } (\mathcal{M}(\alpha))\}$$

be the graph of the respective solution mapping.

Theorem 4.5. *The graph $\mathcal{G}_\mathcal{M}$ is closed in the following sense: Let*

$$\alpha_n \to \alpha \textit{ in } C^1([0, 1]), \alpha_n, \alpha \in \mathcal{U}_{ad} \tag{4.11}$$

and

$$(\pi_{\alpha_n}\boldsymbol{u}_n, p_n^0) \rightharpoonup (\overline{\boldsymbol{u}}, \overline{p}) \text{ in } (H_0^1(\widehat{\Omega}))^2 \times L_0^2(\widehat{\Omega}). \tag{4.12}$$

Then also

$$\langle \sigma_n^\nu, v_\nu\rangle_{\alpha_n} \to \langle \sigma^\nu(\alpha), v_\nu\rangle_\alpha, \tag{4.13}$$

$$(\sigma_n^\tau, v_\tau)_{0,S(\alpha_n)} \to (\sigma^\tau(\alpha), v_\tau)_{0,S(\alpha)} \tag{4.14}$$

holds for every $\boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2$, where $(\boldsymbol{u}_n, p_n, \sigma_n^\nu, \sigma_n^\tau)$ is a solution to $(\mathcal{M}(\alpha_n))$, $n = 1, \dots$ In addition, the quadruplet $(\overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)}, \sigma^\nu(\alpha), \sigma^\tau(\alpha))$ solves $(\mathcal{M}(\alpha))$.

Proof. From Theorem 4.3 we already know that $(\overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)})$ is a solution to $(\mathcal{P}(\alpha))$. To this couple there exists a unique pair $(\sigma^\nu(\alpha), \sigma^\tau(\alpha)) \in H^{-1/2}(S(\alpha)) \times L^{r'}(S(\alpha))$ such that $(\overline{\boldsymbol{u}}_{|\Omega(\alpha)}, \overline{p}_{|\Omega(\alpha)}, \sigma^\nu(\alpha), \sigma^\tau(\alpha))$ is a solution to $(\mathcal{M}(\alpha))$. It remains to prove (4.13) and (4.14) only. We use the formulation of $(\mathcal{M}(\alpha_n))$:

$$\left.\begin{aligned}
&(\boldsymbol{u}_n, p_n, \sigma_n^\nu, \sigma_n^\tau) \in \boldsymbol{W}(\Omega(\alpha_n)) \times L_0^2(\Omega(\alpha_n)) \times H^{-1/2}(S(\alpha_n)) \times K(|u_{n\tau|S(\alpha_n)}|):\\
&\forall \boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2: \qquad \langle \sigma_n^\nu, v_\nu\rangle_{\alpha_n} + (\sigma_n^\tau, v_\tau)_{S(\alpha_n)} = a_{\alpha_n}(\boldsymbol{u}_n, \boldsymbol{v})\\
&\qquad\qquad\qquad\qquad\qquad\qquad -b_{\alpha_n}(\boldsymbol{v}, p_n) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha_n)},\\
&\forall q \in L_0^2(\Omega(\alpha_n)): \qquad b_{\alpha_n}(\boldsymbol{u}_n, q) = 0,\\
&\forall \mu^\nu \in H^{-1/2}(S(\alpha_n)): \langle \mu^\nu, u_{n\nu}\rangle_{\alpha_n} = 0,\\
&\forall \mu^\tau \in K(|u_{n\tau|S(\alpha_n)}|): (\mu^\tau + \sigma_n^\tau, u_{n\tau})_{S(\alpha_n)} \le 0,
\end{aligned}\right\} \tag{4.15}$$

where

$$K(|u_{n\tau|S(\alpha_n)}|) = \{\mu^\tau \in L^{r'}(S(\alpha_n)) \mid |\mu^\tau| \le g(|\boldsymbol{u}_{n|S(\alpha_n)} \cdot \boldsymbol{\tau}^{\alpha_n}|) \text{ a.e. in } S(\alpha_n)\}.$$

Observe that one can use test functions $\boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2$ in $(4.15)_1$ since $\boldsymbol{W}(\Omega(\beta)) = (H_0^1(\widehat{\Omega}))^2_{|\Omega(\beta)}$ $\forall \beta \in \mathcal{U}_{ad}$.

Denote $\boldsymbol{u}(\alpha) := \overline{\boldsymbol{u}}_{|\Omega(\alpha)}$ and $p(\alpha) := \overline{p}_{|\Omega(\alpha)}$. Letting $n \to \infty$ in $(4.15)_1$ we obtain:

$$\begin{aligned}
\lim_{n\to\infty}\left\{\langle \sigma_n^\nu, v_\nu\rangle_{\alpha_n} + (\sigma_n^\tau, v_\tau)_{S(\alpha_n)}\right\} &= a_\alpha(\boldsymbol{u}(\alpha), \boldsymbol{v}) - b_\alpha(\boldsymbol{v}, p(\alpha)) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha)}\\
&= \langle \sigma^\nu, v_\nu\rangle_\alpha + (\sigma^\tau, v_\tau)_{S(\alpha)},
\end{aligned} \tag{4.16}$$

which holds for every $\boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2$. It remains to show that the limit of each term on the left of (4.16) exists. Arguing as in (4.6) we obtain:

$$v_\tau \circ \alpha_n := (\boldsymbol{v}_{|S(\alpha_n)} \cdot \boldsymbol{\tau}^{\alpha_n}) \circ \alpha_n \to (\boldsymbol{v}_{|S(\alpha)} \cdot \boldsymbol{\tau}^\alpha) \circ \alpha =: v_\tau \circ \alpha \text{ in } L^r((0,1)), n \to \infty. \tag{4.17}$$

Therefore if $\boldsymbol{v} \in (H_0^1(\widehat{\Omega}))^2$ in (4.16) is such that $v_\tau = 0$ on $S(\alpha)$ then $v_\tau \circ \alpha_n \to 0$ in $L^r((0,1))$. Hence

$$(\sigma_n^\tau, v_\tau)_{S(\alpha_n)} = \int_0^1 (\sigma_n^\tau \circ \alpha_n)(v_\tau \circ \alpha_n)\sqrt{1 + (\alpha_n')^2}\, dx_1 \to 0, n \to \infty,$$

making use (4.7) and the fact that $|\sigma_n^\tau \circ \alpha_n| \le g(|(\boldsymbol{u}_{n|S(\alpha_n)} \cdot \boldsymbol{\tau}^{\alpha_n}) \circ \alpha_n|)$ a.e. in $(0,1)$. This together with (4.16) gives (4.13) and consequently also (4.14). □

5 Application of the stability property in optimal shape design problems

On the basis of the results of Sect. 4 it is easy to prove the existence of solutions to a class of optimal shape design problems for systems governed by the Stokes equation with a solution dependent slip bound.

Let $J_\mathcal{P} : \mathcal{G}_\mathcal{P} \to \mathbb{R}$ and $J_\mathcal{M} : \mathcal{G}_\mathcal{M} \to \mathbb{R}$ be cost functionals defined on the graphs of the solution mappings corresponding to $(\mathcal{P}(\alpha))$ and $(\mathcal{M}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$, respectively. Shape optimization problems with $(\mathcal{P}(\alpha))$, $(\mathcal{M}(\alpha))$ as the state relation read as follows:

$$\left.\begin{aligned}
&\textit{Find } \boldsymbol{z}^* \in \mathcal{G}_\mathcal{P} \textit{ such that}\\
&J_\mathcal{P}(\boldsymbol{z}^*) \le J_\mathcal{P}(\boldsymbol{z})\ \forall \boldsymbol{z} \in \mathcal{G}_\mathcal{P},
\end{aligned}\right\} \tag{$\mathbb{P}_\mathcal{P}$}$$

and

$$\left.\begin{array}{l}\textit{Find } z^* \in \mathcal{G}_{\mathcal{M}} \textit{ such that}\\ J_{\mathcal{M}}(z^*) \le J_{\mathcal{M}}(z)\ \forall z \in \mathcal{G}_{\mathcal{M}}.\end{array}\right\} \qquad (\mathbb{P}_{\mathcal{M}})$$

If $(\mathcal{P}(\alpha))$ has a unique solution for any $\alpha \in \mathcal{U}_{ad}$ then $(\mathbb{P}_{\mathcal{P}})$ can be written as the following minimization problem on $\mathcal{U}_{ad}$:

$$\left.\begin{array}{l}\textit{Find } \alpha^* \in \mathcal{U}_{ad} \textit{ such that}\\ \mathcal{J}_{\mathcal{P}}(\alpha^*) \le \mathcal{J}_{\mathcal{P}}(\alpha)\ \forall \alpha \in \mathcal{U}_{ad},\end{array}\right\} \qquad (\overline{\mathbb{P}}_{\mathcal{P}})$$

where $\mathcal{J}_{\mathcal{P}}(\alpha) := J_{\mathcal{P}}(\alpha, \boldsymbol{u}(\alpha), p(\alpha))$, $(\alpha, \boldsymbol{u}(\alpha), p(\alpha)) \in \mathcal{G}_{\mathcal{P}}$. The same can be done for $(\mathbb{P}_{\mathcal{M}})$.

To prove the existence of solutions to $(\mathbb{P}_{\mathcal{P}})$ and $(\mathbb{P}_{\mathcal{M}})$ we use compactness and lower semicontinuity arguments.

Convergence in $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{G}_{\mathcal{M}}$ will be introduced using the results of Theorem 4.3 and 4.5. We say that $z_n \to z$, $z_n, z \in \mathcal{G}_{\mathcal{P}}$ if (4.11) and (4.12) hold. Analogously, $z_n \to z$, $z_n, z \in \mathcal{G}_{\mathcal{M}}$ if (4.11)–(4.14) hold.

From the definition of $\mathcal{U}_{ad}$ and the Arzelà-Ascoli theorem we see that $\mathcal{U}_{ad}$ is a compact subset of C^1. This, together with (4.2) and Theorem 4.3 and 4.5 proves the following result.

Theorem 5.1. *The sets $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{G}_{\mathcal{M}}$ are compact with respect to convergences introduced above.*

We say that $J_{\mathcal{P}}$ and $J_{\mathcal{M}}$ are lower semicontinuous functionals on $\mathcal{G}_{\mathcal{P}}$, and $\mathcal{G}_{\mathcal{M}}$, respectively if

$$z_n \to z,\, z_n, z \in \mathcal{G}_{\mathcal{P}} \Rightarrow \liminf_{n\to\infty} J_{\mathcal{P}}(z_n) \ge J_{\mathcal{P}}(z), \tag{5.1}$$

$$z_n \to z,\, z_n, z \in \mathcal{G}_{\mathcal{M}} \Rightarrow \liminf_{n\to\infty} J_{\mathcal{M}}(z_n) \ge J_{\mathcal{M}}(z). \tag{5.2}$$

Theorem 5.2. *Let the functionals $J_{\mathcal{P}}$, $J_{\mathcal{M}}$ satisfy (5.1), and (5.2), respectively. Then there exists a solution to $(\mathbb{P}_{\mathcal{P}})$ and $(\mathbb{P}_{\mathcal{M}})$.*

Proof is straightforward.

Conclusions

In the first part of this paper we analyzed the mathematical model of the Stokes system with a threshold slip boundary condition whose slip bound depends on the solution itself. We used two weak formulations of this problem: the standard velocity-pressure formulation and the extended formulation in terms of the velocity, pressure, shear and normal stress. We proved the existence of a solution for a large class of functions representing the slip bound and studied under which conditions the solution is unique. In the second part of the paper we analyzed how solutions to both weak formulations depend on the geometry of the problem, in particular on the shape of the slip part S of the boundary. Using an appropriate parametrization of S we proved that the graphs of the respective solution mappings are compact in an appropriate topology. This result plays the key role in shape optimization. Let us mention that the same stability result can be proven for the Navier-Stokes system with the same boundary conditions.

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SHAPE OPTIMIZATION FOR STOKES PROBLEM WITH THRESHOLD SLIP BOUNDARY CONDITIONS

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This paper is dedicated to Prof. Tomáš Roubíček in the occasion of his 60th birthday.

Abstract. This paper deals with shape optimization of systems governed by the Stokes flow with threshold slip boundary conditions. The stability of solutions to the state problem with respect to a class of domains is studied. For computational purposes the slip term and impermeability condition are handled by a regularization. To get a finite dimensional optimization problem, the optimized part of the boundary is described by Bézier polynomials. Numerical examples illustrate the computational efficiency.

1. **Introduction.** The standard kinematic boundary condition in mathematical models of fluid mechanics is represented by the no-slip condition, namely the fluid has the zero velocity $\boldsymbol{u}$ on the boundary of a solid impermeable wall. This condition however does not always hold. In many real problems a fluid slip along the boundary has been observed. In particular, this effect occurs on hydrophobic surfaces, i.e. surfaces coated by a thin film of a non-wettable material from which the fluid (water) is repelled [24]. The Navier boundary condition is the classical one which takes into account the fluid slip [20]. It says that the shear stress σ_t is proportional to the tangential velocity u_t: $\sigma_t = -ku_t$, where k is the adhesive coefficient. Consequently,

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a slip occurs whenever σ_t is non-vanishing. This model is not able to describe frequent situations when the slip has a threshold character, i.e. it may come only if the shear stress attains certain value which is either given a priori or depends in some way on the solution itself. For the physical justification of different slip laws we refer to [15, 23]. Due to their non-smooth character, resulting mathematical models lead to an inequality type problem. For the steady Stokes flow with slip conditions and a given slip bound we refer to [6], [25], [17], and to [18] for the steady Navier-Stokes flow. The Stokes problem with a solution dependent slip bound has been studied in [13]. Recently, the stick-slip condition has been considered in [3] as an implicit constitutive equation on the boundary having a monotone 2-graph property. The existence of weak solutions to Bingham and Navier–Stokes fluids is proven there.

The present paper deals with optimal shape design problems governed by the Stokes system subject to the threshold boundary conditions. Such problems are of a great practical importance. Indeed, using appropriate shapes of hydrophobic surfaces, one can control (among others) the velocity profile to reduce the energy losses. The stability of solutions to the state problem with respect to an appropriate class of domains is the key property used in the existence analysis. This subject has been studied in [26] for the Navier boundary condition, in [14] for the slip bound given a priori and in [13] for the solution dependent slip bound. Due to the threshold character of the slip boundary conditions, the respective control-to-state mapping which with any admissible domain associates the solution to the state problem $(\mathcal{M})$ is non-differentiable in the classical sense. Therefore the resulting optimization problem $(\mathbb{P})$ formulated and analyzed in [13] and [14] is generally non-smooth, as well. It can be solved numerically by non-smooth optimization methods. The main drawback (to some extent) of this approach is the fact that it requires knowledge of the non-smooth differential calculus to perform sensitivity analysis ([21]) needed in computations. A possible way how to overcome this difficulty is to approximate the nonsmooth slip term j in $(\mathcal{M})$ by an appropriate sequence of smooth functionals j_ε, $\varepsilon \to 0+$ to get a sequence of smooth nonlinear equations $(\mathcal{M}_\varepsilon)$. Denoting by $(\mathbb{P}_\varepsilon)$ the shape optimization problem with $(\mathcal{M}_\varepsilon)$ as the state problem a natural question arises, namely if there exists a relation between solutions to $(\mathbb{P}_\varepsilon)$ and $(\mathbb{P})$ for $\varepsilon \to 0+$. This is one of subjects analyzed here.

The paper is organized as follows: in Section 2 we present the velocity-pressure formulation $(\mathcal{M})$ of the Stokes system with a class of slip boundary conditions. Besides the regularization of the slip term we use for computational purposes also a penalization of the impermeability condition to define the regularized-penalized problems $(\mathcal{M}_\varepsilon)$. Section 3 is devoted to the stability analysis of solutions to $(\mathcal{M}_\varepsilon)$ with respect to domains Ω and the parameter $\varepsilon \to 0+$. The assumptions are formulated in an abstract way enabling us to use them in other problems, too. On the basis of these results one can easily prove the existence of solutions to $(\mathbb{P}_\varepsilon)$ and $(\mathbb{P})$ and to establish the mutual relation between their solutions if $\varepsilon \to 0+$. This is done in Section 4. Section 5 introduces an appropriate family of admissible domains, and penalty/regularization functionals satisfying all the assumptions formulated in the previous sections. Section 6 deals with numerical aspects. Optimized part of the boundary is described by Bézier polynomials, while the regularized-penalized problem $(\mathcal{M}_\varepsilon)$ is discretized by P1-bubble/P1 elements. Sensitivity analysis uses the standard adjoint state approach. Finally, Section 7 presents numerical results of three model examples.

2. **State problem.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega = \bar{\Gamma} \cup \bar{S}$, where Γ, S are disjoint, non-empty and open in $\partial\Omega$. In Ω we consider the Stokes system with a slip-type boundary condition prescribed on S:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \\ u_\nu = 0 & \text{on } S \\ \sigma_t \in \partial j(-u_t) & \text{on } S, \end{cases} \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2)$ is the velocity field, p is the pressure and $\boldsymbol{f}$ is an external force. Further, $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\boldsymbol{t} = (-\nu_2, \nu_1)$ stand for the unit outward normal, and tangential vector to S, $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $u_t = \boldsymbol{u} \cdot \boldsymbol{t}$ denote the normal, and tangential component of $\boldsymbol{u}$, $\sigma_t = 2\mathbb{D}(\boldsymbol{u})\boldsymbol{\nu} \cdot \boldsymbol{t}$ is the shear stress on S and $\mathbb{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ is the symmetric part of the gradient of $\boldsymbol{u}$. Finally, $\partial j(\bullet)$ stands for the subgradient of a convex functional j at a point $\bullet$.

To give the weak *velocity* and *velocity-pressure* formulation we first introduce several function spaces:

$$\begin{aligned} \mathbb{V}(\Omega) &= \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma,\ v_\nu = 0 \text{ on } S\}, \\ \mathbb{V}_{\mathrm{div}}(\Omega) &= \{\boldsymbol{v} \in \mathbb{V}(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}, \\ L^2_0(\Omega) &= \{q \in L^2(\Omega) \mid \textstyle\int_\Omega q\, dx = 0\}. \end{aligned}$$

The weak velocity formulation of (1) is defined by the following minimization problem:

$$\text{Find } \boldsymbol{u} = \operatorname*{argmin}_{\boldsymbol{v} \in \mathbb{V}_{\mathrm{div}}(\Omega)} \left\{ J(\boldsymbol{v}) := \tfrac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) + j(v_t) - (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \right\}, \tag{$\mathcal{P}(\Omega)$}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2 \int_\Omega \mathbb{D}(\boldsymbol{u}) : \mathbb{D}(\boldsymbol{v})\, dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in (H^1(\Omega))^2.$$

Further $j : L^2(S) \to \mathbb{R}_+$ is a *non-negative, convex, lower semicontinuous* functional, and $(\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \forall \boldsymbol{v} \in (L^2(\Omega))^2$.

It is well-known that ($\mathcal{P}(\Omega)$) has a unique solution $\boldsymbol{u}$ and, in addition, ($\mathcal{P}(\Omega)$) is equivalent to the following variational inequality of the second kind:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \mathbb{V}_{\mathrm{div}}(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(v_t) - j(u_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}_{\mathrm{div}}(\Omega). \end{cases} \tag{$\mathcal{P}'(\Omega)$}$$

The velocity-pressure variational formulation of (1) reads as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in \mathbb{V}(\Omega) \times L^2_0(\Omega) \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}, p) + j(v_t) - j(u_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{0,\Omega}\ \forall \boldsymbol{v} \in \mathbb{V}(\Omega) \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L^2_0(\Omega), \end{cases} \tag{$\mathcal{M}(\Omega)$}$$

where $b : (H^1(\Omega))^2 \times L^2_0(\Omega) \to \mathbb{R}$ is defined by $b(\boldsymbol{v}, q) = \int_\Omega \operatorname{div} \boldsymbol{v}\, q\, dx$. Also ($\mathcal{M}(\Omega)$) has a unique solution $(\boldsymbol{u}, p)$ as a consequence of the inf-sup condition satisfied by b (see [8, Th. 3.7]):

$$\exists \beta = \text{const.} > 0 : \quad \sup_{\boldsymbol{v} \in (H^1_0(\Omega))^2} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega} \quad \forall q \in L^2_0(\Omega). \tag{2}$$

In addition, the first component $\boldsymbol{u}$ solves ($\mathcal{P}(\Omega)$).

Since the functional j is generally non-differentiable, we use a regularization approach together with a penalization of the impermeability condition $v_\nu = 0$ on S. To this end we introduce the spaces

$$\mathbb{W}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\},$$
$$\mathbb{W}_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in \mathbb{W}(\Omega) \mid b(\boldsymbol{v}, q) = 0 \quad \forall q \in L^2_0(\Omega)\}$$

and a system of functionals $\{j_\varepsilon\}, \varepsilon \to 0+$ with the following properties:

– $j_\varepsilon : L^2(S) \to \mathbb{R}_+$ is non-negative, convex, and differentiable $\forall\varepsilon>0$; (3)

– $\lim_{\varepsilon\to 0+} j_\varepsilon(q) = j(q) \quad \forall q \in L^2(S)$; (4)

– $\liminf_{\varepsilon\to 0+} j_\varepsilon(q_\varepsilon) \geq j(q)$ holds for any $\{q_\varepsilon\}$, $q_\varepsilon \in L^2(S)$ s.t. $q_\varepsilon \to q$ in $L^2(S)$. (5)

The condition $v_\nu = 0$ on S will be penalized by the functional

$$g(v_\nu) = \tfrac{1}{2}\int_S (v_\nu)^2\, ds, \quad \boldsymbol{v} \in \mathbb{W}(\Omega). \tag{6}$$

The *penalized-regularized* formulation of $(\mathcal{P}(\Omega)), (\mathcal{M}(\Omega))$ reads as follows:

$$\text{Find } \boldsymbol{u}_\varepsilon = \operatorname*{argmin}_{\boldsymbol{v}\in\mathbb{W}_{\mathrm{div}}(\Omega)} \left\{ J_\varepsilon(\boldsymbol{v}) := \tfrac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) + j_\varepsilon(v_t) + \frac{1}{\varepsilon}g(v_\nu) - (\boldsymbol{f},\boldsymbol{v})_{0,\Omega} \right\} \tag{$\mathcal{P}_\varepsilon(\Omega)$}$$

and

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbb{W}(\Omega) \times L^2_0(\Omega) \text{ such that} \\ a(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) - b(\boldsymbol{v}, p_\varepsilon) + \langle \nabla j_\varepsilon(u_{\varepsilon t}), v_t\rangle \\ \qquad\qquad + \dfrac{1}{\varepsilon}\langle \nabla g(u_{\varepsilon\nu}), v_\nu\rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega) \\ b(\boldsymbol{u}_\varepsilon, q) = 0 \quad \forall q \in L^2_0(\Omega), \end{cases} \tag{$\mathcal{M}_\varepsilon(\Omega)$}$$

respectively. From (6) we see that

$$\langle \nabla g(u_\nu), v_\nu\rangle = \int_S u_\nu v_\nu\, ds. \tag{7}$$

Problems $(\mathcal{P}_\varepsilon(\Omega))$, $(\mathcal{M}_\varepsilon(\Omega))$ have unique solutions $\boldsymbol{u}_\varepsilon$, and $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$, respectively for every $\varepsilon > 0$. In addition, the first component $\boldsymbol{u}_\varepsilon$ of the solution to $(\mathcal{M}_\varepsilon(\Omega))$ solves $(\mathcal{P}_\varepsilon(\Omega))$. Using techniques from [9, Chpt. I, Th. 7.1 and Chpt. II, Th. 6.3] one can show that

$$(\boldsymbol{u}_\varepsilon, p_\varepsilon) \rightharpoonup (\boldsymbol{u}, p) \quad \text{in } (H^1(\Omega))^2 \times L^2(\Omega), \tag{8}$$

as $\varepsilon \to 0+$ where $(\boldsymbol{u}, p)$ is the solution of $(\mathcal{M}(\Omega))$. In the next section we shall study the stability of solutions to $(\mathcal{M}_\varepsilon(\Omega))$ with respect to Ω and $\varepsilon \to 0+$. Convergence (8) will be a special case of this result.

3. **Stability of** $(\mathcal{M}_\varepsilon(\Omega))$ **with respect to** $\varepsilon > 0$ **and** Ω**.** Now we shall consider problems $(\mathcal{P}_\varepsilon(\Omega))$ and $(\mathcal{M}_\varepsilon(\Omega))$ parametrized simultaneously by $\varepsilon > 0$ and Ω. To this end we introduce a system $\mathcal{O}$ of bounded domains Ω with the Lipschitz boundaries $\partial\Omega = \overline{\Gamma^\Omega} \cup \overline{S^\Omega}$, where Γ^Ω, S^Ω are parts of $\partial\Omega$ where the no-slip, and slip boundary conditions, respectively are prescribed. We shall suppose that $|\Gamma^\Omega| \geq \delta$, $|S^\Omega| \geq \delta$, where $\delta > 0$ does not depend on $\Omega \in \mathcal{O}$ and $|S^\Omega|, |\Gamma^\Omega|$ stand for the length of S^Ω, and Γ^Ω, respectively. Furthermore let there exist two bounded domains C, $\hat{\Omega}$ such that $C \subset \overline{\Omega} \subset \hat{\Omega}$ and $\operatorname{dist}(\partial\Omega, \partial\hat{\Omega}) \geq \delta_0$ for all $\Omega \in \mathcal{O}$, where $\delta_0 > 0$ is independent of $\Omega \in \mathcal{O}$.

To analyze the stability with respect to $\Omega \in \mathcal{O}$ one has to define convergence $\xrightarrow{\mathcal{O}}$ in $\mathcal{O}$. The system $\mathcal{O}$ with a concrete choice of $\xrightarrow{\mathcal{O}}$ will be denoted by $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$.

In this abstract setting we do not specify explicitly the choice of $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$. This will be done implicitly, namely we shall consider such $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$ for which the assumptions formulated below will be satisfied. Only what we require a priori is that any subsequence of a convergent sequence of domains from $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$ converges to the same element.

First we suppose that $\mathcal{O}$ possesses a *uniform extension property*: for any $\Omega \in \mathcal{O}$ there exists an extension mapping $E_\Omega \in \mathcal{L}((H^1(\Omega))^2, (H^1(\hat{\Omega}))^2)$ such that

$$\|E_\Omega \boldsymbol{v}\|_{1,\hat{\Omega}} \leq c\|\boldsymbol{v}\|_{1,\Omega} \tag{9}$$

holds for every $\boldsymbol{v} \in (H^1(\Omega))^2$ with a constant $c > 0$ independent of $\Omega \in \mathcal{O}$.

To emphasize the fact that Ω is one of the parameters of the problem, it will be appended to all data as a superscript. Thus we shall write $a^\Omega, b^\Omega, \boldsymbol{u}^\Omega, \boldsymbol{u}_\varepsilon^\Omega, \ldots$ instead of $a, b, \boldsymbol{u}, \boldsymbol{u}_\varepsilon, \ldots$ which are defined in the same way as in Section 2. In particular the functionals, $j^\Omega, j_\varepsilon^\Omega, g^\Omega : L^2(S^\Omega) \to \mathbb{R}_+$. To simplify notation we shall also write $\hat{\boldsymbol{v}}^\Omega := E_\Omega \boldsymbol{v}$, $\boldsymbol{v} \in (H^1(\Omega))^2$, in what follows while $\mathring{q}$ stands for the extension of a function $q \in L^2(\Omega)$ by zero outside of Ω.

To guarantee uniform boundedness of solutions to $(\mathcal{M}_\varepsilon(\Omega))$ with respect to $\Omega \in \mathcal{O}$ and $\varepsilon > 0$, the system $\mathcal{O}$ will be chosen in such a way that the following assumptions are satisfied:

$$\exists\, \alpha = \text{const.} > 0 : \quad a^\Omega(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|_{1,\Omega}^2 \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega)\ \forall \Omega \in \mathcal{O}; \tag{10}$$

$$\exists\, \beta = \text{const.} > 0 : \quad \sup_{\boldsymbol{v} \in (H_0^1(\Omega))^2} \frac{b^\Omega(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta\|q\|_{0,\Omega} \quad \forall q \in L_0^2(\Omega)\ \forall \Omega \in \mathcal{O}, \tag{11}$$

i.e. a^Ω is $\mathbb{W}(\Omega)$-elliptic and b^Ω satisfies the inf-sup condition, both uniformly with respect to $\Omega \in \mathcal{O}$.

Further we shall suppose that

$$\exists\, c = \text{const.} > 0\ \exists\, \varepsilon_0 > 0 : \quad j_\varepsilon^\Omega(0) \leq c \quad \forall \varepsilon \in]0, \varepsilon_0],\ \forall \Omega \in \mathcal{O}, \tag{12}$$

and the right hand side of the Stokes system in Ω is the restriction of a function $\boldsymbol{f} \in (L^2(\hat{\Omega}))^2$.

Lemma 3.1. *Let* (10)–(12) *be satisfied. Then there exists a constant* $c > 0$ *such that*

$$\|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega} + \|p_\varepsilon^\Omega\|_{0,\Omega} \leq c \tag{13}$$

and

$$0 \leq g^\Omega(u_{\varepsilon\nu}^\Omega) \leq c\,\varepsilon \tag{14}$$

hold for any $\varepsilon \in]0, \varepsilon_0]$ *and* $\Omega \in \mathcal{O}$.

Proof. From the definition of $(\mathcal{P}_\varepsilon(\Omega))$ we have:

$$\frac{1}{2}a^\Omega(\boldsymbol{u}_\varepsilon^\Omega, \boldsymbol{u}_\varepsilon^\Omega) + \frac{1}{\varepsilon}g^\Omega(u_{\varepsilon\nu}^\Omega) \leq J_\varepsilon^\Omega(\boldsymbol{u}_\varepsilon^\Omega) + (\boldsymbol{f}, \boldsymbol{u}_\varepsilon^\Omega)_{0,\Omega}$$
$$\leq J_\varepsilon^\Omega(\boldsymbol{0}) + (\boldsymbol{f}, \boldsymbol{u}_\varepsilon^\Omega)_{0,\Omega} \leq j_\varepsilon^\Omega(0) + \|\boldsymbol{f}\|_{0,\hat{\Omega}}\|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega}.$$

From this, (10) and (12) the boundedness of $\{\boldsymbol{u}_\varepsilon^\Omega\}$ and (14) follow. Using test functions $\boldsymbol{v} \in (H_0^1(\Omega))^2$ in $(\mathcal{M}_\varepsilon(\Omega))$ together with (11) we obtain:

$$\beta \|p_\varepsilon^\Omega\|_{0,\Omega} \leq \sup_{\boldsymbol{v}\in(H_0^1(\Omega))^2} \frac{b^\Omega(\boldsymbol{v}, p_\varepsilon^\Omega)}{\|\boldsymbol{v}\|_{1,\Omega}} \leq \|\boldsymbol{u}_\varepsilon^\Omega\|_{1,\Omega} + \|\boldsymbol{f}\|_{0,\hat{\Omega}}$$

and hence (13) holds true. □

Owing to (9) and (13), the extensions of $(\boldsymbol{u}_\varepsilon^\Omega, p_\varepsilon^\Omega)$ from Ω to $\hat{\Omega}$ are bounded, as well:

$$\exists\, c = \text{const.} > 0 : \quad \|\hat{\boldsymbol{u}}_\varepsilon^\Omega\|_{1,\hat{\Omega}} + \|\mathring{p}_\varepsilon^\Omega\|_{0,\hat{\Omega}} \leq c \quad \forall \varepsilon \in]0, \varepsilon_0] \ \forall \Omega \in \mathcal{O}. \tag{15}$$

Let $\{\Omega_k\}, \Omega_k \in \mathcal{O}$ be such that $\Omega_k \xrightarrow{\mathcal{O}} \Omega \in \mathcal{O}$ and consider problems $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$, where $\varepsilon_k \to 0+$ as $k \to \infty$. Next we will study the relation between solutions of $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$ and $(\mathcal{M}(\Omega))$ when $k \to \infty$. To this end we shall suppose that $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$ is chosen in such a way that the following assumptions are satisfied:

– for any $\{\boldsymbol{v}^k\}$ such that $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, $\boldsymbol{v}^k|_{\Omega_k} \in \mathbb{W}(\Omega_k)$ it follows that
$$\boldsymbol{v}|_\Omega \in \mathbb{W}(\Omega); \tag{16}$$

– $\forall \boldsymbol{v} \in \mathbb{V}(\Omega)$ there exists a sequence $\{\boldsymbol{v}^k\}$, $\boldsymbol{v}^k \in (H^1(\hat{\Omega}))^2$ and a function $\bar{\boldsymbol{v}} \in (H^1(\hat{\Omega}))^2$, $\bar{\boldsymbol{v}}|_\Omega = \boldsymbol{v}$ such that
$$\boldsymbol{v}^k \to \bar{\boldsymbol{v}} \quad \text{in } (H^1(\hat{\Omega}))^2 \tag{17}$$
and for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ for which
$$\boldsymbol{v}^k|_{\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k}); \tag{18}$$

– if $\{\boldsymbol{v}^k\}$, $\{\boldsymbol{w}^k\}$, and $\{q_k\}$, $\{z_k\}$ are such that $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$, $\boldsymbol{w}^k \to \boldsymbol{w}$ in $(H^1(\hat{\Omega}))^2$, and $q_k \rightharpoonup q$, $z_k \to z$ in $L^2(\hat{\Omega})$ then
$$\limsup_{k\to\infty} a^{\Omega_k}(\boldsymbol{v}^k|_{\Omega_k}, \boldsymbol{w}^k|_{\Omega_k} - \boldsymbol{v}^k|_{\Omega_k}) \leq a^\Omega(\boldsymbol{v}|_\Omega, \boldsymbol{w}|_\Omega - \boldsymbol{v}|_\Omega) \tag{19}$$
$$\lim_{k\to\infty} b^{\Omega_k}(\boldsymbol{w}^k|_{\Omega_k}, q_k|_{\Omega_k}) = b^\Omega(\boldsymbol{w}|_\Omega, q|_\Omega) \tag{20}$$
$$\lim_{k\to\infty} b^{\Omega_k}(\boldsymbol{v}^k|_{\Omega_k}, z_k|_{\Omega_k}) = b^\Omega(\boldsymbol{v}|_\Omega, z|_\Omega) \tag{21}$$
$$\lim_{k\to\infty} (\boldsymbol{f}, \boldsymbol{v}|_{\Omega_k})_{0,\Omega_k} = (\boldsymbol{f}, \boldsymbol{v}|_\Omega)_{0,\Omega}. \tag{22}$$

– if $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ then[1]
$$g^{\Omega_k}(v_\nu^k) \to g^\Omega(v_\nu) \tag{23}$$
and
$$j_{\varepsilon_k}^{\Omega_k}(v_t^k) \to j^\Omega(v_t), \quad k \to \infty. \tag{24}$$

Theorem 3.2. *Let $\varepsilon_k \to 0+$, $\Omega_k \xrightarrow{\mathcal{O}} \Omega$ as $k \to \infty$, $\Omega_k, \Omega \in \mathcal{O}$ and (16)–(24) be satisfied. Let the sequence of solutions $\{(\boldsymbol{u}_{\varepsilon_k}^{\Omega_k}, p_{\varepsilon_k}^{\Omega_k})\}$ to $(\mathcal{M}_{\varepsilon_k}(\Omega_k))$ be such that*
$$\hat{\boldsymbol{u}}_{\varepsilon_k}^{\Omega_k} \rightharpoonup \bar{\boldsymbol{u}} \quad \textit{in } (H^1(\hat{\Omega}))^2, \tag{25}$$
$$\mathring{p}_{\varepsilon_k}^{\Omega_k} \rightharpoonup \bar{p} \quad \textit{in } L^2(\hat{\Omega}), \quad k \to \infty \tag{26}$$
for some $(\bar{\boldsymbol{u}}, \bar{p}) \in (H^1(\hat{\Omega}))^2 \times L_0^2(\hat{\Omega})$. Then $(\bar{\boldsymbol{u}}|_\Omega, \bar{p}|_\Omega)$ solves $(\mathcal{M}(\Omega))$.

[1] Recall that $v_\nu^k = \boldsymbol{v}^k|_{S^{\Omega_k}} \cdot \boldsymbol{\nu}^k$, $v_\nu = \boldsymbol{v}|_{S^\Omega} \cdot \boldsymbol{\nu}$, and $\boldsymbol{\nu}^k, \boldsymbol{\nu}$ is the outward unit normal vector to S^{Ω_k}, and S^Ω, respectively (similarly v_t^k and v_t).

Proof. To simplify notation, instead of the superscript Ω_k we shall write simply k in what follows. Thus $a^k := a^{\Omega_k}$, $j^k_{\varepsilon_k} := j^{\Omega_k}_{\varepsilon_k}$, etc. Using this convention, $(\boldsymbol{u}^k_{\varepsilon_k}, p^k_{\varepsilon_k})$ satisfies:

$$\begin{cases} a^k(\boldsymbol{u}^k_{\varepsilon_k}, \boldsymbol{v}) - b^k(\boldsymbol{v}, p^k_{\varepsilon_k}) + \langle \nabla j^k_{\varepsilon_k}(u^k_{\varepsilon_k t}), v_t\rangle \\ \qquad\qquad + \dfrac{1}{\varepsilon_k}\langle \nabla g^k(u^k_{\varepsilon_k \nu}), v_\nu\rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega_k} \quad \forall \boldsymbol{v} \in \mathbb{W}(\Omega_k) \\ b^k(\boldsymbol{u}^k_{\varepsilon_k}, q) = 0 \quad \forall q \in L^2_0(\Omega_k). \end{cases} \qquad (\mathcal{M}_{\varepsilon_k}(\Omega_k))$$

First we show that $\bar{\boldsymbol{u}}_{|\Omega} \in \mathbb{V}_{\text{div}}(\Omega)$. The fact that $\bar{\boldsymbol{u}} = \boldsymbol{0}$ on Γ^Ω and $\bar{\boldsymbol{u}}_{|\Omega}\cdot\boldsymbol{\nu} = 0$ on S^Ω follows from (16), and (14)+(23). Let $z \in L^2(\hat{\Omega})$ be arbitrary, denote $z_k := z_{|\Omega_k}$ and decompose z_k: $z_k = \bar{z}_k + c_k$, where $c_k = \left(\int_{\Omega_k} z_k\,dx\right)/\text{meas}\,\Omega_k$. Since $\bar{z}_k \in L^2_0(\Omega_k)$ we have:

$$\int_{\Omega_k} \text{div}\,\boldsymbol{u}^k_{\varepsilon_k} z_k\,dx = \int_{\Omega_k} \text{div}\,\boldsymbol{u}^k_{\varepsilon_k} \bar{z}_k\,dx + c_k\int_{\Omega_k} \text{div}\boldsymbol{u}^k_{\varepsilon_k}\,dx = c_k\int_{S^k} u^k_{\varepsilon_k\nu}\,ds.$$

Passing to the limit with $k \to \infty$, using (14), (21), (25) and the definition of z_k we see that

$$\int_\Omega \text{div}\,\bar{\boldsymbol{u}}\,z\,dx = 0 \quad \forall z \in L^2(\hat{\Omega}), \tag{27}$$

i.e. $\text{div}\,\bar{\boldsymbol{u}}_{|\Omega} = 0$.

The fact that $\bar{p}_{|\Omega} \in L^2_0(\Omega)$ follows from (20), hence $(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) \in \mathbb{V}_{\text{div}}(\Omega)\times L^2_0(\Omega)$.

Let $\boldsymbol{v} \in \mathbb{V}(\Omega)$ be arbitrary and $\{\boldsymbol{v}^k\}$ be a sequence satisfying (17) and (18). Using $\boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_k}$ as a test function in $(\mathcal{M}_{\varepsilon_{n_k}}(\Omega_{n_k}))$[2], where n_k is the filter of indices for which (18) holds, we obtain:

$$a^{n_k}(\boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}) - b^{n_k}(\boldsymbol{v}^k, p^{n_k}_{\varepsilon_{n_k}}) + \\ \langle \nabla j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k}t}), v^k_t - u^{n_k}_{\varepsilon_{n_k}t}\rangle \geq (\boldsymbol{f}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}})_{0,\Omega_{n_k}} \tag{28}$$

taking into account that $\boldsymbol{v}^k{}_{|\Omega_{n_k}} \in \mathbb{V}(\Omega_{n_k})$ and the second equation in $(\mathcal{M}_{\varepsilon_{n_k}}(\Omega_{n_k}))$. Adding the term $j^{n_k}_{\varepsilon_{n_k}}(v^k_t) - j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k}t})$ to both sides of (28) and using convexity of $j^{n_k}_{\varepsilon_{n_k}}$ we arrive at

$$a^{n_k}(\boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}}) - b^{n_k}(\boldsymbol{v}^k, p^{n_k}_{\varepsilon_{n_k}}) \\ + j^{n_k}_{\varepsilon_{n_k}}(v^k_t) - j^{n_k}_{\varepsilon_{n_k}}(u^{n_k}_{\varepsilon_{n_k}t}) \geq (\boldsymbol{f}, \boldsymbol{v}^k - \boldsymbol{u}^{n_k}_{\varepsilon_{n_k}})_{0,\Omega_{n_k}}. \tag{29}$$

If $k \to \infty$ in (29) then

$$a^\Omega(\bar{\boldsymbol{u}}_{|\Omega}, \boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega}) - b^\Omega(\boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) + j^\Omega(v_t) - j^\Omega(\bar{u}_t) \geq (\boldsymbol{f}, \boldsymbol{v} - \bar{\boldsymbol{u}}_{|\Omega})_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V}(\Omega)$$

making use of (19)–(22), (24). From this and the inf-sup condition it follows that $\bar{p}_{|\Omega} = p^\Omega$. Consequently, $(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) := (\boldsymbol{u}^\Omega, p^\Omega)$ is the solution of $(\mathcal{M}(\Omega))$. $\square$

Let us comment on the assumptions formulated above. It is known that if the system $\mathcal{O}$ consists of domains with the *uniform cone property* (it will be denoted

[2] for simplicity of notation we write $\boldsymbol{v}^k$ instead of $\boldsymbol{v}^k{}_{|\Omega_{n_k}}$

by $\mathcal{O}_{\text{cone}}$) then (9) is satisfied ([4]). $\mathcal{O}_{\text{cone}}$ has yet other properties which guarantee that some of the previous assumptions are automatically satisfied. It holds:

– $\mathcal{O}_{\text{cone}}$ is compact with respect to the Hausdorff metric; (30)

– if $\Omega_k \xrightarrow{\mathcal{H}} \Omega$, $\Omega_k \in \mathcal{O}_{\text{cone}}$ then

$$\partial\Omega_k \xrightarrow{\mathcal{H}} \partial\Omega \tag{31}$$

and

$$\chi_k \to \chi \quad \text{in } L^2(\hat{\Omega}), \tag{32}$$

where $\xrightarrow{\mathcal{H}}$ stands for convergence in the Hausdorff metric, χ_k, χ are the characteristic functions of Ω_k and Ω, respectively (see [22], [16]). From (32) we easily obtain (19)–(22). Also the uniform ellipticity of a^Ω with respect to $\Omega \in \mathcal{O}_{\text{cone}}$, i.e. (10), is satisfied (see [11]). The next property is an easy consequence of (32), too:

– if $\Omega_k \xrightarrow{\mathcal{H}} \Omega$, $\Omega_k \in \mathcal{O}_{\text{cone}}$ and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ then

$$\liminf_{k\to\infty} \|\mathbb{D}\boldsymbol{v}^k\|_{0,\Omega_k} \geq \|\mathbb{D}\boldsymbol{v}\|_{0,\Omega}. \tag{33}$$

On the other hand, in order to satisfy (16), (17), (18), (23), and (24) we usually need appropriate subsets of $\mathcal{O}_{\text{cone}}$ which consist of domains with more regular boundaries. One example of such a system will be presented in Section 5.

4. **Optimal shape design problems.** First we present a class of optimal shape design problems we want to solve with the velocity-pressure formulation $(\mathcal{M}(\Omega))$ of (1) as the state relation.

Let $\{\mathcal{O}, \xrightarrow{\mathcal{O}}\}$ be a system of admissible domains introduced in Section 3. Further we choose an objective functional I which depends on $(\Omega, \boldsymbol{u}^\Omega, p^\Omega)$, $\Omega \in \mathcal{O}$, with $(\boldsymbol{u}^\Omega, p^\Omega)$ being the solution to $(\mathcal{M}(\Omega))$ and denote $\mathcal{J}(\Omega) := I(\Omega, \boldsymbol{u}^\Omega, p^\Omega)$.

Optimal shape design problems we shall deal with read as follows:

$$\text{Find } \Omega^* \in \arg\min \{\mathcal{J}(\Omega) \mid \Omega \in \mathcal{O}\}. \tag{$\mathbb{P}$}$$

To prove the existence of solutions to $(\mathbb{P})$ we shall need the following assumptions:

– (*sequential compactness of* $\mathcal{O}$)
in any sequence $\{\Omega_k\}$, $\Omega_k \in \mathcal{O}$ there exist: a subsequence $\{\Omega_{k_j}\}$ and $\Omega \in \mathcal{O}$ such that

$$\Omega_{k_j} \xrightarrow{\mathcal{O}} \Omega \in \mathcal{O}, \quad j \to \infty; \tag{34}$$

– (*uniform boundedness of* $\{(\boldsymbol{u}^\Omega, p^\Omega)\}$)

$$\exists\, c = \text{const.} > 0 : \quad \|\hat{\boldsymbol{u}}^\Omega\|_{1,\hat{\Omega}} + \|\mathring{p}^\Omega\|_{0,\hat{\Omega}} \leq c \quad \forall \Omega \in \mathcal{O}; \tag{35}$$

– (*lower semicontinuity of* I)
if $\Omega_k \xrightarrow{\mathcal{O}} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$, $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, and $q_k \rightharpoonup q$ in $L^2(\hat{\Omega})$ then

$$\liminf_{k\to\infty} I(\Omega_k, \boldsymbol{v}^k|_{\Omega_k}, q_k|_{\Omega_k}) \geq I(\Omega, \boldsymbol{v}|_\Omega, q|_\Omega). \tag{36}$$

Finally we suppose that the following assumption is satisfied:

– if $\Omega_k \xrightarrow{\mathcal{O}} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$ and $\{(\boldsymbol{u}^{\Omega_k}, p^{\Omega_k})\}$ is the sequence of solutions to $(\mathcal{M}(\Omega_k))$ such that $(\hat{\boldsymbol{u}}^{\Omega_k}, \mathring{p}^{\Omega_k}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ then

$$(\bar{\boldsymbol{u}}|_\Omega, \bar{p}|_\Omega) \text{ solves } (\mathcal{M}(\Omega)). \tag{37}$$

Theorem 4.1. *Let (34)–(37) be satisfied. Then* ($\mathbb{P}$) *has a solution.*

Proof. Let $\{\Omega_k\}$, $\Omega_k \in \mathcal{O}$ be a minimizing sequence in ($\mathbb{P}$):

$$\lim_{k\to\infty} \mathcal{J}(\Omega_k) = \inf_{\mathcal{O}} \mathcal{J}(\Omega).$$

Owing to (34), (35), and (37) we may pass to a subsequence (denoted by the same symbol) such that $\Omega_k \overset{\mathcal{O}}{\to} \Omega^* \in \mathcal{O}$, $(\hat{\boldsymbol{u}}^{\Omega_k}, \mathring{p}^{\Omega_k}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$ and $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ solves $(\mathcal{M}(\Omega^*))$. The rest of the proof follows from (36). $\square$

As we have mentioned in Introduction, problem ($\mathbb{P}$) is generally non-smooth due to a possible non-differentiability of the control-to-state mapping $\phi : \Omega \mapsto (\boldsymbol{u}^\Omega, p^\Omega)$. For this reason we shall approximate problem ($\mathbb{P}$) by a sequence of problems $(\mathbb{P}_\varepsilon)$, $\varepsilon \to 0+$ which utilize $(\mathcal{M}_\varepsilon(\Omega))$ as the state problem.

Problem $(\mathbb{P}_\varepsilon), \varepsilon > 0$, reads as follows:

$$\text{Find } \Omega^*_\varepsilon \in \arg\min \{\mathcal{J}_\varepsilon(\Omega) \mid \Omega \in \mathcal{O}\}, \tag{$\mathbb{P}_\varepsilon$}$$

where $\mathcal{J}_\varepsilon(\Omega) := I(\Omega, \boldsymbol{u}^\Omega_\varepsilon, p^\Omega_\varepsilon)$ and $(\boldsymbol{u}^\Omega_\varepsilon, p^\Omega_\varepsilon)$ solves $(\mathcal{M}_\varepsilon(\Omega))$.

Next we shall analyze if and under which conditions there exists a relation between solutions to ($\mathbb{P}$) and $(\mathbb{P}_\varepsilon)$ when $\varepsilon \to 0+$.

First of all we have to guarantee that $(\mathbb{P}_\varepsilon)$ has a solution for any $\varepsilon \in]0, \varepsilon_0]$, $\varepsilon_0 > 0$ sufficiently small. To this end we need the following minor modification of (37):

– if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$ and $\{(\boldsymbol{u}^{\Omega_k}_{\bar{\varepsilon}}, p^{\Omega_k}_{\bar{\varepsilon}})\}$ is the sequence of solutions to $(\mathcal{M}_{\bar{\varepsilon}}(\Omega_k))$ such that $(\hat{\boldsymbol{u}}^{\Omega_k}_{\bar{\varepsilon}}, \mathring{p}^{\Omega_k}_{\bar{\varepsilon}}) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p})$ in $(H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega})$, $k \to \infty$, then

$$(\bar{\boldsymbol{u}}_{|\Omega}, \bar{p}_{|\Omega}) \text{ solves } (\mathcal{M}_{\bar{\varepsilon}}(\Omega)) \tag{38}$$

and this holds for any $\bar{\varepsilon} \in]0, \varepsilon_0]$.

Theorem 4.2. *Let (15), (34), (36), and (38) be satisfied. Then* $(\mathbb{P}_\varepsilon)$ *has a solution for any* $\varepsilon > 0$.

Proof. It can be omitted. $\square$

To prove the next theorem we have to replace (36) by the following stronger *continuity assumption*:

– if $\Omega_k \overset{\mathcal{O}}{\to} \Omega$, $\Omega_k, \Omega \in \mathcal{O}$, and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, $q_k \rightharpoonup q$ in $L^2(\hat{\Omega})$, $k \to \infty$, then

$$\lim_{k\to\infty} I(\Omega_k, \boldsymbol{v}^k{}_{|\Omega_k}, q_k{}_{|\Omega_k}) = I(\Omega, \boldsymbol{v}_{|\Omega}, q_{|\Omega}). \tag{39}$$

Theorem 4.3. *Let (15), (34), (39) and all the assumptions of Theorem 3.2 be satisfied. Then for any sequence of solutions* $\{\Omega^*_{\varepsilon_k}\}$ *to* $(\mathbb{P}_{\varepsilon_k})$, $\varepsilon_k \to 0+$ *as* $k \to \infty$, *there exist: its subsequence (denoted by the same symbol) and* $\Omega^* \in \mathcal{O}$ *such that*

$$\begin{cases} \Omega^*_{\varepsilon_k} \overset{\mathcal{O}}{\to} \Omega^*, \\ (\hat{\boldsymbol{u}}^*_k, \mathring{p}^*_k) \rightharpoonup (\bar{\boldsymbol{u}}, \bar{p}) \quad \text{in } (H^1(\hat{\Omega}))^2 \times L^2(\hat{\Omega}), \quad k \to \infty, \end{cases} \tag{40}$$

where $(\boldsymbol{u}^*_k, p^*_k)$ *is the solution of* $(\mathcal{M}_{\varepsilon_k}(\Omega^*_{\varepsilon_k}))$. *In addition,* Ω^* *is a solution to* ($\mathbb{P}$) *and* $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ *solves* $(\mathcal{M}(\Omega^*))$. *Any accumulation point of* $\{(\boldsymbol{u}^*_k, p^*_k)\}$ *in the sense of (40) is a solution to* ($\mathbb{P}$).

Proof. From (15), (34) and Theorem 3.2 it follows that there exists $\Omega^* \in \mathcal{O}$ and a subsequence of $\{(\Omega^*_{\varepsilon_k}, \hat{\boldsymbol{u}}^*_k, \mathring{p}^*_k)\}$ (denoted by the same symbol) satisfying (40) and such that $(\bar{\boldsymbol{u}}_{|\Omega^*}, \bar{p}_{|\Omega^*})$ solves $(\mathcal{M}(\Omega^*))$. It remains to show that Ω^* solves $(\mathbb{P})$. Indeed, let $\widetilde{\Omega} \in \mathcal{O}$ be arbitrary but fixed and $\{(\widetilde{\boldsymbol{u}}_k, \widetilde{p}_k)\}$ be the sequence of solutions to $(\mathcal{M}_{\varepsilon_k}(\widetilde{\Omega}))$, $k \to \infty$. Since

$$(\widetilde{\boldsymbol{u}}_k, \widetilde{p}_k) \rightharpoonup (\widetilde{\boldsymbol{u}}, \widetilde{p}) \quad \text{in } (H^1(\widetilde{\Omega}))^2 \times L^2(\widetilde{\Omega}), \quad k \to \infty, \tag{41}$$

where $(\widetilde{\boldsymbol{u}}, \widetilde{p})$ is a solution of $(\mathcal{M}(\widetilde{\Omega}))$, the definition of $(\mathbb{P}_{\varepsilon_k})$ yields:

$$I(\Omega^*_{\varepsilon_k}, \boldsymbol{u}^*_k, p^*_k) \le I(\widetilde{\Omega}, \widetilde{\boldsymbol{u}}_k, \widetilde{p}_k).$$

Letting $k \to \infty$ and using (39), (40), and (41) we obtain that $\mathcal{J}(\Omega^*) \le \mathcal{J}(\widetilde{\Omega}) \quad \forall \widetilde{\Omega} \in \mathcal{O}$. $\square$

To conclude the theoretical part we formulate assumptions under which (37) and (38) are satisfied. Since the proof is only a minor modification of the one of Theorem 3.2 it will be omitted.

Theorem 4.4. *Let (16) and (19)–(22) be satisfied. If, in addition*

a) (17), (18) are satisfied and from $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$, $\Omega_k \overset{\mathcal{O}}{\to} \Omega$ it follows that $j^{\Omega_k}(v^k_t) \to j^{\Omega}(v_t)$, then (37) holds;

b) (23) and (24) with $\varepsilon_k = \bar{\varepsilon} \in]0, \varepsilon_0]$ $\forall k \in \mathbb{N}$ are satisfied, then (38) holds.

5. **Model shape optimization problems.** The aim of this section is to apply the previous theoretical results to a class of optimization problems that will be used in numerical experiments. The system $\mathcal{O}$ consists of domains with a simple shape, namely a part of the boundary to be optimized with the prescribed stick-slip condition is represented by the graph of a function.

The system $\mathcal{O}$ is defined as follows:

$$\mathcal{O} = \{\Omega(\alpha) \mid \alpha \in \mathcal{U}_{ad}\},$$

where

$$\Omega(\alpha) = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in]0, 1[,\ x_2 \in]\alpha(x_1), 1[\big\}$$

and

$$\mathcal{U}_{ad} = \Big\{\alpha \in C^{1,1}([0,1]) \mid \alpha_{\min} \le \alpha \le \alpha_{\max} < 1,\ |\alpha^{(j)}| \le C_j, j{=}1,2 \text{ a.e. in }]0,1[\Big\}, \tag{42}$$

i.e. $\mathcal{U}_{ad}$ is the set of functions which are together with their first derivatives equi-bounded and equi-Lipschitz continuous in $[0, 1]$. The constants $\alpha_{\min}$, $\alpha_{\max}$, C_1, and C_2 are chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. The boundary $\partial\Omega(\alpha) = \overline{\Gamma(\alpha)} \cup \overline{S(\alpha)}$, and $S(\alpha)$ is the graph of $\alpha \in \mathcal{U}_{ad}$ (see Figure 1).

In $\mathcal{O}$ we introduce convergence as follows:

$$\Omega_k := \Omega(\alpha_k) \overset{\mathcal{O}}{\to} \Omega(\alpha),\ \alpha_k \in \mathcal{U}_{ad} \iff \alpha_k \to \alpha \text{ in } C^1([0,1]).$$

On any $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ we shall consider the Stokes system with the no-slip, stick-slip boundary condition prescribed on $\Gamma(\alpha)$, and $S(\alpha)$, respectively. The fact that the shape of $\Omega(\alpha)$ is fully determined by the function $\alpha \in \mathcal{U}_{ad}$ enables us to simplify notation. Instead of $\mathbb{V}(\Omega(\alpha))$, $\mathbb{V}_{div}(\Omega(\alpha))$, $L^2_0(\Omega(\alpha))$,... we shall write $\mathbb{V}(\alpha)$, $\mathbb{V}_{div}(\alpha)$, $L^2_0(\alpha)$,... Similarly, $a^\alpha, b^\alpha, j^\alpha$,... is used in place of $a^{\Omega(\alpha)}$, $b^{\Omega(\alpha)}$, $j^{\Omega(\alpha)}$,...

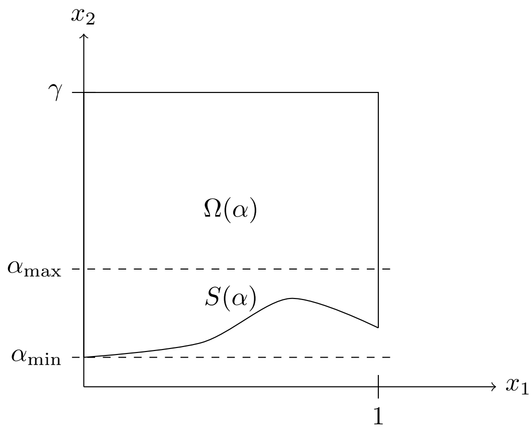

FIGURE 1. Shape of admissible domains.

Using this convention of notation, the velocity-pressure formulation of (1) reads as follows:

$$\begin{cases} \text{Find } (\boldsymbol{u}^\alpha, p^\alpha) \in \mathbb{V}(\alpha) \times L_0^2(\alpha) \text{ such that} \\ a^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v} - \boldsymbol{u}^\alpha) - b^\alpha(\boldsymbol{v} - \boldsymbol{u}^\alpha, p^\alpha) \\ \qquad + j^\alpha(v_t) - j^\alpha(u_t^\alpha) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^\alpha)_{0,\Omega(\alpha)} \quad \forall \boldsymbol{v} \in \mathbb{V}(\alpha) \\ b^\alpha(\boldsymbol{u}^\alpha, q) = 0 \quad \forall q \in L_0^2(\alpha). \end{cases} \tag{$\mathcal{M}(\alpha)$}$$

Now we shall verify all the assumptions of Section 3 and 4. Owing to the definition of $\mathcal{U}_{ad}$, all domains from $\mathcal{O}$ enjoy the uniform cone property and consequently (9), (10), (19)–(22) are satisfied. The constant β in the inf-sup condition (11) depends only on $\|\alpha\|_{W^{1,\infty}(0,1)}$ and it can be chosen independently of $\alpha \in \mathcal{U}_{ad}$ (see [2], [7]). Clearly, (16) is satisfied as well due to the special shape of $\Omega(\alpha), \alpha \in \mathcal{U}_{ad}$. Let us notice that all these assumptions hold true for any α belonging to an appropriate subset of $C^{0,1}([0,1])$. The reason why we ask $\alpha \in C^{1,1}([0,1])$ is to satisfy the remaining conditions of Section 3 and 4. Some of them have been already proven in [14], namely:

– (17) and (18) (Lemma 3 in [14]) ;
– (23) for the penalty functional

$$g^\alpha(v_\nu) = \int_0^1 (v_\nu \circ \alpha)^2 \, dx_1 = \int_0^1 (\boldsymbol{v}(x_1, \alpha(x_1)) \cdot \boldsymbol{\nu}^\alpha)^2 \, dx_1,$$

where $\boldsymbol{\nu}^\alpha$ stands for the unit outward normal vector to $S(\alpha)$.

In the next section we use the slip functional j^α, $\alpha \in \mathcal{U}_{ad}$, of the following form:

$$j^\alpha(q) = \int_{S(\alpha)} \varphi(q) \, ds, \quad \varphi(q) = \sigma_0 |q| + \frac{\sigma_1}{2}|q|^2, \; q \in L^2(S(\alpha)), \tag{43}$$

where σ_0 and σ_1 are given non-negative constants such that $\sigma_0 + \sigma_1 > 0$. With this choice of φ, the boundary condition $(1)_5$ can be rewritten as follows:

$$\left. \begin{aligned} &|\sigma_t| < \sigma_0 \Rightarrow u_t = 0, \\ &|\sigma_t| \geq \sigma_0 \Rightarrow -\sigma_t = \sigma_0 \frac{u_t}{|u_t|} + \sigma_1 u_t \end{aligned} \right\} \text{ on } S(\alpha). \tag{44}$$

If $\sigma_0 = 0$ then (44) is usually referred to as the Navier boundary condition, while for $\sigma_1 = 0$ it reminds the Tresca friction law known from solid mechanics.

Lemma 5.1. *The functional j^α defined by (43) is non-negative, convex, continuous and weakly lower semicontinuous in $L^2(S(\alpha))$ $\forall \alpha \in \mathcal{U}_{ad}$.*

Proof. Clearly, φ is non-negative and convex, so is j^α. Since

$$j^\alpha(q) = \sigma_0 \|q\|_{L^1(S(\alpha))} + \frac{\sigma_1}{2}\|q\|^2_{0,S(\alpha)},$$

its continuity and the weak lower semicontinuity, respectively, follows from the corresponding properties of the norms $\|\cdot\|_{L^1(S(\alpha))}$ and $\|\cdot\|_{0,S(\alpha)}$. □

For any $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon > 0$ we define the regularization functional

$$j^\alpha_\varepsilon(q) = \int_{S(\alpha)} \varphi_\varepsilon(q)\, ds,$$

where

$$\varphi_\varepsilon(q) = \begin{cases} \varphi(q) & \text{if } |q| \geq \varepsilon, \\ \sigma_0 \frac{|q|^2+\varepsilon^2}{2\varepsilon} + \frac{\sigma_1}{2}|q|^2 & \text{if } |q| < \varepsilon, \end{cases} \quad q \in L^2(S(\alpha)). \tag{45}$$

The behavior of j^α_ε and j^α with respect to $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon \to 0+$ is summarized in the next lemma.

Lemma 5.2. *The functionals $j^\alpha, j^\alpha_\varepsilon$ defined by (43), and (45), respectively, have the following properties:*

(i) for every $\alpha \in \mathcal{U}_{ad}$ and $\varepsilon > 0$, j^α_ε is non-negative, convex and continuously differentiable in $L^2(S(\alpha))$;

(ii) condition (12) is satisfied;

(iii) for every $\alpha \in \mathcal{U}_{ad}$ it holds:

$$q_\varepsilon \to q \text{ in } L^2(S(\alpha)) \Rightarrow j^\alpha_\varepsilon(q_\varepsilon) \to j^\alpha(q),\ \varepsilon \to 0+; \tag{46}$$

(iv) for every $\alpha \in \mathcal{U}_{ad}$ it holds:

$$q_\varepsilon \rightharpoonup q \text{ weakly in } L^2(S(\alpha)) \Rightarrow \liminf_{\varepsilon \to 0+} j^\alpha_\varepsilon(q_\varepsilon) \geq j^\alpha(q); \tag{47}$$

(v) if $\alpha_k \to \alpha$ in $C^1([0,1])$, $\alpha_k, \alpha \in \mathcal{U}_{ad}$, $\varepsilon_k \to 0+$, and $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat\Omega))^2$, then

$$j^{\alpha_k}_{\varepsilon_k}(v^k_t) \to j^\alpha(v_t), \quad k \to \infty$$

and

$$j^{\alpha_k}(v^k_t) \to j^\alpha(v_t), \quad k \to \infty.$$

Proof. It is readily seen that *(i)* holds. From the definition of φ_ε and φ we have:

$$\|\varphi_\varepsilon - \varphi\|_{\infty,\mathbb{R}} \leq \varphi_\varepsilon(0) = \frac{\sigma_0 \varepsilon}{2}, \tag{48}$$

and

$$\varphi_\varepsilon(x) \geq \varphi(x)\ \forall x \in \mathbb{R}; \tag{49}$$

ad (ii) From (48) it follows that

$$\forall \alpha \in \mathcal{U}_{ad}\ j^\alpha_\varepsilon(0) \leq \sqrt{1 + C_1^2}\,\frac{\sigma_0}{2}\varepsilon, \tag{50}$$

making use of the definition of $\mathcal{U}_{ad}$, i.e. (12) holds with $c := \sqrt{1+C_1^2}\frac{\sigma_0}{2}\varepsilon_0$;

ad (iii) If $q_\varepsilon \to q$ in $L^2(S(\alpha))$, $\varepsilon \to 0+$, $\alpha \in \mathcal{U}_{ad}$, then

$$|j^\alpha_\varepsilon(q_\varepsilon) - j^\alpha(q)| \leq |j^\alpha_\varepsilon(q_\varepsilon) - j^\alpha(q_\varepsilon)| + |j^\alpha(q_\varepsilon) - j^\alpha(q)| =: J_1 + J_2. \tag{51}$$

But
$$J_1 \le \int_{S(\alpha)} |\varphi_\varepsilon(q_\varepsilon) - \varphi(q_\varepsilon)|\, ds \le |S(\alpha)| \|\varphi_\varepsilon - \varphi\|_{\infty,\mathbb{R}} \overset{(48)}{\le} \frac{|S(\alpha)|\sigma_0\varepsilon}{2}$$
and
$$J_2 \to 0 \text{ for } \varepsilon \to 0+,$$
as follows from continuity of j^α, i.e. (46) holds true;

ad (iv) Let $q_\varepsilon \rightharpoonup q$ weakly in $L^2(S(\alpha))$, $\varepsilon \to 0+$, $\alpha \in \mathcal{U}_{ad}$. Then
$$\liminf_{\varepsilon \to 0+} j_\varepsilon^\alpha(q_\varepsilon) \overset{(49)}{\ge} \liminf_{\varepsilon \to 0+} j^\alpha(q_\varepsilon) \ge j^\alpha(q),$$
making use of the weak lower semicontinuity of j^α;

ad (v) It holds:
$$\begin{aligned} |j_{\varepsilon_k}^{\alpha_k}(v_t^k) - j^\alpha(v_t)| &= \left| \int_0^1 \varphi_{\varepsilon_k}(v_t^k \circ \alpha_k)\sqrt{1+\alpha_k'^2} - \varphi(v_t \circ \alpha)\sqrt{1+\alpha'^2} \right| dx_1 \\ &\le \int_0^1 \left|\varphi_{\varepsilon_k}(v_t^k \circ \alpha_k) - \varphi(v_t^k \circ \alpha_k)\right| \sqrt{1+\alpha_k'^2}\, dx_1 \\ &+ \int_0^1 \left| \varphi(v_t^k \circ \alpha_k)\sqrt{1+\alpha_k'^2} - \varphi(v_t \circ \alpha)\sqrt{1+\alpha'^2} \right| dx_1 =: J_3 + J_4. \end{aligned}$$
Clearly
$$J_3 \le \|\sqrt{1+\alpha_k'^2}\|_{\infty,(0,1)} \|\varphi_{\varepsilon_k} - \varphi\|_{\infty,\mathbb{R}} \overset{(48)}{\le} \sqrt{1+C_1^2}\frac{\sigma_0}{2}\varepsilon_k. \tag{52}$$
From $\boldsymbol{v}^k \rightharpoonup \boldsymbol{v}$ in $(H^1(\hat{\Omega}))^2$ and $\alpha_k \to \alpha$ in $C^1([0,1])$ it follows that (see Theorem 3 in [14])
$$v_t^k \circ \alpha_k \to v_t \circ \alpha \quad \text{in } L^2(0,1), \quad k \to \infty$$
and also
$$\varphi(v_t^k \circ \alpha_k) \to \varphi(v_t \circ \alpha) \quad \text{in } L^1(0,1).$$
Hence
$$J_4 \to 0 \text{ as } k \to \infty.$$
From this and (52), the first limit in *(v)* follows. The second limit can be proven analogously. $\square$

Owing to the definition of $\mathcal{U}_{ad}$, the system $\mathcal{O}$ is compact with respect to the C^1 norm, i.e. (34) holds. If the cost functional I is lower semicontinous as in (36) then all the assumptions of Theorem 4.1 and 4.2 are satisfied. Consequently, problems $(\mathbb{P})$ and $(\mathbb{P}_\varepsilon)$ have a solution. If, in addition, I satisfies (39), then Theorem 4.3 can be applied. It says that solutions to $(\mathbb{P}_\varepsilon)$ for $\varepsilon \to 0+$ are close to the ones of $(\mathbb{P})$ in the sense of (40).

6. **Approximation and numerical realization of** $(\mathbb{P}_\varepsilon)$**.** In this section we describe how to discretize and realize shape optimization problems governed by the Stokes system with the regularized, penalized threshold, and impermeablity condition, respectively. The system of admissible domains Ω is as in Section 5, i.e. the shapes of Ω are uniquely determined by functions $\alpha \in \mathcal{U}_{ad}$ defined by (42). The control variables $\alpha \in \mathcal{U}_{ad}$ will be discretized by Bézier functions, while a finite element method will be used to discretize the state equation $(\mathcal{M}_\varepsilon(\alpha))$.

6.1. **Discrete design parametrization and a finite element approximation of the state problem.** We define the following finite dimensional parametrization of the slip boundary $S(\alpha) = \{(x_1, x_2) \mid x_1 \in [0,1],\ x_2{=}\alpha(x_1)\}$, $\alpha \in \mathcal{U}_{ad}$ using a Bézier polynomial of degree m:

$$\alpha_m(x_1) = \sum_{i=0}^{m} a_i B_i^{(m)}(x_1), \quad x_1 \in [0,1], \tag{53}$$

where $B_i^{(m)}(t) = \binom{m}{i} t^i (1-t)^{(m-i)}$, $i{=}0,...,m$ are the Bernstein polynomials on $[0,1]$. Thus, the discrete design variable is the vector of the x_2-coordinates $\mathbf{a} = (a_0, a_1, ..., a_m)$ of the Bézier control points $(\frac{i}{m}, a_i)$, $i{=}0,...,m$.

Next we discretize the state problem $(\mathcal{M}_\varepsilon(\alpha_m))$ by the P1-bubble/P1 elements satisfying the Ladyzhenskaya-Babuška-Brezzi condition [1]. Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}_h(\alpha_m)$ (a polygonal approximation of $\Omega(\alpha_m)$) and

$$\mathcal{V}_h(\alpha_m) = \{v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_h\}$$

be the space of piecewise linear finite elements of Lagrange type. The space of bubble functions is defined by

$$B_h(\alpha_m) = \left\{ v_h \in C(\overline{\Omega}_h(\alpha_m)) \mid v_{h|T} \in \operatorname{span}(b_T)\ \forall T \in \mathcal{T}_h \right\},$$

where $b_T = \lambda_{1,T}\lambda_{2,T}\lambda_{3,T} \in P_3(T)$ is the "bubble" function and $\lambda_{1,T}$, $\lambda_{2,T}$, and $\lambda_{3,T}$ are the barycentric coordinates of points with respect to the vertices of T.

Then we introduce the following finite element spaces:

$$\begin{aligned} &\mathbb{W}_h(\alpha_m) = [\mathcal{V}_h(\alpha_m) + B_h(\alpha_m)]^2 \\ &Q_h(\alpha_m) = \left\{ q_h \in C(\overline{\Omega}_h(\alpha_m)) \mid q_{h|T} \in P_1(T)\ \forall T \in \mathcal{T}_h,\ \textstyle\int_{\Omega_h(\alpha_m)} q_h\, dx = 0 \right\}, \end{aligned}$$

which are the discretizations of the spaces $\mathbb{W}(\Omega(\alpha_m))$ and $L_0^2(\Omega(\alpha_m))$, respectively.

The finite element approximation of the state problem in the parametrized domain $\Omega(\alpha_m)$ then reads (for simplicity of notation, the superscript α_m is omitted):

$$\begin{cases} \text{Find } (\boldsymbol{u}_\varepsilon^h, p_\varepsilon^h) \in \mathbb{W}_h(\alpha_m) \times Q_h(\alpha_m) \text{ such that} \\ a(\boldsymbol{u}_\varepsilon^h, \boldsymbol{v}) - b(\boldsymbol{v}, p_\varepsilon^h) + \langle \nabla j_\varepsilon(u_{\varepsilon t}^h), v_t \rangle \\ \qquad + \dfrac{1}{\varepsilon}\langle \nabla g(u_{\varepsilon\nu}^h), v_\nu \rangle = (\boldsymbol{f}, \boldsymbol{v})_{0,\Omega(\alpha_m)} \quad \forall \boldsymbol{v} \in \mathbb{W}_h(\alpha_m) \\ b(\boldsymbol{u}_\varepsilon^h, q) = 0 \quad \forall q \in Q_h(\alpha_m). \end{cases} \tag{$\mathcal{M}_\varepsilon^h(\alpha_m)$}$$

Finally we present a way how to construct a finite element mesh $\mathcal{T}_h$ in $\overline{\Omega}_h(\alpha_m)$ in such a way that the coordinates of its nodes $\{N^{(i)}\}_{i=1}^{n_p}$ depend smoothly on the design parameter vector $\mathbf{a}$. Let $\widehat{\mathcal{T}_h}$ be a (not necessarily structured) reference triangulation of the square $[0,1] \times [0,1]$ with the nodes $\{\hat{N}^{(i)}\}_{i=1}^{n_p}$. Then we set

$$N_1^{(i)} = \hat{N}_1^{(i)}, \quad N_2^{(i)} = \hat{N}_2^{(i)} + \alpha_m(\hat{N}_1^{(i)})(1 - \hat{N}_2^{(i)}),\ i = 1, ..., n_p,$$

where $N^{(i)} = (N_1^{(i)}, N_2^{(i)})$ and similarly for $\hat{N}^{(i)}$. This simple transformation is efficient and works well in case of moderate mesh deformations as shown in Figure 2.

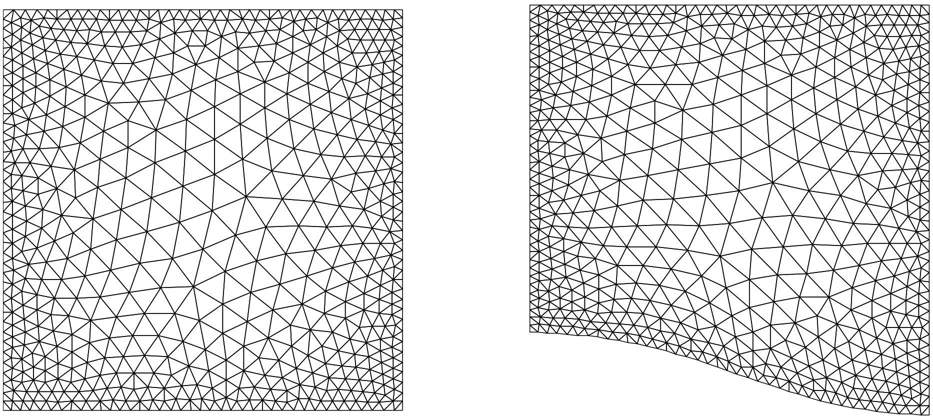

FIGURE 2. Left: reference triangulation $\widehat{\mathcal{T}_h}$. Right: Mapped triangulation $\mathcal{T}_h$.

6.2. **Nonlinear programming problem and sensitivity analysis.** After performing the finite element discretization of $(\mathcal{M}_\varepsilon^h(\alpha_m))$, the algebraic form of the state problem is given by the following system of nonlinear algebraic equations:

$$\mathbf{r}([\mathbf{u},\mathbf{p}]^{\mathrm{T}}) := \begin{bmatrix} \mathbf{A}+\mathbf{C}_\varepsilon(\mathbf{u})+\frac{1}{\varepsilon}\mathbf{G} & -\mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \tag{54}$$

where $\mathbf{u} \in \mathbb{R}^{n_u}, \mathbf{p} \in \mathbb{R}^{n_p}$ is the vector of the nodal values of the velocity $\boldsymbol{u}$ and the pressure p, respectively, $\mathbf{A} \in \mathbb{R}^{n_u \times n_u}$ is a symmetric and positive definite matrix, $\mathbf{B} \in \mathbb{R}^{n_p \times n_u}$ is the velocity-pressure coupling matrix, $\frac{1}{\varepsilon}\mathbf{G} \in \mathbb{R}^{n_c \times n_u}$ is a matrix representation of the penalized impermeability condition, and $\mathbf{C}_\varepsilon(\mathbf{u}) \in \mathbb{R}^{n_c \times n_u}$ is a matrix function representation of the smoothed slip term. Further n_p is the total number of the nodes in $\mathcal{T}_h$, n_c is the number of the nodes lying on the slip boundary $\overline{S}(\alpha_m)$, and n_u is the dimension of the solution component representing the velocity. The system (54) can be solved iteratively by a standard way by using e.g. Newton's method.

Let

$$\boldsymbol{\mathcal{U}} = \Big\{\mathbf{a}\in\mathbb{R}^{m+1} \mid \alpha_{\min}\le a_i \le \alpha_{\max},\ i=0,\dots,m;\ |a_{i+1}-a_i| \le \tfrac{C_1}{m},\ i=0,\dots,m-1, \\ |a_{i+2}-2a_{i+1}+a_i| \le \tfrac{C_2}{m^2},\ i=0,\dots,m-2\Big\},$$

where C_1, C_2 are the same as in (42), be the set of admissible discrete design variables. From the properties of the Bernstein polynomials ([5]) it easily follows that if $\mathbf{a} \in \boldsymbol{\mathcal{U}}$ then $\alpha_m \in \mathcal{U}_{ad}$, where α_m is defined by (53).

As the residual vector $\mathbf{r}$ in (54) depends also on the design variable $\mathbf{a}$, we write the algebraic state problem (54) in the form

$$\mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a})) = \mathbf{0}, \quad \mathbf{q}(\mathbf{a}) = [\mathbf{u}(\mathbf{a}),\mathbf{p}(\mathbf{a})]^{\mathrm{T}}.$$

Denote $\mathfrak{J} : \boldsymbol{\mathcal{U}} \to \mathbb{R}$, $\mathfrak{J}(\mathbf{a}) := \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a}))$, where $\mathcal{I}$ is a discretization of the cost functional I. Then the discrete optimization problem to be realized reads as follows:

$$\mathbf{a}^* \in \underset{\mathbf{a}\in\boldsymbol{\mathcal{U}}}{\operatorname{argmin}} \left\{\mathfrak{J}(\mathbf{a}) \mid \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a})) = \mathbf{0}\right\}. \tag{55}$$

In order to be able to use gradient-based nonlinear programming algorithms for solving (55) we need to evaluate the gradient of $\mathfrak{J}$ with respect to the design variable vector $\mathbf{a}$. The cost function $\mathfrak{J}$ is continuously differentiable provided that $\mathcal{I}$ is so owing to the fact that $\mathcal{T}_h$ is a smooth topologically equivalent deformation of $\widehat{\mathcal{T}}_h$ (see [12]). Then, it is well-known that the partial derivatives of $\mathfrak{J}$ with respect to the design variables are given by

$$\frac{\mathrm{d}\mathfrak{J}(\mathbf{a})}{\mathrm{d}a_i} = \frac{\partial \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_i} + \boldsymbol{\eta}^{\mathrm{T}} \left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial a_i}\right], \quad i = 0, ..., m, \tag{56}$$

where $\boldsymbol{\eta}$ is the solution to the adjoint equation

$$\left[\frac{\partial \mathbf{r}(\mathbf{a},\mathbf{q}(\mathbf{a}))}{\partial \mathbf{q}}\right]^{\mathrm{T}} \boldsymbol{\eta} = \nabla_{\mathbf{q}} \mathcal{I}(\mathbf{a},\mathbf{q}(\mathbf{a})). \tag{57}$$

The partial derivatives in (56), (57) can be computed by hand or using automatic differentiation of computer programs. For details we refer to [10] and [12].

Remark 1. The evaluation of the Jacobian matrix on the left hand side of (57) requires in fact C^2 continuity of φ_ε and this is not the case when φ_ε is defined by (45). To get such smoothness, the piecewise quadratic approximation of the absolute value function in $(45)_2$ has to be replaced by a piecewise quartic approximation resulting in

$$\varphi_\varepsilon(q) = \begin{cases} \varphi(q) & \text{if } |q| \geq \varepsilon, \\ \sigma_0 \left[-\frac{1}{8\varepsilon^3}|q|^4 + \frac{3}{4\varepsilon}|q|^2 + \frac{3}{8}\varepsilon\right] + \frac{1}{2}\sigma_1 |q|^2 & \text{if } |q| < \varepsilon. \end{cases}$$

The functional j_ε defined using this approximation clearly satisfies all the assumptions of Lemma 5.2.

7. **Numerical examples.** In this section we present numerical results of three model examples in which for the sake of simplicity of computations we use the bilinear form a^Ω defined by the full velocity gradients, i.e. $a^\Omega(\boldsymbol{u},\boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{0,\Omega}$. Let us mention that for this definition of a^Ω, the assumptions (10) and (19) remain valid. We consider the following cost functionals of the least squares type:

$$I_1(\Omega(\alpha), \boldsymbol{u}^\alpha, p^\alpha) = \tfrac{1}{2} \int_0^1 (u_t^\alpha \circ \alpha - u_t^0)^2 \, dx_1$$

and

$$I_2(\Omega(\alpha), \boldsymbol{u}^\alpha, p^\alpha) = \tfrac{1}{2} \int_{\Omega(\alpha)} (p^\alpha - p_0)^2 \, d\boldsymbol{x},$$

where $u_t^0 \in C([0,1]),\ p_0 \in L^2(\hat{\Omega})$ are given.

The state solver as well as the cost function evaluation were implemented using MATLAB [19]. The partial derivatives in (56), (57) of the MATLAB code were easy enough to be computed by hand. Minimization was carried out by `fmincon` with `'interior-point'` option from the MATLAB Optimization Toolbox. The parameters defining the stopping criterion were chosen as `TolX`$=10^{-4}$, `TolFun`$=10^{-4}$, `TolCon`$=10^{-5}$.

Example 1. (Tresca) Let σ_0=1, σ_1=0 in (45) and

$$\boldsymbol{f}(\boldsymbol{x}) = (10 \sin(2\pi(\tfrac{1}{2} - x_2)),\, 0).$$

Our aim is to minimize the objective functional I_1 with

$$u_t^0(x_1) = 0.036 \cdot \left[\max\{\sin(2\pi x_1 - \tfrac{\pi}{5}), 0\}\right]^2.$$

The parameters defining the set $\mathcal{U}$ are m=10, $\alpha_{\min}$=−0.05, $\alpha_{\max}$=0.25, C_1=1, and C_2=10.

We solved the shape optimization problem using a reference mesh $\widehat{\mathcal{T}}_h$ consisting of 4759 elements for three different penalty and smoothing parameters ε=10^{-3}, 10^{-4}, 10^{-5} to implement the non-penetration and slip terms. In all cases $\mathbf{a}^0$=$\mathbf{0} \in \mathbb{R}^{m+1}$ was used as the initial guess.

The zoomed optimized shapes of the slip boundaries and convergence histories of the objective function values are shown in Figure 3. Gradient based (descent) optimization methods are guaranteed to find only local minima. In this case the found local minima are close to the global ones, too. Moreover, the behaviour is stable with respect to ε. There is almost no difference between the optimized shapes for ε=10^{-4} and ε=10^{-5}.

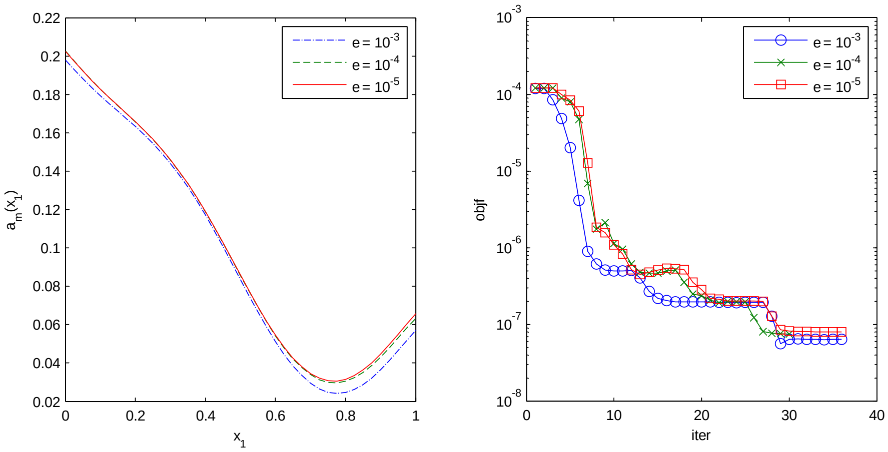

FIGURE 3. Optimized shapes (left) and convergence histories (right) for different values of the penalty/smoothing parameter ε.

The streamlines and pressure contours as well as the distributions of the tangential velocity u_t and the shear stress σ_t on $S(\alpha_{opt})$ corresponding to the state solution in the optimized domain for $\varepsilon = 10^{-5}$ are shown in Figures 4 and 5.

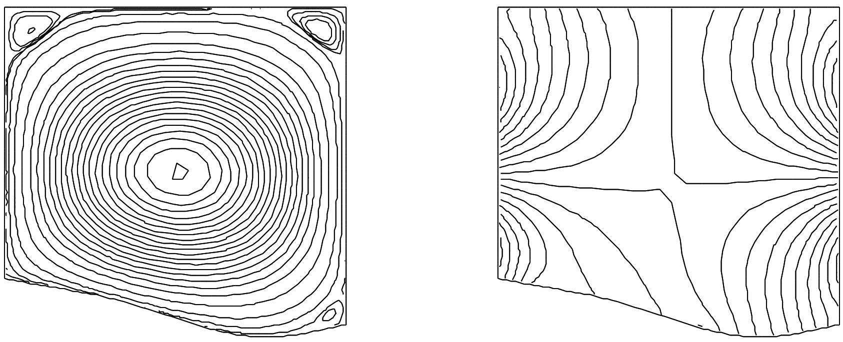

FIGURE 4. Streamlines (left) and pressure contours (right) for $\varepsilon = 10^{-5}$.

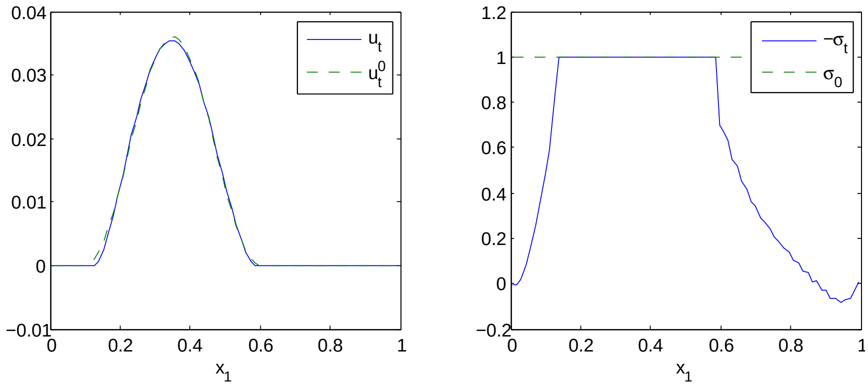

FIGURE 5. Tangential velocity and shear stress for $\varepsilon = 10^{-5}$

Example 2. (Threshold Navier) In this example we assume the threshold Navier boundary condition (44) with $\sigma_0 = 1, \sigma_1 = 10$, and $\varepsilon = 10^{-5}$. The cost functional I_1, the external force $\boldsymbol{f}$, and the reference mesh are the same as in Example 1.

The parameters defining the set $\boldsymbol{\mathcal{U}}$ are $m = 10, \alpha_{\min}{=}{-0.05}, \alpha_{\max} = 0.25, C_1 = 1$, and $C_2 = 10$. The value of the cost functional corresponding to the initial guess $\mathbf{a}^0 = \mathbf{0}$ was 1.21×10^{-4}. After 28 optimization iterations (29 function evaluations) it was reduced to 1.30×10^{-7}. The streamlines and pressure contours as well as the tangential velocity and shear stress distributions on $S(\alpha_{opt})$ in the optimized domain are shown in Figures 6 and 7.

The computed optimal shapes corresponding to this example and the previous one (for $\varepsilon = 10^{-5}$) are compared in Figure 8.

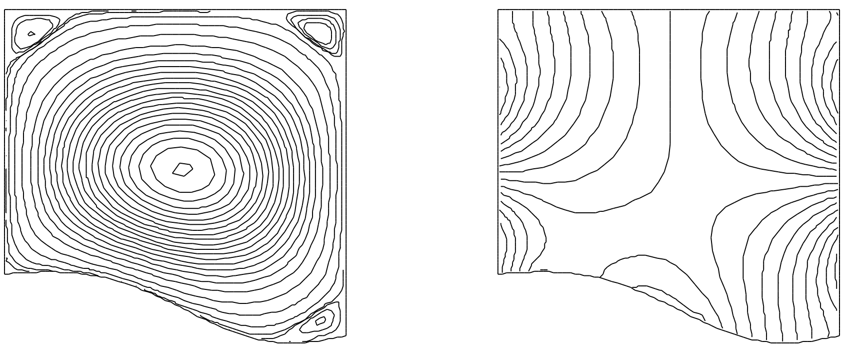

FIGURE 6. Streamlines (left) and pressure contours (right).

Example 3. In this example we wish to solve a pressure reconstruction problem, by minimizing the cost functional I_2. As the pressure is uniquely determined up to a constant, we set $p(1, 1) = 0$.

We consider the Tresca-type model with $\sigma_0{=}10$, $\sigma_1{=}0$, and $\varepsilon{=}10^{-5}$. The parameters defining $\boldsymbol{\mathcal{U}}$ are $m{=}10$, $\alpha_{\min}{=}{-0.05}$, $\alpha_{\max}{=}0.25$, $C_1{=}2$, and $C_2{=}10$. Further,

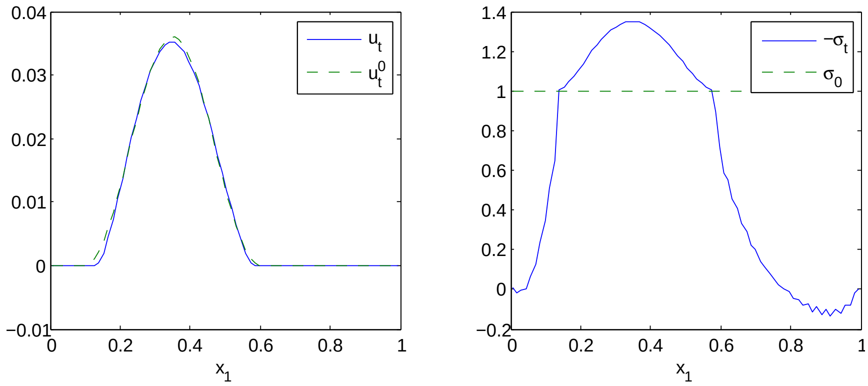

FIGURE 7. Tangential velocity and shear stress.

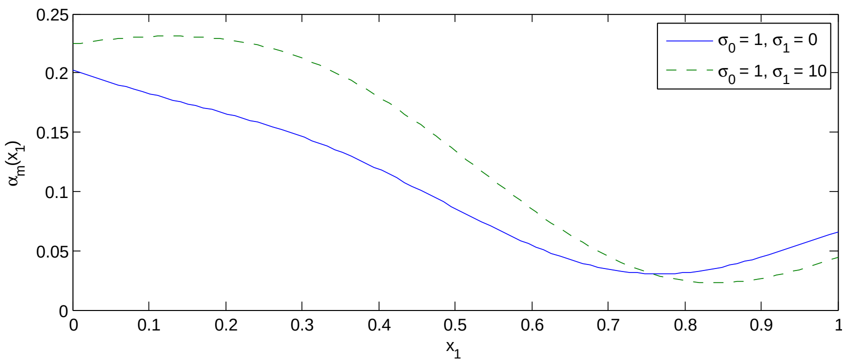

FIGURE 8. Optimized Bézier functions α_m for two different values of σ_1.

let $\boldsymbol{f} = (f_1, f_2)$, where

$$f_1(x_1, x_2) = 4\pi^2(\sin(2\pi x_1) + \sin(2\pi x_2) - 2\cos(2\pi x_1)\sin(2\pi x_2)),$$
$$f_2(x_1, x_2) = -4\pi^2(\sin(2\pi x_1) + \sin(2\pi x_2) - 2\cos(2\pi x_2)\sin(2\pi x_1)),$$
$$p_0 = 2\pi(\cos(2\pi x_2) - \cos(2\pi x_1))$$

is the external force and the target pressure.

The objective function value corresponding to the initial guess $\alpha_i = 0.1$, $i = 0, ..., 10$ was 1.26×10^0. After 31 optimization iterations (48 function evaluations) it was reduced to 1.06×10^{-2}.

The contours of the target pressure p_0 and the computed pressure in the optimized geometry are shown in Figure 9. The shear stress and tangential velocity distributions are shown in Figure 10.

8. **Conclusions.** In this paper we have considered shape optimization with the state constraint given by the Stokes system with the threshold slip boundary conditions on a part of the computational domain. In numerical realization, the part of boundary to be optimized is parametrized using a Bézier function. The problem is discretized using stable P1-bubble/P1 elements. The slip boundary condition

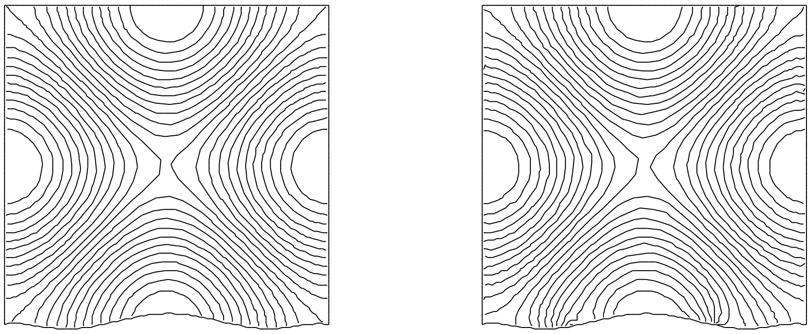

FIGURE 9. Contours of the target pressure p_0 (left) and computed pressure (right).

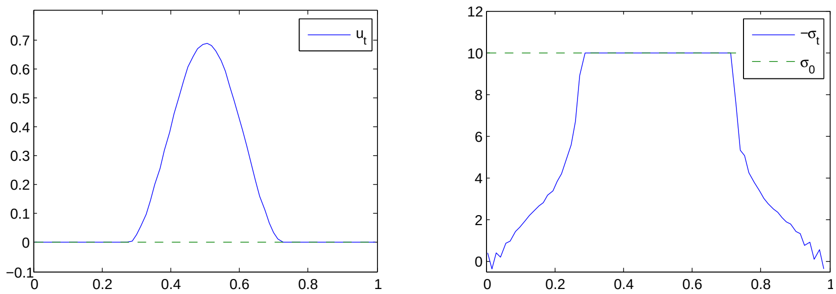

FIGURE 10. Tangential velocity and shear stress on $S(\alpha_{opt})$.

is realized approximately using a combination of the penalty method and smoothing of the nondifferentiable slip term. The numerical examples demonstrate the effectiveness of our approach.

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4.4 Sensitivity analysis for non-Newtonian fluids

Shape optimization for non-Newtonian fluids is not so often considered in theoretical and computational studies. Some works has been published on optimal control [3, 4, 42, 51], however for boundary control there is still lack of results.

In the papers [43, 44] we deal with the characterization of shape derivative for a class of models with shear-rate-dependent viscosity. We consider the minimization of the drag functional for the obstacle problem. The formal derivation of the linearized problem as well as the formula for the shape gradient of the cost function is justified by the material derivative approach. Hence it is feasible to use the obtained formulas e.g. to approximate the shape gradient in numerical simulations.

We present the reprint of the paper [44], where the results and main steps in the proofs are shown for the stationary problem. For the complete proofs and the unsteady problem we refer to [43].

Reprint

- J. Sokolowski, J. Stebel. Shape Sensitivity Analysis of Incompressible Non-Newtonian Fluids. In *System Modeling and Optimization*, pages 427–436. IFIP Advances in Information and Communication Technology, vol 391. Springer, 2013.

Shape Sensitivity Analysis of Incompressible Non-newtonian Fluids

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Abstract. We study the shape differentiability of a cost function for the steady flow of an incompressible viscous fluid of power-law type. The fluid is confined to a bounded planar domain surrounding an obstacle. For smooth perturbations of the shape of the obstacle we express the shape gradient of the cost function which can be subsequently used to improve the initial design.

Keywords: shape optimization, shape gradient, incompressible fluid, non-Newtonian fluid, Navier-Stokes equations.

1 Introduction

Shape optimization for nonlinear partial differential equations is a growing field in the contemporary optimum design of structures. In this field systems of the solid and fluid mechanics as well as e.g., the coupled models of fluid-structure interaction are included for real life problems. The main difficulty associated with the mathematical analysis of nonlinear state equations is the lack of existence of global strong solutions for mathematical models in three spatial dimensions.

In numerical methods of shape optimization the common approach is the discretization of continuous shape gradient. Therefore, the proper derivation and analysis of the regularity properties of the shape gradient is crucial for numerical solution of the shape optimization problem. The shape sensitivity analysis requires, in particular, the proof of the Lipschitz continuity of solutions the the state equations with respect to the boundary variations. This property of solutions can be obtained e.g. by analysis of the state equation transported to the fixed reference domain which is explained in the case of linear elliptic boundary value problems in monograph [11]. For the nonlinear problems the Lipschitz continuity is not obvious and it requires the additional regularity of solutions to the state equation. In addition, for the applications of levelset method of shape optimization it is required that the obtained shape gradient of the cost functional is given by a function while the general theory gives only the existence

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of a distribution. In conlusion, it seems that the shape sensitivity analysis in the case of a nonlinear state equation is the main step towards the numerical solution of the shape optimization problems.

In gas dynamics described by the compressible Navier-Stokes there is the existence of weak global solutions. However, the shape sensitivity analysis can be performed only for specific local solutions. The state of art in shape optimization for compressible Navier-Stokes equations is presented in the forthcoming monograph [8], see also [7]. For incompressible Navier-Stokes equations, the sensitivity analysis of shape functionals is performed e.g. in [2] and [6]. In this paper we are concerned with the non-Newtonian model where the stress is a (nonlinear) function of the velocity gradient. Optimal control problem for this model was studied in [9, 13]. Numerical shape optimization was done in [1], see also [3]. We present new results on the existence of the shape gradient.

We consider the steady flow of an incompressible fluid in a bounded domain $\Omega := B \setminus S$ in $\mathbb{R}^2$, where B is a container and S is an obstacle. Motion of the fluid is described by the system of equations

$$\begin{aligned} \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbb{S}(\mathbb{D}\mathbf{v}) + \nabla p + \mathbb{C}\mathbf{v} &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega, \\ \mathbf{v} &= \mathbf{g} && \text{on } \partial\Omega. \end{aligned} \tag{$P(\Omega)$}$$

Here $\mathbf{v}$, p, $\mathbb{C}$, $\mathbf{f}$ stands for the velocity, the pressure, the constant skew-symmetric Coriolis tensor and the body force, respectively. The traceless part $\mathbb{S}$ of the Cauchy stress can depend on the symmetric part $\mathbb{D}\mathbf{v}$ of the velocity gradient in the following way:

$$\mathbb{S}(\mathbb{D}\mathbf{v}) = \nu(|\mathbb{D}\mathbf{v}|^2)\mathbb{D}\mathbf{v}, \tag{1}$$

where ν, $|\mathbb{D}\mathbf{v}|^2$ is the viscosity and the shear rate, respectively. In particular, we assume that ν has a polynomial growth (see Section 2.1 below), which includes e.g. the Carreau and the power-law model.

In the model the term of Coriolis type is present. This term appears e.g. when the change of variables is performed in order to take into account the flight scenario of the obstacle in the fluid.

The aim of this paper is to investigate differentiability of a shape functional depending on the solution to $(P(\Omega))$ with respect to the variations of the shape of the obstacle. We consider a model problem with the drag functional

$$J(\Omega) := \int_{\partial S} (\mathbb{S}(\mathbb{D}\mathbf{v}) - p\mathbb{I})\mathbf{n}\cdot\mathbf{d}, \tag{2}$$

with a given constant unit vector $\mathbf{d}$. Instead of J one could take other type of functional, since our method does not rely on its specific form.

Our main interest is the rigorous analysis of the shape differentiability for $(P(\Omega))$ and (2). We follow the general framework developed in [11] using the speed method and the notion of the material derivative. Let us point out that due to (1) the state problem is nonlinear in its nature. We refer the reader to [12] for an introduction to optimization problems for nonlinear partial differential equations.

1.1 Shape Derivatives

We start by the description of the framework for the shape sensitivity analysis. For this reason, we introduce a vector field $\mathbf{T} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ vanishing in the vicinity of ∂B and define the mapping

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}).$$

For small $\varepsilon > 0$ the mapping $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ takes diffeomorphically the region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$ where $S_\varepsilon = \mathbf{y}(S)$. We consider the counterpart of problem $(P(\Omega))$ in Ω_ε, with the data $\mathbf{f}_{|\Omega_\varepsilon}$ and $\mathbf{g}_{|\Omega_\varepsilon}$. The new problem will be denoted by $(P(\Omega_\varepsilon))$ and its solution by $(\bar{\mathbf{v}}_\varepsilon, \bar{p}_\varepsilon)$.

For the nonlinear system $(P(\Omega))$ we introduce the shape derivatives of solutions. To this end we need the linearized system of the form:

Find the couple $(\mathbf{u}, \pi)$ *such that*

$$\begin{aligned} \operatorname{div}\,[\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} - \mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\mathbf{u}] + \nabla \pi + \mathbb{C}\mathbf{u} = \mathbf{F} \qquad & \textit{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \qquad & \textit{in } \Omega, \\ \mathbf{u} = \mathbf{h} \qquad & \textit{on } \partial\Omega, \end{aligned} \qquad (P_{\text{lin}}(\Omega))$$

where $\mathbf{F}$ *and* $\mathbf{h}$ *are given elements.*

The shape derivative $\mathbf{v}'$ and the material derivative $\dot{\mathbf{v}}$ of solutions are formally introduced by

$$\mathbf{v}' := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon - \mathbf{v}}{\varepsilon}, \quad \dot{\mathbf{v}} := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y} - \mathbf{v}}{\varepsilon},$$

where $\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y}(\mathbf{x}) := \bar{\mathbf{v}}_\varepsilon(\mathbf{y}(\mathbf{x}))$, and are related to each other as follows:

$$\dot{\mathbf{v}} = \mathbf{v}' + (\nabla \mathbf{v})\mathbf{T}.$$

The standard calculus for differentiating with respect to shape yields that $\mathbf{v}'$ is the solution of $(P_{\text{lin}}(\Omega))$ with the data $\mathbf{F} = \mathbf{0}$ and $\mathbf{h} = -\partial \mathbf{v}/\partial \mathbf{n}(\mathbf{T} \cdot \mathbf{n})$. Using (7) as the definition of J we obtain the expression for the shape gradient:

$$\begin{aligned} dJ(\Omega; \mathbf{T}) &:= \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} \\ &= \int_\Omega [(\mathbb{C}\mathbf{v}') \cdot \boldsymbol{\xi} + (\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\mathbf{v}' - \mathbf{v}' \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{v}') : \nabla \boldsymbol{\xi}] - \int_{\partial S} (\mathbf{f} \cdot \mathbf{d})\mathbf{T} \cdot \mathbf{n}. \end{aligned} \qquad (3)$$

In the above formula, the part containing $\mathbf{v}'$ depends implicitly on the direction $\mathbf{T}$. This is not convenient for practical use, hence we introduce the adjoint problem for further simplification of (3):

Find the couple $(\mathbf{w}, s)$ *such that*

$$\begin{aligned} -2(\mathbb{D}\mathbf{w})\mathbf{v} - \operatorname{div}\,\left[\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w}\right] + \nabla s - \mathbb{C}\mathbf{w} = \mathbf{0} \qquad & \textit{in } \Omega, \\ \operatorname{div} \mathbf{w} = 0 \qquad & \textit{in } \Omega, \\ \mathbf{w} = \mathbf{d} \qquad & \textit{on } \partial\Omega. \end{aligned} \qquad (P_{\text{adj}}(\Omega))$$

Consequently, the expression for dJ reduces to

$$dJ(\Omega;\mathbf{T}) = -\int_{\partial S}\left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top\mathbb{D}\mathbf{w} - s\mathbb{I}\right) : \frac{\partial\mathbf{v}}{\partial\mathbf{n}}\otimes\mathbf{n} + \mathbf{f}\cdot\mathbf{d}\right]\mathbf{T}\cdot\mathbf{n}. \tag{4}$$

In order to prove the result given by (3) and (4) we need the material derivatives. In particular, it is sufficient to show that the linear mapping

$$\mathbf{T} \mapsto dJ(\Omega;\mathbf{T})$$

is continuous in an appropriate topology, see the structure Theorem in the book [11] for details.

2 Preliminaries

We impose the structural assumptions on the data, state the known results on well-posedness of $(P(\Omega))$ and introduce the elementary notation for shape sensitivity analysis.

2.1 Structural Assumptions

We require that $\mathbb{S}$ has a potential $\Phi : [0,\infty) \to [0,\infty)$, i.e. $\mathbb{S}_{ij}(\mathbb{D}) = \partial\Phi(|\mathbb{D}|^2)/\partial\mathbb{D}_{ij}$. Further we assume that Φ is a $\mathcal{C}^3$ function with $\Phi(0) = 0$ and that there exist constants $C_1, C_2, C_3 > 0$ and $r \geq 2$ such that

$$C_1(1+|\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \mathbb{S}'(\mathbb{A}) :: (\mathbb{B}\otimes\mathbb{B}) \leq C_2(1+|\mathbb{A}|^{r-2})|\mathbb{B}|^2, \tag{5a}$$

$$|\mathbb{S}''(\mathbb{A})| \leq C_3(1+|\mathbb{A}|^{r-3}) \tag{5b}$$

for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{2\times 2}_{sym}$. Here the symbol :: stands for the usual scalar product in $\mathbb{R}^{2^4}$. The above inequalities imply the monotone structure of $\mathbb{S}$, see e.g. [5].

2.2 Weak Formulation

For the definition of the weak solution we will use the space

$$\mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega) := \{\boldsymbol{\phi} \in \mathbf{W}^{1,r}_0(\Omega);\ \operatorname{div}\boldsymbol{\phi} = 0\}.$$

Let $\mathbf{f} \in (\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega))^*$ and $\mathbf{g} \in \mathbf{W}^{1,r}(\Omega)$ with $\operatorname{div}\mathbf{g} = 0$. Then a function $\mathbf{v} \in \mathbf{g} + \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$ is said to be a weak solution to the problem $(P(\Omega))$ if

$$\int_\Omega \left[\mathbb{S}(\mathbb{D}\mathbf{v}) : \mathbb{D}\boldsymbol{\phi} - \mathbf{v}\otimes\mathbf{v} : \nabla\boldsymbol{\phi} + \mathbb{C}\mathbf{v}\cdot\boldsymbol{\phi}\right] = \int_\Omega \mathbf{f}\cdot\boldsymbol{\phi} \tag{6}$$

for every $\boldsymbol{\phi} \in \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$. Note that the pressure is eliminated since test functions are divergence free.

The following result was shown in [4].

Theorem 1 (Kaplický et al. [4]). *Let $\Omega \in \mathcal{C}^2$, $\mathbf{f} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$, $\epsilon_0 > 0$ and* (5a)–(5b) *hold with $r > \frac{3}{2}$. Then there exists a constant $\delta > 0$ such that for every $\mathbf{g}$ satisfying*

$$\|\mathbf{g}\|_{3,q} \leq \delta, \ (q > 2),$$

problem $(P(\Omega))$ has a weak solution satisfying $\mathbf{v} \in \mathbf{W}^{2,2+\epsilon}(\Omega)$, $p \in W^{1,2+\epsilon}(\Omega)$, $\epsilon > 0$.

Note that the above result applies only to the unperturbed domain, i.e. $\varepsilon = 0$. Assuming smallness of $\|\mathbf{f}\|_{2,B}$ and $\|\mathbf{g}\|_{3,q,B}$, one can prove that $(P(\Omega))$, $(P(\Omega_\varepsilon)$ has a unique weak solution satisfying

$$\|\mathbf{v}\| \leq C_E(\|\mathbf{f}\|_{2,B}, \|\mathbf{g}\|_{3,q,B}) \quad \text{and} \quad \|\bar{\mathbf{v}}_\varepsilon\| \leq C_E(\|\mathbf{f}\|_{2,B}, \|\mathbf{g}\|_{3,q,B}),$$

respectively, where C_E is independent of ε. At this point we summarize the main hypotheses.

Assumption 1. *In what follows, $\Omega \in \mathcal{C}^2$ is a bounded planar domain of the form $\Omega = B \setminus S$, $\mathbf{f} \in \mathbf{L}^{2+\epsilon_0}(B)$, $\epsilon_0 > 0$, $\mathbf{g} \in \mathbf{W}^{3,q}(B)$ $(q > 2)$ is supported in the vicinity of ∂B,* (5a)–(5b) *hold with $r \in [2, 4)$ and $\|\mathbf{f}\|_{2,B}$, $\|\mathbf{g}\|_{3,q,B}$ are small enough.*

Let us point out that equation (2) which defines J is not suitable for weak solutions in general, since the energy inequality does not provide enough information about the trace of p and $\mathbb{D}\mathbf{v}$. We therefore introduce an alternative definition that requires less regularity. Let us fix an arbitrary divergence free function $\boldsymbol{\xi} \in \mathcal{C}_c^\infty(B, \mathbb{R}^2)$ such that $\boldsymbol{\xi} = \mathbf{d}$ in a vicinity of S. Then, integrating (2) by parts and using $(P(\Omega))$ yields:

$$J(\Omega) = \int_\Omega [(\mathbb{C}\mathbf{v} - \mathbf{f}) \cdot \boldsymbol{\xi} + (\mathbb{S}(\mathbb{D}\mathbf{v}) - \mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\xi}] \,. \tag{7}$$

Note that this identity is finite for any $\mathbf{v} \in \mathbf{W}^{1,2}(\Omega)$.

2.3 Deformation of the Shape

Let us introduce the following notation: We will denote by $\mathrm{D}\,\mathbf{T}$ the Jacobian matrix whose components are $(\mathrm{D}\,\mathbf{T})_{ij} = (\nabla \mathbf{T})_{ji} = \partial_i T_j$. Further,

$$\mathbb{N}(\mathbf{x}) := \mathfrak{g}(\mathbf{x})\mathbb{M}^{-1}(\mathbf{x}), \quad \mathbb{M}(\mathbf{x}) := \mathbb{I} + \varepsilon\,\mathrm{D}\,\mathbf{T}(\mathbf{x}), \quad \mathfrak{g}(\mathbf{x}) := \det \mathbb{M}(\mathbf{x}).$$

One can easily check that the matrix $\mathbb{N}$ and the determinant $\mathfrak{g}$ admit the expansions:

$$\mathfrak{g} = 1 + \varepsilon \mathrm{div}\,\mathbf{T} + O(\varepsilon^2), \quad \mathbb{N} = \mathbb{I} + \varepsilon\mathbb{N}' + O(\varepsilon^2), \quad \mathbb{N}' = (\mathrm{div}\,\mathbf{T})\mathbb{I} - \mathrm{D}\,\mathbf{T}, \tag{8}$$

where the symbol $O(\varepsilon^2)$ denotes a function whose norm in $\mathcal{C}^1(\overline{\Omega})$ is bounded by $C\varepsilon^2$, see [11].

The value of the shape functional for Ω_ε is given by

$$J(\Omega_\varepsilon) := \int_{\Omega_\varepsilon} \left[(\mathbb{C}\bar{\mathbf{v}}_\varepsilon - \mathbf{f}) \cdot \boldsymbol{\xi}_\varepsilon + (\mathbb{S}(\mathbb{D}\bar{\mathbf{v}}_\varepsilon) - \bar{\mathbf{v}}_\varepsilon \otimes \bar{\mathbf{v}}_\varepsilon) : \nabla \boldsymbol{\xi}_\varepsilon \right],$$

where $\boldsymbol{\xi}_\varepsilon := (\mathbb{N}^{-\top}\boldsymbol{\xi}) \circ \mathbf{y}^{-1}$. Using the properties of the Piola transform one can check that $\operatorname{div} \boldsymbol{\xi}_\varepsilon = 0$. If $\bar{\mathbf{v}}_\varepsilon$ and $\bar{p}_\varepsilon$ were sufficiently smooth, it would hold that

$$J(\Omega_\varepsilon) = \int_{\partial S_\varepsilon} (\mathbb{S}(\mathbb{D}\bar{\mathbf{v}}_\varepsilon) - \bar{p}_\varepsilon \mathbb{I})\mathbf{n}_\varepsilon \cdot \mathbf{d}. \tag{9}$$

Nevertheless, as opposed to $(P(\Omega))$, we do not require any additional regularity of the solution to the perturbed problem $(P(\Omega_\varepsilon))$ and hence the expression in (9) need not be well defined.

We introduce the auxiliary function $\tilde{\mathbf{v}}$:

$$\tilde{\mathbf{v}} := \lim_{\varepsilon \to 0} \frac{\mathbb{N}^\top \bar{\mathbf{v}}_\varepsilon \circ \mathbf{y} - \mathbf{v}}{\varepsilon},$$

which is related to the material derivative $\dot{\mathbf{v}}$ by the identity

$$\tilde{\mathbf{v}} = \mathbb{N}'^\top \mathbf{v} + \dot{\mathbf{v}}.$$

For the justification of the results of the paper we will use $\tilde{\mathbf{v}}$ since, unlike the material derivative, it preserves the divergence free condition.

3 Main Results

The first result is the existence of $\tilde{\mathbf{v}}$ and hence also of the material derivative.

Theorem 2. *Let Assumption 1 be satisfied. Then the function $\tilde{\mathbf{v}}$ exists and is the unique weak solution of $(P_{\mathrm{lin}}(\Omega))$ with the data*

$$\begin{aligned} \mathbf{F} = \mathbf{A}_0' &:= \operatorname{div}(\mathbf{v} \otimes \mathbb{N}'^\top \mathbf{v}) + \mathbb{N}' \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) \\ &\quad + \operatorname{div} \left[\mathbb{S}'(\mathbb{D}\mathbf{v}) \left(((\mathbb{N}' - \mathbb{I} \operatorname{tr} \mathbb{N}')\nabla \mathbf{v})_{sym} - \mathbb{D}(\mathbb{N}'^\top \mathbf{v}) \right) + \mathbb{N}'^\top \mathbb{S}(\mathbb{D}\mathbf{v}) \right] \\ &- \mathbb{N}' \operatorname{div} \mathbb{S}(\mathbb{D}\mathbf{v}) + \left((\mathbb{N}' - \mathbb{I} \operatorname{tr} \mathbb{N}')\mathbb{C} + \mathbb{C}\mathbb{N}'^\top \right) \mathbf{v} + (\mathbb{I} \operatorname{tr} \mathbb{N}' - \mathbb{N}')\mathbf{f} + (\nabla \mathbf{f})\mathbf{T}, \end{aligned} \tag{10a}$$

$$\mathbf{h} = \mathbf{0}. \tag{10b}$$

The following estimate holds:

$$\|\tilde{\mathbf{v}}\|_{1,2,\Omega} \le C \|\mathbf{A}_0'\|_{\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)^*} \le C \|\mathbf{T}\|_{\mathcal{C}^2(\overline{\Omega})}. \tag{11}$$

Next we establish the existence of the shape gradient of J.

Theorem 3. *Let Assumption 1 be satisfied and* $\mathbf{f} \in \mathbf{W}^{1,2}(\Omega)$. *Then the shape gradient of* J *reads*

$$dJ(\Omega, \mathbf{T}) = J_{\mathbf{v}}(\tilde{\mathbf{v}}) + J_e(\mathbf{T}),$$

where the dynamical part $J_{\mathbf{v}}$ *and the geometrical part* J_e *is given by*

$$J_{\mathbf{v}}(\tilde{\mathbf{v}}) = \int_{\Omega} \left[(\mathbb{C}\tilde{\mathbf{v}}) \cdot \boldsymbol{\xi} + (\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\tilde{\mathbf{v}} - \tilde{\mathbf{v}} \otimes \mathbf{v} - \mathbf{v} \otimes \tilde{\mathbf{v}}) : \nabla\boldsymbol{\xi} \right],$$

$$\begin{aligned} J_e(\mathbf{T}) = \int_{\Omega} \Big\{ & \big[(\mathbb{I} \operatorname{tr} \mathbb{N}' - \mathbb{N}') \, \mathbb{C}\mathbf{v} - \mathbb{C}\mathbb{N}'^{\top}\mathbf{v} - (\mathbb{I} \operatorname{tr} \mathbb{N}' - \mathbb{N}') \, \mathbf{f} - (\nabla\mathbf{f})\mathbf{T} \big] \cdot \boldsymbol{\xi} \\ &+ \big[\mathbf{v} \otimes \mathbb{N}'^{\top}\mathbf{v} + \mathbb{S}'(\mathbb{D}\mathbf{v}) \big((\mathbb{N}'\nabla\mathbf{v} - \nabla(\mathbb{N}'^{\top}\mathbf{v}))_{sym} - (\operatorname{tr}\mathbb{N}')\mathbb{D}\mathbf{v} \big) + \mathbb{N}'^{\top}\mathbb{S}(\mathbb{D}\mathbf{v}) \big] : \nabla\boldsymbol{\xi} \\ &+ \big[\mathbf{v} \otimes \mathbf{v} - \mathbb{S}(\mathbb{D}\mathbf{v}) \big] : \nabla(\mathbb{N}'^{\top}\boldsymbol{\xi}) \Big\}, \end{aligned}$$

respectively. In particular, as $\tilde{\mathbf{v}}$ *depends continuously on* $\mathbf{T}$, *the mapping*

$$\mathbf{T} \mapsto dJ(\Omega, \mathbf{T})$$

is a bounded linear functional on $\mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$.

Based on the previous result we can deduce that the shape gradient has the form of a distribution supported on the boundary of the obstacle. Since this representation is unique, the formal results derived in Section 1.1 are justified provided that the shape derivatives and adjoints exist and are sufficiently regular.

Corollary 1. *Let Assumption 1 be satisfied. Then*

(i) the shape derivative $\mathbf{v}'$ *exists and is the unique weak solution to* $(P_{\text{lin}}(\Omega))$ *with* $\mathbf{F} = \mathbf{0}$, $\mathbf{h} = -\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}(\mathbf{T} \cdot \mathbf{n})$;

(ii) the adjoint problem $(P_{\text{adj}}(\Omega))$ *has a unique weak solution that satisfies:* $\mathbf{w} \in \mathbf{W}^{2,2}(\Omega)$ *and* $s \in W^{1,2}(\Omega)$.

If in addition $\mathbf{f} \in \mathbf{W}^{1,2}(\Omega)$, *then*

(iii) the shape gradient of J *satisfies* (3);

(iv) the representation (4) *is satisfied in the following sense:*

$$dJ(\Omega; \mathbf{T}) = -\int_{\partial S} \left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^{\top}\mathbb{D}\mathbf{w} - s\mathbb{I} \right) : \frac{\partial\mathbf{v}}{\partial\mathbf{n}} \otimes \mathbf{n} + \mathbf{f} \cdot \mathbf{d} \right] \mathbf{T} \cdot \mathbf{n}. \qquad (12)$$

In the remaining part we show the main steps of the proof of Theorem 3. Details can be found in [10], where the time-dependent problem is treated.

4 Formulation in the Fixed Domain

In this section we transform the problem $(P(\Omega_\varepsilon))$ to the fixed domain Ω. Let us introduce the following notation:

$$\mathbf{v}_\varepsilon(\mathbf{x}) := \mathbb{N}^\top(\mathbf{x})\bar{\mathbf{v}}_\varepsilon(\mathbf{y}(\mathbf{x})),\ \mathbf{x} \in \Omega.$$

Note that the definition of $\mathbf{v}_\varepsilon$ implies that $\operatorname{div}\mathbf{v}_\varepsilon = 0$. The new function $\mathbf{v}_\varepsilon \in \mathbf{g} + \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$ satisfies the equality

$$\int_\Omega \Big[\mathfrak{g}\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon) : \mathbb{D}_\varepsilon\boldsymbol{\phi} - \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon : \nabla\boldsymbol{\phi} + \mathbb{C}\mathbf{v}_\varepsilon \cdot \boldsymbol{\phi}\Big] = \int_\Omega \mathbf{f}\cdot\boldsymbol{\phi} + \langle \mathbf{A}^1_\varepsilon, \boldsymbol{\phi}\rangle_{\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)} \quad \text{for all } \boldsymbol{\phi} \in \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega), \tag{13}$$

where the term $\mathbf{A}^1_\varepsilon$ on the right hand side is defined for $\boldsymbol{\phi} \in \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)$ by

$$\langle \mathbf{A}^1_\varepsilon, \boldsymbol{\phi}\rangle_{\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)} = \int_\Omega \Big[\mathbf{v}_\varepsilon \otimes \mathbb{N}^{-\top}\mathbf{v}_\varepsilon : \nabla(\mathbb{N}^{-\top}\boldsymbol{\phi}) - \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon : \nabla\boldsymbol{\phi} + (\mathbb{C} - \mathfrak{g}\mathbb{N}^{-1}\mathbb{C}\mathbb{N}^{-\top})\mathbf{v}_\varepsilon \cdot \boldsymbol{\phi} + (\mathfrak{g}\mathbb{N}^{-1}\mathbf{f}\circ\mathbf{y} - \mathbf{f})\cdot\boldsymbol{\phi}\Big]. \tag{14}$$

Here $\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon := \mathfrak{g}^{-1}(\mathbb{N}\nabla(\mathbb{N}^{-\top}\mathbf{v}_\varepsilon))_{sym}$.

Applying change of coordinates we further get:

$$J(\Omega_\varepsilon) = \int_\Omega \Big[\mathfrak{g}\left(\mathbb{N}^{-1}\mathbb{C}\mathbb{N}^{-\top}\mathbf{v}_\varepsilon - \mathbb{N}^{-1}\mathbf{f}\circ\mathbf{y}\right)\cdot\boldsymbol{\xi} + \left(\mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon) - \mathbf{v}_\varepsilon \otimes (\mathbb{N}^{-\top}\mathbf{v}_\varepsilon)\right) : \nabla(\mathbb{N}^{-\top}\boldsymbol{\xi})\Big]. \tag{15}$$

Now after all quantities and equations have been transformed to the fixed domain Ω, we can analyze the limit $\varepsilon \to 0$.

Lemma 1. *The sequence $\{\mathbf{v}_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$ and satisfies:*

$$\begin{aligned} \mathbf{v}_\varepsilon &\rightharpoonup \mathbf{v} && \text{weakly in } \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega), \\ \mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon) &\rightharpoonup \mathbb{S}(\mathbb{D}\mathbf{v}) && \text{weakly in } L^{r'}(\Omega,\mathbb{R}^{2\times2}), \\ \mathbf{A}^1_\varepsilon &\rightharpoonup \mathbf{0} && \text{weakly in } \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)^*. \end{aligned}$$

In particular, $\mathbf{v}$ is the unique weak solution to $(P(\Omega))$.

5 Existence of Material Derivative

Our next task is to identify $\tilde{\mathbf{v}}$ as the limit of the sequence $\{\mathbf{u}_\varepsilon\}$, where

$$\mathbf{u}_\varepsilon := \frac{\mathbf{v}_\varepsilon - \mathbf{v}}{\varepsilon}.$$

First we write down the system for the differences $\mathbf{u}_\varepsilon$. Subtracting (13) and (6) we find that $\mathbf{u}_\varepsilon \in \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$ satisfies the equality

$$\int_\Omega \Big[\frac{1}{\varepsilon}\mathfrak{g}(\mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v}_\varepsilon) - \mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v})) : \mathbb{D}_\varepsilon \boldsymbol{\phi} + \mathbb{C}\mathbf{u}_\varepsilon \cdot \boldsymbol{\phi} - (\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \otimes \mathbf{v}) : \nabla \boldsymbol{\phi}\Big] = \frac{1}{\varepsilon}\langle \mathbf{A}_\varepsilon, \boldsymbol{\phi}\rangle_{\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)} \qquad (16)$$

for all $\boldsymbol{\phi} \in \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)$. The term $\mathbf{A}_\varepsilon \in \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)^*$ on the right hand side is defined as follows:

$$\mathbf{A}_\varepsilon := \mathbf{A}^1_\varepsilon + \mathbf{A}^2_\varepsilon,$$

$$\mathbf{A}^1_\varepsilon \text{ is given by (14)},$$

$$\langle \mathbf{A}^2_\varepsilon, \boldsymbol{\phi}\rangle_{\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)} := \int_\Omega \Big[\mathbb{N}^\top \mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v}) : \nabla(\mathbb{N}^{-\top}\boldsymbol{\phi}) - \mathbb{S}(\mathbb{D}\mathbf{v}) : \mathbb{D}\boldsymbol{\phi}\Big].$$

Next we state the properties of the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$.

Lemma 2. *The sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)$. Further it holds:*

$$\begin{aligned} \frac{\mathbf{A}_\varepsilon}{\varepsilon} &\rightharpoonup \mathbf{A}'_0 && \textit{weakly in } \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega)^*,\\ \mathbf{u}_\varepsilon &\rightharpoonup \tilde{\mathbf{v}} && \textit{weakly in } \mathbf{W}^{1,2}_{0,\mathrm{div}}(\Omega),\\ \frac{1}{\varepsilon}(\mathfrak{g}(\mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v}_\varepsilon) - \mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v})), \mathbb{D}_\varepsilon\boldsymbol{\phi}) &\to (\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\tilde{\mathbf{v}}, \mathbb{D}\boldsymbol{\phi}) && \textit{for all } \boldsymbol{\phi} \in \mathbf{W}^{1,\frac{2r}{4-r}}(\Omega), \end{aligned}$$

where $\mathbf{A}'_0$ is defined in (10a) *and $\tilde{\mathbf{v}}$ is the solution of* $(P_{\mathrm{lin}}(\Omega))$ *with $\mathbf{F} := \mathbf{A}'_0$ and $\mathbf{h} = \mathbf{0}$.*

This completes the proof of Theorem 2.

6 Shape Gradient of J

To prove Theorem 3, we decompose the fraction

$$\frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon} = J^\varepsilon_1 + J^\varepsilon_2$$

in a suitable way. Using Lemma 1 and Lemma 2 and the properties of $\mathfrak{g}$ and $\mathbb{N}'$, it is then possible to show that

$$J^\varepsilon_1 \to J_{\mathbf{v}}(\tilde{\mathbf{v}}) \quad \text{and} \quad J^\varepsilon_2 \to J_e(\mathbf{T}).$$

The continuity of the map $\mathbf{T} \mapsto dJ(\Omega; \mathbf{T})$ follows from the estimate (11).

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Chapter 5

Conclusion

The field of non-Newtonian fluids is very broad and far from being completely understood both from theoretical and practical point of view. In this thesis we tried to demonstrate some achievements in the mathematical and numerical analysis of non-Newtonian fluids, in particular the existence, uniqueness, convergence and error estimates for the solutions of piezoviscous fluid models under a physically motivated boundary conditions. Further developments in this field are required in order to justify or improve existing numerical approaches.

Shape optimization in fluid mechanics is also an important discipline with practical impact in engineering. The complex structure of these problems is possibly the reason for lack of stronger general results such as uniqueness or convergence of approximate optimal solutions. The specific properties of the geometric description and specific features of the state problem often mean that each optimization problem has to be studied on its own, requiring to adopt appropriate knowledge and tools. The works on shape optimization presented in this thesis can give guidelines to studies of related problems. We believe that they can contribute to a wider spread of shape optimization to more particular applications.

Ivo Babuška, a worldwide-known mathematician recognized for his work in numerical mathematics, is known for his famous question *"Will you sign the blueprint?"* [41] It is directed at the issue of robustness and reliability of numerical simulations, which is extremely important in fluid mechanics and shape optimization since the simulation results are usually extremely sensitive to input parameters as well as various sources of error. We hope that the need for reliable computational results will also lead to a growing interest in rigorous theoretical studies in this field.

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