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WEAK REGULARIZABILITY AND POLE ASSIGNMENT FOR NON-SQUARE LINEAR SYSTEMS

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The problem of pole assignment by state feedback in the class of non-square linear systems is considered in the paper. It is shown that the problem is solvable under the assumption of weak regularizability, a newly introduced concept that can be viewed as a generalization of the regularizability of square systems. Necessary conditions of solvability for the problem of pole assignment are established. It is also shown that sufficient conditions can be derived in some special cases. Some conclusions and prospects for further studies are drawn in the last section.

Keywords: linear systems, linear state feedback, pole assignment

Classification: 93C05, 93B52, 93B55

1. INTRODUCTION

We consider a linear, time-invariant, continuous system of the form

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

where $E, A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$ with $\mathbb{R}, x(t)$, and $u(t)$ denoting the field of real numbers, state, and input of the system, respectively. It is called *non-square* since q, in general, does not equal n. The system (1) will frequently be referred to as the triple (E, A, B).

Applying the (linear and proportional) state feedback

$$u(t) \;=\; Fx(t) + v(t), \tag{2}$$

where $F \in \mathbb{R}^{m\times n}$ and v is a new control input, to the system (1) gives the closed-loop system

$$E\dot{x}(t) \;=\; [A + BF]x(t) + Bv(t). \tag{3}$$

The system (3) differs from (1) just by the term BF added to A, which means that the differences between these two systems are mainly given by the changes in the (finite and infinite) zero structure of $sE - A - BF$ when varying F. Finite zeros of $sE - A - BF$ correspond to exponential free-response modes of the system, while zeros at infinity represent impulsive modes. By choosing different state feedback gains we alter the zero structure of $sE - A - BF$, and consequently the response of the closed-loop system. Such a problem is called the pole structure assignment, here by state feedback (2).

It constitutes one of the fundamental problems of control as it aims at shaping the desired system response by assigning a closed-loop pole structure. Thus, the pole structure assignment techniques belong to the basic tools for the controller design. One can meet, see [4, 10, 12] and the references therein, many modifications of this problem. Among them, in the case of square systems, the so-called eigenstructure assignment comprises, in addition to the eigenvalues, an deliberate assignment of eigenvectors [8], too. The case when just a finite pole structure is assigned to the system (1), which results in the elimination of the system impulsive behavior, is likely of main practical interest [6]. But the assignment of the infinite eigenvalue to the closed-loop system is important, too. Consider for example the design of perfect observers [3]. A special case of the pole structure assignment problem is known as the pole assignment where just a modification of the (finite and infinite) eigenvalues of $sE - A - BF$ is of concern.

The pole and pole structure assignment problems have been widely studied by many authors in the case of square systems. The seminal work in this direction belongs to Rosenbrock. In [12], he gives necessary and sufficient conditions of solvability for the case of the explicit (E is invertible) and controllable systems. This result has then been generalized to the explicit and uncontrollable systems, see [14]. In the case of implicit and square systems the problem is solved in [6] for controllable systems, while in [10] the problem is considered in the case of uncontrollable systems where necessary and sufficient conditions of solvability to the pole assignment and necessary conditions to the pole structure assignment can be found.

The situation is different as far as the non-square systems are concerned. The literature on this topic is not very extensive [2, 5], especially that devoted to the pole assignment. This is of course caused by the fact that the field of possible applications is not very large yet. Nevertheless, some applications exist, see [1] and the references therein. For example, the non-square difference systems arise quite naturally in the graph theory (Petri nets) when writing down balance equations for each node and where (oriented) edges are characterized by delays [1, 13]. Such systems can be applied to some problems of supervisory control.

The paper is an attempt to solve the pole assignment problem when the system under consideration is implicit and non-square and is organized as follows. In the section 2, the basic concepts and definitions are given. In particular, the feedback canonical form and normal external description of the system (1) are introduced. Some mathematical tricks used for solving the problem, such as the extension of (E, A, B) and the conformal mapping, are considered therein, too. A solution to the problem of pole assignment to square systems is also presented here. In the section 3, conditions of solvability to the full rank assignment in non-square systems, see Theorem 3.1, are presented and a concept of weak regularizability is defined. This concept can be viewed as a generalization of the regularizability known in the theory of square systems. The problem of pole assignment to weakly regularizable systems is restated in terms of the greatest common divisor of all minors of the maximal order $(\min(q, n))$ in the section 4. Then, under the assumption of weak regularizability, necessary conditions of solvability are derived, see Theorems 4.2 and 4.6. Just sufficient conditions of solvability in some special cases are given by Theorems 4.4 and 4.7. Moreover, Theorem 4.8 gives a complete solution to the problem in which the maximal number of poles is assigned. An example illustrating the methods developed in the paper is provided in that section, too. The last section is devoted to concluding remarks on the achieved results.

2. BACKGROUND

As far as the notation is concerned, standard symbols, see [4] for instance, are basically used. The divisibility of the polynomials $\alpha(s), \beta(s) \in \mathbb{R}[s]$ is denoted by the symbol $\lhd$, i.e $\alpha(s) \lhd \beta(s)$ $(\beta(s) \rhd \alpha(s))$ means $\alpha(s)$ divides $\beta(s)$. The degree of a polynomial vector $x(s) \in \mathbb{R}^k[s]$, $\deg x(s)$, stands for the greatest degree of all its entries $x_i(s)$. Accordingly, the degree of the ith column of a polynomial matrix $M(s) \in \mathbb{R}^{p\times m}[s]$ is denoted by $\deg_{ci} M(s)$. Such a matrix is called *column reduced* if it can be written in the form $M(s) = M_{lc} \operatorname{diag}\left\{s^{c_i}\right\}_{i=1}^{m} + \bar{M}(s)$, where $M_{lc} \in \mathbb{R}^{p\times m}$ is of full column rank and $\bar{M}(s) \in \mathbb{R}^{p\times m}[s]$ is such that $\deg_{ci} \bar{M}(s) < c_i := \deg_{ci} M(s)$. Matrices $A(s)$ and $B(s)$ are said to be *equivalent* $(A(s) \cong B(s))$ if there exist unimodular matrices $M(s)$ and $N(s)$ over $\mathbb{R}[s]$ such that $A(s) = M(s)B(s)N(s)$. A polynomial matrix of degree 1 is called a *matrix pencil.* Pencils $A(s)$ and $B(s)$ are said to be *strictly (pencil) equivalent* $(A(s) \sim B(s))$ if there exist invertible matrices M and N over $\mathbb{R}$ such that $A(s) = MB(s)N$. Let $\mathbb{S}_t^k$ denote the set of all k-tuples $\{j_1, j_2, \ldots, j_k\}$, $j_1 < j_2 < \cdots < j_k$, $j_i \le t$, $j_i, t \in \mathbb{N}$, the set of natural numbers, $i = 1, 2, \ldots, k$, $k \le t$. Let further $P^{[\boldsymbol{\alpha}]}$ and $P_{[\boldsymbol{\beta}]}$, $\boldsymbol{\alpha} \in \mathbb{S}_m^j$, $\boldsymbol{\beta} \in \mathbb{S}_n^k$, denote submatrices of an $m \times n$ matrix P consisting of rows $i_1, i_2, \ldots, i_j$ and columns $j_1, j_2, \ldots, j_k$ of P, respectively. For example, $P_{[\boldsymbol{\beta}]}^{[/\boldsymbol{\alpha}]}$, $\boldsymbol{\alpha} \in \mathbb{S}_m^j$, $\boldsymbol{\beta} \in \mathbb{S}_n^k$, where $/\boldsymbol{\alpha} := \{1, 2, \ldots, m\} - \boldsymbol{\alpha}$, denotes a submatrix of P obtained by eliminating rows $i_1, i_2, \ldots, i_j$ of P and having columns $j_1, j_2, \ldots, j_k$ of P.

The pencil $sE-A$, and analogously the system (1), is called *regular* if the matrices E and A are square and $\det[sE-A]$ is not identically equal to zero. The regularity of (1) guarantees the existence of its transfer function. Closely related to this concept is the notion of regularizability. The system (1) is called *regularizable* (by state feedback) if it is square and there exists an F such that the pencil $sE - A - BF$ is regular. In the case of non-square systems an analogous concept, weak regularizability, is defined in the section 3. It should be noted that the assumption of weak regularizability plays a central role in the problems of pole (structure) assignment.

The *pole structure* of the system (E, A, B) is defined [10] by the zero structure of the pencil $sE - A$. More particularly, the finite zero structure of $sE - A$ is given [12] by the invariant polynomials of $sE - A$, say $\psi_i(s) \rhd \psi_{i+1}(s)$, $i = 1, \ldots, r-1$, while the infinite zero structure is defined [13] by the terms s^{-d_i}, $d_i > 0$, $i = 1, \ldots, k_d$, occurring in the Smith–McMillan form at infinity of $sE - A$ (or by the infinite elementary divisors of $sE - A$ of the orders $\mu_i := d_i + 1$, $d_i > 0$).

The problem of pole structure assignment by state feedback (2) to a square system (1), see [10] and the references therein, consists of finding conditions (necessary and sufficient, if possible) under which there exists an F in (2) such that prescribed monic polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \ldots \rhd \psi_r(s)$ and positive integers $d_1 \ge d_2 \ge \cdots \ge d_{k_d}$ will define the invariant polynomials and infinite zero orders of $sE-A-BF$. It can be easily seen that the formulation remains valid even in the case of the non-square systems (1).

The problem of pole assignment mentioned in Introduction can be stated in a similar way: Find conditions (necessary and sufficient, if possible) under which there exists an F

in (2) such that the roots of a prescribed monic polynomial, say $\psi(s)$, and a positive integer, say d, will define finite and infinite eigenvalues (with multiplicities included) of $sE - A - BF$.

2.1. Feedback Canonical Form

Under the action of the feedback group, which consists of quadruples (P, Q, G, F), where $P,\ Q,\ G$ are invertible matrices and F is an $m \times n$ matrix over $\mathbb{R}$, each system (E, A, B) can be brought into the *feedback canonical form* (FCF)[9],

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P[A + BF]Q, PBG) =: (E_C,\ A_C,\ B_C)$$

with $(E_C,\ A_C,\ B_C)$ denoting the feedback canonical form. The pencil $sE_C - A_C$ is a block diagonal matrix,

$$sE_C - A_C := \text{block diag}\,\{sE_t - A_t\}\,,\ t \in \{\epsilon, \sigma, q, p, l, \eta\},$$

where $sE_t - A_t$ is again a block diagonal matrix consisting of the blocks, non-increasingly ordered by size, of type (b_t), $t \in \{\epsilon, \sigma, q, p, l, \eta\}$,

$$(b_\epsilon)\quad \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}}^{\epsilon_i+1}\right\}\epsilon_i \qquad (b_\sigma)\quad \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s \end{bmatrix}}^{\sigma_i}\right\}\sigma_i$$

$$(b_q)\quad \left.\overbrace{\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}}^{q_i}\right\}q_i{+}1 \qquad (b_p)\quad \left.\overbrace{\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}}^{p_i+1}\right\}p_i{+}1$$

$$(b_l)\quad \left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -a_{i0} & -a_{i1} & \cdots & s - a_{il_i} \end{bmatrix}}^{l_i}\right\}l_i \qquad (b_\eta)\quad \left.\overbrace{\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}}^{\eta_i}\right\}\eta_i{+}1,$$

$i = 1, \ldots, k_t$, with k_t denoting the number of the corresponding blocks. The values describing these blocks are called:

- the nonproper controllability indices, $\epsilon_1 \geq \cdots \geq \epsilon_{k_\epsilon} \geq 0$;

- the proper controllability indices, $\sigma_1 \geq \cdots \geq \sigma_{k_\sigma} > 0$;
- the almost proper controllability indices, $q_1 \geq \cdots \geq q_{k_q} \geq 0$;
- the almost nonproper controllability indices, $p_1 \geq \cdots \geq p_{k_p} \geq 0$;
- the fixed invariant polynomials of $[sE_C - A_C,\ -B_C]$ represented by the polynomials $\alpha_i(s) = s^{l_i} + a_{il_i}s^{l_i-1} + \cdots + a_{i1}s + a_{i0}$, $l_i > 0$, $\alpha_1(s) \triangleright \alpha_2(s) \triangleright \cdots \triangleright \alpha_{k_l}(s)$;
- the row minimal indices of $[sE_C - A_C,\ -B_C]$, $\eta_1 \geq \cdots \geq \eta_{k_\eta} \geq 0$.

The matrix B_C is of the form

$$B_C := \begin{bmatrix} 0 & 0 \\ B_\sigma & 0 \\ 0 & B_q \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \begin{cases} B_\sigma := \text{block diag}\left\{[0\ldots 0\ 1]^T \in \mathbb{R}^{\sigma_i}\right\}_{i=1}^{k_\sigma}, \\ B_q := \text{block diag}\left\{[0\ldots 0\ 1]^T \in \mathbb{R}^{q_i+1}\right\}_{i=1}^{k_q}. \end{cases}$$

2.2. Normal External Description

Definition 2.1. Polynomial matrices $N(s)$, $D(s)$ are said to form a *normal external description* (NED) of the system (E, A, B) if they satisfy the following conditions:

- $\left[\begin{smallmatrix} N(s) \\ D(s) \end{smallmatrix}\right]$ forms a minimal polynomial basis for $\mathrm{Ker}[sE - A,\ -B]$,

$$[sE - A,\ -B] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0,$$

- $N(s)$ forms a minimal polynomial basis for $\mathrm{Ker}\Pi[sE - A]$, where Π is a maximal left annihilator of B,

$$\Pi[sE - A]N(s) = 0.$$

It should be noted that a normal external description always exists and is not unique. However, when the above polynomial bases are brought into their canonical forms [4], then such a normal external description is unique.

Let F define a state feedback (2). Then the formulas

$$[sE - A,\ -B] \begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -F & I_m \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0,$$

$$[sE - A - BF,\ -B] \begin{bmatrix} N(s) \\ D(s) - FN(s) \end{bmatrix} = 0$$

show how the state feedback acts upon the NED of (E, A, B) and hint a relationship between the pencils $[sE - A - BF,\ -B]$ and their NEDs. To enlighten this relationship a bit more, consider a system (E_C, A_C, B_C) and its canonical NED. It can be seen that the NED is given just by the ϵ_i- and σ_i-blocks of (E_C, A_C, B_C) and does not depend on the quantities $q_i,\ p_i,\ \eta_i$ and polynomials $\alpha_i(s)$, which means that the relationships between these two groups of quantities would not be reflected by the NED. So, if we want to use the NED for studying the effect of state feedback upon the system, we should find a way to bring the lost information back into the play. To that end the matrix B_C is extended by the matrix $\overline{B}_C$ (see [10] for detail) in such a way that the lost quantities of the system will appear in its NED. Such a system $(E_C, A_C, [B_C\ \overline{B}_C])$ obtained from (E_C, A_C, B_C) by this trick is said to be an *extended system* of (E_C, A_C, B_C). Its NED is of the form $\begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix}$, where $N_E(s) := \text{block diag}\,\{N_t(s)\}\,,\ t \in \{\epsilon, \sigma, q, p, l, \eta\}$, with

$$
\begin{aligned}
N_\epsilon(s) &:= \text{block diag}\left\{[1\ s \ldots s^{\epsilon_i}]^T\right\}_{i=1}^{k_\epsilon}, \quad & N_\sigma(s) &:= \text{block diag}\left\{\left[1\ s \ldots s^{\sigma_i-1}\right]^T\right\}_{i=1}^{k_\sigma},\\
N_q(s) &:= \text{block diag}\left\{\left[1\ s \ldots s^{q_i-1}\right]^T\right\}_{i=1}^{k_q}, & N_p(s) &:= \text{block diag}\left\{[s^{p_i} \ldots s\ 1]^T\right\}_{i=1}^{k_p},\\
N_l(s) &:= \text{block diag}\left\{\left[1\ s \ldots s^{l_i-1}\right]^T\right\}_{i=1}^{k_l}, & N_\eta(s) &:= \text{block diag}\left\{\left[s^{\eta_i-1} \ldots s\ 1\right]^T\right\}_{i=1}^{k_\eta},
\end{aligned}
$$

and

$$
D_E(s) := \begin{bmatrix} D_{E1}(s) \\ --- \\ D_{E2}(s) \end{bmatrix} := \begin{bmatrix}
0 & S_\sigma & 0 & 0 & 0 & 0 \\
0 & 0 & S_q & 0 & 0 & 0 \\
\hline
0 & 0 & -I_{k_q} & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{k_p} & 0 & 0 \\
0 & 0 & 0 & 0 & S_\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & S_\eta
\end{bmatrix}
$$

with

$$
\begin{aligned}
S_\sigma &:= \text{diag}\,\{s^{\sigma_i}\}_{i=1}^{k_\sigma}\,, \quad & S_q &:= \text{diag}\,\{s^{q_i}\}_{i=1}^{k_q}\,,\\
S_\alpha &:= \text{diag}\,\{\alpha_i(s)\}_{i=1}^{k_l}\,, & S_\eta &:= \text{block diag}\left\{\begin{bmatrix} s^{\eta_i} \\ -1 \end{bmatrix}\right\}_{i=1}^{k_\eta}.
\end{aligned}
$$

The “feedback” matrix describing the effect of state feedback upon the system,

$$
D_{EF}(s) := \begin{bmatrix} D_{EF1}(s) \\ --- \\ D_{EF2}(s) \end{bmatrix} = \begin{bmatrix} D_{E1}(s) \\ --- \\ D_{E2}(s) \end{bmatrix} - \begin{bmatrix} F \\ -- \\ 0 \end{bmatrix} N_E(s), \tag{4}
$$

or explicitely

$$
D_{EF}(s) = \begin{bmatrix}
D_{1\epsilon}(s) & S_\sigma + D_{1\sigma}(s) & D_{1q}(s) & D_{1p}(s) & D_{1l}(s) & D_{1\eta}(s) \\
D_{2\epsilon}(s) & D_{2\sigma}(s) & S_q + D_{2q}(s) & D_{2p}(s) & D_{2l}(s) & D_{2\eta}(s) \\
\hline
0 & 0 & -I_{k_q} & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{k_p} & 0 & 0 \\
0 & 0 & 0 & 0 & S_\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & S_\eta
\end{bmatrix}, \tag{5}
$$

where $D_{ij}(s)$ are arbitrary matrices satisfying the conditions

$$\deg_{ci}\begin{bmatrix} D_{1j}(s) \\ D_{2j}(s) \end{bmatrix} \le \deg_{ci} N_j(s), \quad i = 1, 2, \ldots, j \in \{\epsilon, \sigma, q, p, l, \eta\}. \tag{6}$$

The matrices $D_{ij}(s)$ represent the change to the system, that can be done using a state feedback. In particular, using the concept of the NED, the problem of finding an F such that the prescribed pole structure will be assigned to the system (1) lies in the finding of the appropriate matrices $D_{ij}(s)$. Let $D_{NF}(s) := [D_{ij}(s)]$. Then, matrix $D_{NF}(s)$ being at the disposal, the feedback gain F can be calculated using (4), i. e.

$$D_{NF}(s) = -FN_E(s). \tag{7}$$

2.3. Conformal Mapping

To handle with the finite and infinite zeros in a unified way, we use the conformal mapping

$$s = \frac{1 + aw}{w}, \tag{8}$$

where $a \in \mathbb{R}$, and is not a pole of (E, A, B). This is done as follows.

- Perform first the substitution given by (8) to the equation

$$\begin{bmatrix} sE - A, & -B \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0. \tag{9}$$

- Then premultiply (9) by the matrix diag $\{w^{\nu_i}\}$, $\nu_i := \deg_{ri}\begin{bmatrix} sE - A, & -B \end{bmatrix}$. At the end a w-analogue, $\begin{bmatrix} w\tilde{E} - \tilde{A}, & -\tilde{B}(w) \end{bmatrix}$, of the pencil $\begin{bmatrix} sE - A, & -B \end{bmatrix}$ is obtained. Let $(\tilde{E}, \tilde{A}, \tilde{B})$ denote the associated system.

- Postmultiply further (9) by diag $\{w^{\mu_i}\}$, $\mu_i := \deg_{ci}\left[\begin{smallmatrix} N(s) \\ D(s) \end{smallmatrix}\right]$, to get (9) in the form

$$\begin{bmatrix} w\tilde{E} - \tilde{A}, & -\tilde{B}(w) \end{bmatrix} \begin{bmatrix} \tilde{N}(w) \\ \tilde{D}(w) \end{bmatrix} = 0,$$

  where both $\begin{bmatrix} w\tilde{E} - \tilde{A}, & -\tilde{B}(w) \end{bmatrix}$ and $\left[\begin{smallmatrix} \tilde{N}(w) \\ \tilde{D}(w) \end{smallmatrix}\right]$ are polynomial matrices over $\mathbb{R}[w]$.

The conformal mapping (8) transforms the s-plane into itself moving the points $s = \infty$ to $w = 0$ and $s = a$ to $w = \infty$, while keeping all the finite zeros at finite positions. Consequently, the infinite pole structure of the system (E, A, B) is determined as the finite pole substructure of the system $(\tilde{E}, \tilde{A}, \tilde{B})$ at $w = 0$.

Based on the above formulas, a w-analogue, $\tilde{D}_{EF}(w)$, of the matrix $D_{EF}(s)$ (defined in (5)) is of the form

$$\tilde{D}_{EF}(w) := \left[\begin{array}{cccccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p} & \tilde{D}_{1l} & \tilde{D}_{1\eta} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q + \tilde{D}_{2q} & \tilde{D}_{2p} & \tilde{D}_{2l} & \tilde{D}_{2\eta} \\ \hdashline 0 & 0 & \mathrm{diag}\{-w^{q_i}\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{diag}\{-w^{p_i}\} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_\eta \end{array}\right], \tag{10}$$

where

$$\tilde{S}_\sigma := \mathrm{diag}\left\{(1+aw)^{\sigma_i}\right\}_{i=1}^{k_\sigma}, \qquad \tilde{S}_q := \mathrm{diag}\left\{(1+aw)^{q_i}\right\}_{i=1}^{k_q},$$

$$\tilde{S}_\alpha := \mathrm{diag}\left\{\tilde{\alpha}_i(w)\right\}_{i=1}^{k_l}, \qquad \tilde{S}_\eta := \mathrm{blockdiag}\left\{\begin{bmatrix} (1+aw)^{\eta_i} \\ -w^{\eta_i} \end{bmatrix}\right\}_{i=1}^{k_\eta},$$

$\tilde{D}_{ij}$ are polynomial matrices satisfying the condition, similar to (6), that is

$$\deg_{ci} \begin{bmatrix} \tilde{D}_{1j} \\ \tilde{D}_{2j} \end{bmatrix} \le \deg_{ci} \tilde{N}_j, \quad i = 1, 2, \ldots, j \in \{\epsilon, \sigma, q, p, l, \eta\}. \tag{11}$$

It should be checked whether state feedback (2) does not assign a zero to the resulting matrix at the point $s = a$. If not, then such a state feedback is termed *admissible*. In terms of the matrices $\tilde{D}_{ij}$, the condition of admissibility of a state feedback is satisfied if the matrix $\left[\begin{smallmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} \end{smallmatrix}\right]$ or at least one of its submatrices $\begin{bmatrix} \tilde{D}_{1\epsilon[\boldsymbol{k_q}]} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon[\boldsymbol{k_q}]} & \tilde{D}_{2\sigma} \end{bmatrix}$, $\boldsymbol{k_q} \in \mathbb{S}_{k_\epsilon}^{k_q}$, if $k_\epsilon > k_q$, is column reduced with the column degrees $\epsilon_i,\ i \in \mathbb{S}_{k_\epsilon}^{k_q}$, and $\sigma_i,\ i = 1, 2, \ldots, k_\sigma$.

It appears that the NED of the extended system of (1) is a very useful tool when treating the problems like changing the pole structure by state feedback. The proposition below summarizes some properties of the system (1) that will be used in the sequel.

Proposition 2.2. (Loiseau and Zagalak [10]) The following holds:

(a) The system (E, A, B) is regularizable if and only if $k_\epsilon = k_q$ and $k_\eta = 0$.

(b) The non unit invariant factors of both $w\tilde{E} - \tilde{A} - \tilde{B}(w)F$ and $\tilde{D}_{EF}(w)$ coincide for any admissible F.

(c) The infinite zero structure of $sE - A - BF$ and finite substructure of $\tilde{D}_{EF}(w)$ at $w = 0$ coincide for any admissible F.

As the main problem considered in the paper is the problem of pole assignment to the system (E, A, B), we recall the known result in the case of regularizable systems.

Proposition 2.3. (Loiseau and Zagalak [10]) Given a regularizable system (1) ($k_\epsilon = k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \ge 0$, then there exists a matrix F in (2) such that $\det[sE - A - BF] = \psi(s)$ and the sum of the infinite zero orders of

$sE - A - BF$ equals d if and only if the conditions (12)–(14) (and (15) if $k_\epsilon = 0$) are satisfied:

$$\begin{aligned} \deg \psi(s) + d &= \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, && (12) \\ \psi(s) &\rhd \alpha_1(s)\alpha_2(s)\ldots\alpha_{k_l}(s), && (13) \\ d &\geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i, && (14) \\ \deg \psi(s) &= \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i. && (15) \end{aligned}$$

Remark 2.4. The matrix $D_{EF}(s)$ or its w-analogue $\tilde{D}_{EF}(w)$ shows that the quantities $\alpha_i(s), p_i, q_i$ can not be changed by state feedback (2) while the sum of the indices ϵ_i, σ_i is the number of the poles that can freely be assigned either to finite or infinite locations [10].

3. POLE ASSIGNMENT IN NON-SQUARE SYSTEMS

When dealing with the non-square systems (1), a natural question arising here is under what conditions there exists a state feedback (2) yielding a full rank pencil $sE-A-BF$. The following theorem gives an answer to this question.

Theorem 3.1. (Korotka, Loiseau and Zagalak [5]) There exists a matrix $F \in \mathbb{R}^{m\times n}$ such that pencil $sE - A - BF$ is of

(a) full row rank if and only if

$$k_\epsilon \geq k_q \text{ and } k_\eta = 0, \tag{16}$$

(b) full column rank if and only if

$$k_q \geq k_\epsilon. \tag{17}$$

P r o o f. For pencils having more columns than rows it easily follows, from the form of $D_{EF}(s)$, that $sE-A-BF$ is of full row rank if and only if (16) holds. In the case there are more rows than columns and rank $sE - A - BF$ is full, say n,

$$\operatorname{rank}[sE - A - BF] \leq \operatorname{rank} \Pi[sE - A] + \operatorname{rank} B$$

where Π is a maximal annihilator of B. This condition is equivalent to

$$n - \operatorname{rank}[sE - A - BF] \geq n - \operatorname{rank} \Pi[sE - A] - \operatorname{rank} B. \tag{18}$$

Then, as $n - \operatorname{rank} \Pi[sE - A]$ is equal to the number of the column minimal indices of $[sE - A - B]$, that is to say $k_\epsilon + k_\sigma$, and rank $B = k_\sigma + k_q$, it follows from (18) that $0 \geq k_\epsilon + k_\sigma - k_\sigma - k_q$ and consequently follows (17).

Conversely, if (16), or (17), holds for a pencil $[sE-A,\ -B]$, then it is always possible to find an F such that $sE - A - BF$ will be of full row or column rank. □

Example 3.2. Let the system with $k_\epsilon = 2,\ k_q = 1$ ($\epsilon_1 = \epsilon_2 = q_1 = 1$) be given,

$$[sE{-}A,\ \ -B] := \left[\begin{array}{ccccc|c} s & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & s & -1 \end{array}\right].$$

Evidently, the pencil $sE - A$ is not of full rank but, since the system satisfies (16), a state feedback can always be found such that $sE - A - BF$ will be of full row rank. For example, $F := [1\ 0\ 0\ 0]$ yields

$$sE - A - BF = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & s \end{bmatrix}$$

that is of full row rank.

If the conditions (16) and (17) are satisfied simultaneously then the system is square and regularizable.

The systems satisfying either (16) or (17) might be called *weakly (row or column) regularizable* since we cannot speak of the characteristic polynomial assignment but just of the full rank assignment, which means that (at least) one of the minors of largest possible order $\min\{q, n\}$ of $sE - A - BF$ is not zero. The minors of $P(s) \in \mathbb{R}^{q\times n}[s]$ of order $\min\{q, n\}$ will be called *dominant* and denoted as dm $P(s)$. The greatest common divisor of all dominant minors of $sE{-}A{-}BF$ (hereafter denoted by gcddm$[sE{-}A{-}BF]$) plays a similar role as the determinant of the regular pencils, which means that the zeros of $sE - A - BF$ are given by the zeros of gcddm$[sE - A - BF]$. Clearly, if the system is not weakly regularizable, it would be difficult to speak about the pole assignment in terms of gcddm$[sE - A - BF]$.

Example 3.3. Let $\epsilon_1 = \eta_1 = 1$ and $\sigma_1 = 3$. Then the matrix $D_{EF}(s)$ is of the form

$$D_{EF}(s) = \begin{bmatrix} \alpha_0 + \alpha_1 s & s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 & \gamma \\ 0 & 0 & s \\ 0 & 0 & -1 \end{bmatrix},$$

which shows that there is no F resulting in $D_{EF}(s)$ (and hence $sE{-}A{-}BF$) nonsingular (or at least a dominant minor is nonzero).

It is of course convenient to use the conformal mapping (8) and work on the pencil $w\tilde{E} - \tilde{A} - \tilde{B}(w)F$. Then the roots of $\tilde{\psi}(w)w^d :=$ gcddm$\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$ are the transformed zeros (finite and infinite) of $sE - A - BF$.

4. POLE ASSIGNMENT IN WEAKLY REGULARIZABLE SYSTEMS

When using the above introduced w-notation, it follows, from the previous section, that for the systems satisfying the conditions of Theorem 3.1, the pole assignment is a well defined problem in terms of gcddm$\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$. The problem formulation introduced below is just a rephrased version of that given in Background.

Given a weakly regularizable system (1), a monic polynomial $\psi(s)$, and integer $d > 0$, find conditions under which there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that, using the w-notation, $\tilde{\psi}(w)w^d$ will be a $\operatorname{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$.

By Proposition 2.2, it follows that $\tilde{\psi}(w)w^d$ is also $\operatorname{gcddm}\tilde{D}_{EF}(w)$. Thus, $\operatorname{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$ can be replaced by $\operatorname{gcddm}\tilde{D}_{EF}(w)$ in the above formulation.

4.1. Row regularizable Systems

The following lemma will be useful in the sequel.

Lemma 4.1. Let $P(s) = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ be an $(m+p)\times(n+p)$ polynomial matrix of full row rank with Z nonsingular. Then

$$\operatorname{gcddm} P(s) = \operatorname{gcddm} X \det Z. \tag{19}$$

P r o o f of L e m m a 4.1. It is clear that any $(m+p)\times(m+p)$ nonsingular submatrix, say $P_m(s)$, of $P(s)$ is of the form $\begin{bmatrix} X_m & Y \\ 0 & Z \end{bmatrix}$, where X_m denotes any $m\times m$ nonsingular submatrix of X. Clearly, $\det P_m(s) = \det X_m(s)\det Z(s)$, which implies (19). □

The necessary conditions of solvability to the problem of pole assignment to the row regularizable systems are given in the next theorem.

Theorem 4.2. Let a row regularizable system (1) ($k_\epsilon \ge k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \ge 0$ be given. If there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that a $\tilde{\psi}(w)w^d = \operatorname{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$, then the conditions (20)–(22) (and (23) if $k_q = 0$) are satisfied:

$$\begin{aligned}
\deg\psi(s) + d &\le \sum_{i=1}^{k_q}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, && (20)\\
\psi(s) &\rhd \alpha_1(s)\alpha_2(s)\cdots\alpha_{k_l}(s), && (21)\\
d &\ge \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i, && (22)\\
d &= \sum_{i=1}^{k_p} p_i && (23)
\end{aligned}$$

with equality in (20) if $k_\epsilon = k_q$.

P r o o f. Let F be a matrix such that $sE - A - BF$ is of full row rank and consider the matrix $\tilde{D}_{EF}(w)$ in (10) (without the rows and columns corresponding to $\tilde{S}_\eta$ since

$k_\eta = 0$). It can be seen that this matrix is of the form of the matrix $P(s)$ in Lemma 4.1, $X := \left[\begin{smallmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} \end{smallmatrix}\right]$, which implies that

$$\operatorname{gcddm} \tilde{D}_{EF}(w) = \tilde{\psi}'(w) w^{\sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i} \tilde{\alpha}_1(w) \cdots \tilde{\alpha}_{k_l}(w), \tag{24}$$

where $\tilde{\psi}'(w) := \operatorname{gcddm} X$. The relationship (24) then implies the conditions (21) and (22) and, since

$$\begin{aligned} \deg \tilde{\psi}'(w) &\leq \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i, \quad \text{for } k_\epsilon > k_q, \\ \deg \tilde{\psi}'(w) &= \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i, \quad \text{for } k_\epsilon = k_q, \end{aligned} \tag{25}$$

in view of (11) and the assumption on the admissibility of F, the condition (20) is obtained from (24) and (25).

If $k_q = 0$ then $\tilde{\psi}'(w)$ is always coprime with w. This fact and the w-coprimeness of polynomials $\tilde{\alpha}_i(w)$, $i = 1, \ldots, k_l$, (by assumption on the conformal mapping) imply, in view of (24), the condition (23). □

The following example shows that the conditions of Theorem 4.2 are not sufficient in general.

Example 4.3. Let $\epsilon_1 = 0$ and $\sigma_1 = 3$. Then the matrix $D_{EF}(s)$ is of the form

$$D_{EF}(s) = \begin{bmatrix} \alpha_0 & s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 \end{bmatrix}$$

and the degrees of the gcddm $D_{EF}(s)$ are either 0 or 3, but never 1 or 2 although they satisfy (20). □

When just one nonzero dominant minor of $sE - A - BF$ is to be assigned, a more complete answer to the problem of pole assignment to a row regularizable system can be given.

Theorem 4.4. Let a row regularizable system (1) ($k_\epsilon \geq k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. Let further $\{\epsilon_i'\}_{i=1}^{k_q}$ be a subset of $\{\epsilon_i\}_{i=1}^{k_\epsilon}$. Then there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that a $\tilde{\psi}(w) w^d = \operatorname{gcddm}\left[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\right]$, if the conditions (26) – (28) (and (29) if $k_q = 0$) are satisfied:

$$\deg \psi(s) + d = \sum_{i=1}^{k_q} \epsilon_i' + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, \tag{26}$$

$$\psi(s) \rhd \alpha_1(s)\alpha_2(s) \ldots \alpha_{k_l}(s), \tag{27}$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i, \tag{28}$$

$$\deg \psi(s) = \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i. \tag{29}$$

P r o o f . The conditions (26) – (28) imply that F can be chosen such that $\tilde{D}_{EF}(w)$ is of form (10) (without the rows and columns corresponding to $\tilde{S}_\eta$) with

$$\operatorname{gcddm} \tilde{D}_{EF}(w) = \tilde{\psi}'(w) w^{\sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i} \tilde{\alpha}_1(w) \dots \tilde{\alpha}_{k_l}(w), \tag{30}$$

where $\tilde{\psi}'(w) := \operatorname{gcddm} \begin{bmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} \end{bmatrix}$, and

$$\deg \tilde{\psi}'(w) = \sum_{i=1}^{k_q} \epsilon_i' + \sum_{i=1}^{k_\sigma} \sigma_i.$$

Such matrices $\tilde{D}_{ij}$ always exist because of (11). For instance, choose $\left[\begin{smallmatrix} \tilde{D}_{1\epsilon} \\ \tilde{D}_{2\epsilon} \end{smallmatrix}\right] := \left[\begin{smallmatrix} \tilde{D}_{1\epsilon[\boldsymbol{k}_{\boldsymbol{q}}^\star]} & 0 \\ \tilde{D}_{2\epsilon[\boldsymbol{k}_{\boldsymbol{q}}^\star]} & 0 \end{smallmatrix}\right]$, $\boldsymbol{k}_{\boldsymbol{q}}^\star \in \mathbb{S}_{k_\epsilon}^{k_q}$, such that $\left[\begin{smallmatrix} \tilde{D}_{1\epsilon[\boldsymbol{k}_{\boldsymbol{q}}^\star]} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon[\boldsymbol{k}_{\boldsymbol{q}}^\star]} & \tilde{D}_{2\sigma} \end{smallmatrix}\right]$ is column reduced with column degrees ϵ_i', $i = 1, 2, \dots, k_q$, and σ_i, $i = 1, 2, \dots, k_\sigma$.

If $k_q = 0$ then $\tilde{\psi}'(w)$ is always coprime with w, and (29) should be added. □

4.2. Column Regularizable Systems

Let us consider a column regularizable system (1).

Lemma 4.5. Let $P(s) = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ be an $(m+p) \times (n+p)$, $m - n \le p$, polynomial matrix of full column rank where Z is $p \times p$, nonsingular, and diagonal. Then

$$\operatorname{gcddm} P(s) \lhd \gcd \left\{ \operatorname{gcddm} X \det Z;\ \ \det[X \ Y_{[/\boldsymbol{k}]}] \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]},\ \boldsymbol{k} \in \mathbb{S}_p^{p-m+n} \right\}. \tag{31}$$

P r o o f . Clearly, the dominant minors dm $P(s)$ are determinants of $(n+p) \times (n+p)$ submatrices of $P(s)$, i. e.

$$\operatorname{dm}\ P(s) = \det P^{[\boldsymbol{j}]}(s),\ \boldsymbol{j} \in \mathbb{S}_{m+p}^{n+p}.$$

More particularly,

$$\operatorname{dm}\ P(s) = \det \begin{bmatrix} X^{[\boldsymbol{j}]} & Y^{[\boldsymbol{j}]} \\ 0 & Z^{[\boldsymbol{k}]} \end{bmatrix},\ \ \boldsymbol{j} \in \mathbb{S}_m^{n+i},\ \boldsymbol{k} \in \mathbb{S}_p^{p-i},\ i = 0, 1, \dots, m-n.$$

Then, as a consequence of the diagonal form of Z,

$$\operatorname{dm}\ P(s) = \det[X \ Y_{[/\boldsymbol{k}]}]^{[\boldsymbol{j}]} \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]},\ \ \boldsymbol{k} \in \mathbb{S}_p^{p-i},\ \boldsymbol{j} \in \mathbb{S}_m^{n+i}. \tag{32}$$

Consider now the relationship (32) for $i = 0$ and $i = m - n$. In the former case, the relationship (32) describes the dominant minors of $P(s)$ obtained when eliminating $m - n$ rows of $[X \ \ Y]$, i. e.

$$\operatorname{dm}\ P(s) = \det X^{[\boldsymbol{j}]} \det Z,\ \ \boldsymbol{j} \in \mathbb{S}_m^n.$$

Now, since $\operatorname{gcddm} X = \gcd\{\det X^{[\boldsymbol{j}]},\ \boldsymbol{j} \in \mathbb{S}^n_m\}$, it follows that

$$\operatorname{gcddm} X \det Z = \gcd\{\det X^{[\boldsymbol{j}]} \det Z,\ \boldsymbol{j} \in \mathbb{S}^n_m\}. \tag{33}$$

The case $i = m - n$ corresponds to eliminating $m - n$ rows of $[0\ Z]$, which means that the corresponding minors are

$$\operatorname{dm} P(s) = \det[X\ Y_{[/\boldsymbol{k}]}] \det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]}, \quad \boldsymbol{k} \in \mathbb{S}^{p-m+n}_p. \tag{34}$$

The $\operatorname{gcddm} P(s)$ divides the terms (33), (34) by definition, which implies (31). □

In the following theorem necessary conditions of solvability to the pole assignment problem in a column regularizable system are given.

Theorem 4.6. Let a column regularizable system (1) ($k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. If there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $\tilde{\psi}(w)w^d = \operatorname{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$, then the conditions (35) – (39) (and (40) if $k_\epsilon = 0$) are satisfied:

$$\deg \psi(s) + d \ \leq\ \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_\epsilon} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, \tag{35}$$

$$\psi(s) \ \triangleright\ \prod_{i=k_q-k_\epsilon+1}^{k_l} \alpha_i(s), \tag{36}$$

$$d \ \geq\ \sum_{i=1}^{k_\epsilon+k_p} z_i, \tag{37}$$

$$\deg \psi(s) \ \leq\ \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i, \tag{38}$$

$$d \ \leq\ \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_\epsilon} q_i + \sum_{i=1}^{k_p} p_i, \tag{39}$$

$$d \ \leq\ \sum_{i=1}^{k_p} p_i, \tag{40}$$

where equality holds in (35) for $k_\epsilon = k_q$, $\{z_i\}_{i=1}^{k_\epsilon+k_p}$ denotes the set of the first $k_\epsilon + k_p$ indices of the non-decreasingly ordered set $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$, and $\alpha_i(s) := 1$ for $k_l \leq k_q - k_\epsilon$.

P r o o f. Let $sE - A - BF$ be of full column rank, which implies that the matrix $\tilde{D}_{EF}(w)$ (see (10)) also has full column rank. Bringing, by elementary operations, the matrix $\tilde{S}_\eta$ to the form, $S_{\tilde{\eta}} \cong \begin{bmatrix} I_{k_\eta} \\ 0 \end{bmatrix}$, the matrix $\tilde{D}_{EF}(w)$ can further be simplified. Particularly, the matrices $\tilde{D}_{1\eta}, \tilde{D}_{2\eta}$ can be zeroed, which means that we can study just a submatrix of $\tilde{D}_{EF}(w)$, hereafter denoted by $P(w)$, that does not contain the rows and columns corresponding to the η-blocks. It is evident that $\operatorname{gcddm} \tilde{D}_{EF}(w) = \operatorname{gcddm} P(w)$ as the only nonzero dominant minors of $\tilde{D}_{EF}(w)$ are those of $P(w)$.

Further, it can be seen that the matrix $P(w)$ assumes the form of the matrix $P(s)$ in Lemma 4.5. Let $m := k_\sigma + k_q$, $n := k_\epsilon + k_\sigma$, and $p := k_q + k_p + k_l$,

$$P(w) = \left[\begin{array}{cc|ccc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & \tilde{D}_{1q} & \tilde{D}_{1p} & \tilde{D}_{1l} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q + \tilde{D}_{2q} & \tilde{D}_{2p} & \tilde{D}_{2l} \\ \hline 0 & 0 & \operatorname{diag}\{-w^{q_i}\} & 0 & 0 \\ 0 & 0 & 0 & \operatorname{diag}\{-w^{p_i}\} & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_\alpha \end{array}\right] := \left[\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array}\right]$$

with $Z := \operatorname{diag}\{Z_q, Z_p, \tilde{S}_\alpha\}$, where $Z_q := \operatorname{diag}\{-w^{q_i}\}$ and $Z_p := \operatorname{diag}\{-w^{p_i}\}$. Then, similarly as in Lemma 4.5, the dominant minors of $P(w)$ are formed by eliminating $m - n = k_\sigma + k_q - (k_\epsilon + k_\sigma) = k_q - k_\epsilon$ rows of $P(w)$, i. e.

$$\operatorname{dm} P(w) = \det[X \ Y_{[/\boldsymbol{k}]}]^{[\boldsymbol{j}]} \det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]}, \quad \boldsymbol{j} \in \mathbb{S}^{n+i}_m, \ \boldsymbol{k} \in \mathbb{S}^{p-i}_p, \ i = 0, 1, \ldots, k_q - k_\epsilon.$$

Begin with considering the dominant minors for $i = k_q - k_\epsilon$, i. e.

$$\det[X \ Y_{[/\boldsymbol{k}]}] \det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]}, \quad \boldsymbol{k} \in \mathbb{S}^{k_\epsilon + k_p + k_l}_p.$$

Let further k'_q, k'_p, k'_l be nonnegative numbers satisfying

$$k_\epsilon \leq \quad k'_q \quad \leq k_q, \tag{41}$$

$$k_p - (k_q - k_\epsilon) \leq \quad k'_p \quad \leq k_p, \tag{42}$$

$$k_l - (k_q - k_\epsilon) \leq \quad k'_l \quad \leq k_l, \tag{43}$$

$$k'_q + k'_p + k'_l = k_\epsilon + k_p + k_l. \tag{44}$$

Then, the $\det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]}$ can be written in the form

$$\det Z^{[\boldsymbol{k}]}_{[\boldsymbol{k}]} = \det Z_{q[\boldsymbol{k'_q}]}^{\ [\boldsymbol{k'_q}]} \det Z_{p[\boldsymbol{k'_p}]}^{\ [\boldsymbol{k'_p}]} \det \tilde{S}_{\alpha[\boldsymbol{k'_l}]}^{\ [\boldsymbol{k'_l}]},$$

where $\det \tilde{S}_{\alpha[\boldsymbol{k'_l}]}^{\ [\boldsymbol{k'_l}]} := 1$ for $k'_l = 0$, and $\det Z_{p[\boldsymbol{k'_p}]}^{\ [\boldsymbol{k'_p}]} := 1$ for $k'_p = 0$.

The polynomial $\det \tilde{S}_{\alpha[\boldsymbol{k'_l}]}^{\ [\boldsymbol{k'_l}]}$ of the smallest degree, which is defined for the smallest k'_l, is

$$\det \tilde{S}_{\alpha[\boldsymbol{k'_l}]}^{\ [\boldsymbol{k'_l}]} = \prod_{i=k_q-k_\epsilon+1}^{k_l} \tilde{\alpha}_i(w),$$

and the condition (36) follows.

Consider now the polynomial of the smallest degree of the

$$\det Z_{q[\boldsymbol{k'_q}]}^{\ [\boldsymbol{k'_q}]} \det Z_{p[\boldsymbol{k'_p}]}^{\ [\boldsymbol{k'_p}]} = w^{k_s},$$

where $k_s := k'_q + k'_p$. Since $\det \tilde{S}_{\alpha[\boldsymbol{k_l'}]}^{[\boldsymbol{k_l'}]}$ is always coprime with w ($\tilde{\alpha}_i(w)$ are coprime with w by assumption on the conformal mapping), the smallest k_s is defined by (44) for $k_l{}' = k_l$,

$$k_s = k_\epsilon + k_p.$$

Let $\{z_i\}_{i=1}^{k_s}$ be the set of the k_s smallest indices of the set $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$. In view of (11) there exist matrices $\tilde{D}_{ij}$ such that

$$\det[X \;\; Y_{[/\boldsymbol{k_s}]}] = \det \left[\begin{bmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} \end{bmatrix} \left[\begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k_{q'}}]} \begin{bmatrix} \tilde{D}_{1p} \\ \tilde{D}_{2p} \end{bmatrix}_{[/\boldsymbol{k_{p'}}]} \right] \right]$$

and w are coprime. This implies the condition (37).

By lemma 4.5, the gcddm $P(w)$ satisfies

$$\operatorname{gcddm} P(w) \lhd \gcd \left\{ \operatorname{gcddm} X \det Z; \;\; \det[X \;\; Y_{[/\boldsymbol{k}]}] \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}, \;\; \boldsymbol{k} \in \mathbb{S}_p^{k_\epsilon + k_p + k_l} \right\}. \tag{45}$$

But at first it should be noted that from the full column rank hypothesis of $P(w)$ it follows that at least one minor from the below set is nonzero,

$$\det \left[X \;\; \begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k_\epsilon}]} \right] \det Z_{q[\boldsymbol{k_\epsilon}]}^{[\boldsymbol{k_\epsilon}]} \det Z_p \det \tilde{S}_\alpha, \quad \boldsymbol{k_\epsilon} \in \mathbb{S}_{k_q}^{k_\epsilon}. \tag{46}$$

Hence, (46) limits the largest degree of $\det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}$, i. e.

$$\deg \gcd\{\det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}, \; \boldsymbol{k} \in \mathbb{S}_p^{p-i}, \; i = 0, 1, \ldots, k_q - k_\epsilon\} \leq \sum_{i=1}^{k_\epsilon} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i. \tag{47}$$

Finally, as the matrix $\tilde{S}_q + \tilde{D}_{2q}$ (i. e. all the entries of the matrix) is not divisible by w, then

$$\begin{aligned} \deg \gcd & \left\{ \operatorname{gcddm} X \; \frac{\det Z_q}{\det Z_{q[\boldsymbol{k_\epsilon}]}^{[\boldsymbol{k_\epsilon}]}}; \;\; \det \left[X \;\; \begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k_\epsilon}]} \right], \; \boldsymbol{k_\epsilon} \in \mathbb{S}_{k_q}^{k_\epsilon} \right\} \\ & \leq \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i. \end{aligned} \tag{48}$$

The condition (35) then follows from (45), (47) and (48). It should be noted that if $k_\epsilon = k_q$ then (45) comes to

$$\operatorname{gcddm} P(w) = \det P(w) = \det X \det Z$$

and the equality in (35) holds.

Since $\det Z_{q[\boldsymbol{k_\epsilon}]}^{[\boldsymbol{k_\epsilon}]} \det Z_p$ in (46) is of the form w^j, $j \geq 0$, which means that the q, p-blocks can't contribute to $\tilde{\psi}(w)$, then (38) should satisfy. The w-coprimeness of $\det \tilde{S}_\alpha$

implies analogously (39). If $k_\epsilon = 0$ then the largest degree of $\det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}$ in (47) is limited by $\sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i$. In addition the gcd of the bracketed expression in (48) is not divisible by w. These facts imply (40). □

The conditions of Theorem 4.6 are believed to be also sufficient, but the proof should be completed. Below, sufficient conditions are given for some special cases. More particularly, in the column regularizable systems, it is always possible to assign the gcddm $\tilde{D}_{EF}(w)$ of the form

$$\operatorname{gcddm} \tilde{D}_{EF}(w) = \operatorname{gcddm} X \det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} \det Z_p \det \tilde{S}_\alpha, \tag{49}$$

where $\boldsymbol{k}_\epsilon^\star$ is a specific element from $\mathbb{S}_{k_q}^{k_\epsilon}$.

In order to satisfy (49), let the corresponding minor be a nonzero one from the set (46),

$$\operatorname{dm} \tilde{D}_{EF}(w) = \det \begin{bmatrix} X & \begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k}_\epsilon^\star]} \end{bmatrix} \det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} \det Z_p \det \tilde{S}_\alpha, \quad \text{i.e.}$$

$$\det \begin{bmatrix} X & \begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k}_\epsilon^\star]} \end{bmatrix} \neq 0. \tag{50}$$

This minor belongs to the set

$$\det[X \ Y_{[/\boldsymbol{k}]}] \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}, \quad \boldsymbol{k} \in \mathbb{S}_p^{k_\epsilon + k_p + k_l},$$

and is the only nonzero minor from this set if the following conditions are satisfied:

$$\det \begin{bmatrix} X & \begin{bmatrix} \tilde{D}_{1q} \\ \tilde{S}_q + \tilde{D}_{2q} \end{bmatrix}_{[/\boldsymbol{k}'_q]} & \begin{bmatrix} \tilde{D}_{1p} \\ \tilde{D}_{2p} \end{bmatrix}_{[/\boldsymbol{k}'_p]} & \begin{bmatrix} \tilde{D}_{1l} \\ \tilde{D}_{2l} \end{bmatrix}_{[/\boldsymbol{k}'_l]} \end{bmatrix} = 0 \tag{51}$$

for all $\boldsymbol{k}'_q \neq \boldsymbol{k}_\epsilon^\star$, $k'_p \neq k_p$, $k'_l \neq k_l$, where k'_q, k'_p, k'_l are the same as in (41)–(44).

This condition holds for all $k'_p \neq k_p$, $k'_l \neq k_l$, if $\begin{bmatrix} \tilde{D}_{1p} & \tilde{D}_{1l} \\ \tilde{D}_{2p} & \tilde{D}_{2l} \end{bmatrix} = 0$. Then

$$\gcd(\det[X \quad Y_{[/\boldsymbol{k}]}] \det Z_{[\boldsymbol{k}]}^{[\boldsymbol{k}]}, \quad \boldsymbol{k} \in \mathbb{S}_p^{k_\epsilon + k_p + k_l}) = \operatorname{gcddm} \tilde{D}_{xq} \det Z_p \det \tilde{S}_\alpha,$$

where

$$\tilde{D}_{xq} := \left[\begin{array}{cc|c} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma} & \tilde{D}_{1q} \\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma} & \tilde{S}_q + \tilde{D}_{2q} \\ \hline 0 & 0 & Z_q \end{array} \right].$$

Now, if the matrix $\tilde{D}_{xq}$ is partitioned such that

$$\tilde{D}_{xq}=\left[\begin{array}{ccc|cc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} & & \tilde{D}_{1q[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{1q[/\boldsymbol{k}_\epsilon^\star]} \\ \tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]} & & \left[\tilde{S}_q+\tilde{D}_{2q}\right]^{[\boldsymbol{k}_\epsilon^\star]}_{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2q[/\boldsymbol{k}_\epsilon^\star]}^{[\ \boldsymbol{k}_\epsilon^\star]} \\ \tilde{D}_{2\epsilon}^{[/\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[/\boldsymbol{k}_\epsilon^\star]} & & \tilde{D}_{2q[\ \boldsymbol{k}_\epsilon^\star]}^{[/\boldsymbol{k}_\epsilon^\star]} & \left[\tilde{S}_q+\tilde{D}_{2q}\right]^{[/\boldsymbol{k}_\epsilon^\star]}_{[/\boldsymbol{k}_\epsilon^\star]} \\ \hdashline 0 & 0 & & Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} & 0 \\ 0 & 0 & & 0 & Z_{q[/\boldsymbol{k}_\epsilon^\star]}^{[/\boldsymbol{k}_\epsilon^\star]} \end{array}\right],$$

then (50) can be written in the form

$$\det\begin{bmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} & \tilde{D}_{1q[/\boldsymbol{k}_\epsilon^\star]} \\ \tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2q[/\boldsymbol{k}_\epsilon^\star]}^{[\ \boldsymbol{k}_\epsilon^\star]} \\ \tilde{D}_{2\epsilon}^{[/\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[/\boldsymbol{k}_\epsilon^\star]} & \left[\tilde{S}_q+\tilde{D}_{2q}\right]^{[/\boldsymbol{k}_\epsilon^\star]}_{[/\boldsymbol{k}_\epsilon^\star]} \end{bmatrix}\neq 0.$$

The matrices $\tilde{S}_\sigma+\tilde{D}_{1\sigma}$, $\left[\tilde{S}_q+\tilde{D}_{2q}\right]^{[/\boldsymbol{k}_\epsilon^\star]}_{[/\boldsymbol{k}_\epsilon^\star]}$ are always nonsingular, therefore the condition $\det\tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]}\neq 0$ guaranties (50). Similarly, if the matrices $\tilde{D}_{2\epsilon}^{[/\boldsymbol{k}_\epsilon^\star]}$, $\tilde{D}_{2\sigma}^{[/\boldsymbol{k}_\epsilon^\star]}$, $\tilde{D}_{2q[\ \boldsymbol{k}_\epsilon^\star]}^{[/\boldsymbol{k}_\epsilon^\star]}$ are zero matrices, then (51) holds for all $\boldsymbol{k}_{\boldsymbol{q}}'\neq\boldsymbol{k}_\epsilon^\star$. Indeed, using row and column elementary operations, the matrix $\tilde{D}_{xq}$ is brought to the form

$$\tilde{D}_{xq}\simeq\left[\begin{array}{ccc|cc} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} & & \tilde{D}_{1q[\boldsymbol{k}_\epsilon^\star]} & 0 \\ \tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]} & & \left[\tilde{S}_q+\tilde{D}_{2q}\right]^{[\boldsymbol{k}_\epsilon^\star]}_{[\boldsymbol{k}_\epsilon^\star]} & 0 \\ 0 & 0 & & 0 & I_{k_q-k_\epsilon} \\ \hdashline 0 & 0 & & Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} & 0 \\ 0 & 0 & & 0 & 0_{k_q-k_\epsilon} \end{array}\right],$$

which shows that the only nonzero dominant minor of the matrix $\tilde{D}_{xq}$ is the minor corresponding to (50). Since

$$\operatorname{gcddm}\tilde{D}_{xq}=\det\begin{bmatrix} \tilde{D}_{1\epsilon} & \tilde{S}_\sigma+\tilde{D}_{1\sigma} \\ \tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]} \end{bmatrix}\det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]}=\operatorname{gcddm}X\det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]},$$

the relationship (49) follows.

The above described procedure is not unique but seems to be the simplest one for the construction of the matrices $\tilde{D}_{ij}$. It will also be used in the following theorem.

Theorem 4.7. Let a column regularizable system (1) ($k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$ be given. Let further $\{q_i'\}_{i=1}^{k_\epsilon}$ be a subset of $\{q_i\}_{i=1}^{k_q}$. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that a $\tilde{\psi}(w)w^d = \operatorname{gcddm}\big[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\big]$ if the conditions (52) – (54) (and (55) if $k_\epsilon = 0$) are satisfied:

$$\deg\psi(s) + d = \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon}q_i' + \sum_{i=1}^{k_p}p_i + \sum_{i=1}^{k_l}l_i, \tag{52}$$

$$\psi(s) \rhd \alpha_1(s)\alpha_2(s)\ldots\alpha_{k_l}(s), \tag{53}$$

$$d \geq \sum_{i=1}^{k_\epsilon}q_i' + \sum_{i=1}^{k_p}p_i, \tag{54}$$

$$\deg\psi(s) = \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_l}l_i. \tag{55}$$

P r o o f. Suppose (52) – (54) hold. This implies that

$$\sum_{i=1}^{k_l}l_i \leq \deg\tilde{\psi}(w) \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_l}l_i,$$

$$\sum_{i=1}^{k_\epsilon}q_i' + \sum_{i=1}^{k_p}p_i \leq d \leq \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_\epsilon}q_i' + \sum_{i=1}^{k_p}p_i,$$

which means that an F can be chosen such that

$$\operatorname{gcddm}\tilde{D}_{EF}(w) = \operatorname{gcddm}\begin{bmatrix}\tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma}\\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma}\end{bmatrix}\det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]}\; w^{\sum_{i=1}^{k_p}p_i}\;\alpha_1(s)\ldots\alpha_{k_l}(s),$$

where

$$\deg\operatorname{gcddm}\begin{bmatrix}\tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma}\\ \tilde{D}_{2\epsilon} & \tilde{D}_{2\sigma}\end{bmatrix} = \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i,$$

$\det Z_{q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]}$ denotes $w^{\sum_{i=1}^{k_\epsilon}q_i'}$, and $\boldsymbol{k}_\epsilon^\star$ corresponds to the positions of terms $w^{q_i'}$ in (10). If $k_\epsilon = 0$ then $\operatorname{gcddm}\left[\begin{smallmatrix}\tilde{S}_\sigma+\tilde{D}_{1\sigma}\\ \tilde{D}_{2\sigma}\end{smallmatrix}\right]$ is always coprime with w, which is reflected by (55). Such a matrix F, and consequently matrices $\tilde{D}_{ij}$, always exists because of (11). For instance,

$$\tilde{D}_{2\epsilon} := \begin{bmatrix}\tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]}\\ 0\end{bmatrix}, \qquad \tilde{D}_{2\sigma} := \begin{bmatrix}\tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]}\\ 0\end{bmatrix}, \tag{56}$$

such that

$$\deg\det\begin{bmatrix}\tilde{D}_{1\epsilon} & \tilde{S}_\sigma + \tilde{D}_{1\sigma}\\ \tilde{D}_{2\epsilon}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2\sigma}^{[\boldsymbol{k}_\epsilon^\star]}\end{bmatrix} = \sum_{i=1}^{k_\epsilon}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i,$$

and

$$\tilde{D}_{2q} := \begin{bmatrix} \tilde{D}_{2q[\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} & \tilde{D}_{2q[/\boldsymbol{k}_\epsilon^\star]}^{[\boldsymbol{k}_\epsilon^\star]} \\ 0 & \tilde{D}_{2q[/\boldsymbol{k}_\epsilon^\star]}^{[/\boldsymbol{k}_\epsilon^\star]} \end{bmatrix}, \quad \begin{bmatrix} \tilde{D}_{1p} & \tilde{D}_{1l} \\ \tilde{D}_{2p} & \tilde{D}_{2l} \end{bmatrix} := 0. \tag{57}$$

□

It should be noted that $\deg \psi(s) + d$ is the number of poles (multiplicities included) of system (1) (i. e. $\deg \psi(s)$ is the number of finite poles, multiplicities included, and d is the multiplicity of the pole at infinity). In the case of regularizable system this value is constant (see (12)). The situation is different as far as weakly regularizable systems (see (20) and (35)) are concerned. In the following theorem the problem of pole assignment there is considered in the case when the maximal number of poles is to be assigned.

Theorem 4.8. [5] Given a weakly (row or column) regularizable system (1) (i. e. $k_\epsilon \geq k_q$ and $k_\eta = 0$ or $k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$. If

$$\deg \psi(s) + d = \sum_{i=1}^{k_r} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i, \tag{58}$$

then there exists a state feedback (2) such that $\operatorname{gcddm}\left[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\right] = \tilde{\psi}(w)w^d$ if and only if the conditions (59), (60) (and (61) if $k_r = 0$) are satisfied:

$$\psi(s) \quad \triangleright \quad \alpha_1(s)\alpha_2(s)\ldots\alpha_{k_l}(s), \tag{59}$$

$$d \quad \geq \quad \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i, \tag{60}$$

$$\deg \psi(s) \quad = \quad \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i, \tag{61}$$

where $k_r := \min\{k_\epsilon, k_q\}$.

Proof. The necessity of (58) – (61) follows directly from Theorem 4.2 and Theorem 4.6 for the row and column regularizable systems, respectively. These conditions are also sufficient by Theorem 4.4 and Theorem 4.7. □

Example 4.9. Consider a system (E, A, B) given by

$$[sE - A, \;\; -B] := \left[\begin{array}{cccc|cc} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & -1 \end{array}\right],$$

i. e. with $\epsilon_1 = 0$, $q_1 = 2$, $q_2 = 1$. The system is column regularizable and, as follows from Theorem 4.8, the maximal number of the poles that can be assigned is $d = 2$. We are going to show how to construct a matrix F such that $\operatorname{gcddm}\tilde{D}_{EF}(w) = w^2$. To that end, the extended system of (E, A, B) and its NED are constructed first.

$$[sE-A, \quad -[B\ \bar{B}]\,] := \left[\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & s & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & s & 0 & -1 & 0 & 0 \end{array}\right].$$

$$\left[\begin{array}{c} N_E(s) \\ --- \\ D_E(s) \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \\ \hline 0 & s^2 & 0 \\ 0 & 0 & s \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right], \quad \left[\begin{array}{c} \tilde{N}_E(w) \\ --- \\ \tilde{D}_E(w) \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & (1+aw)w & 0 \\ 0 & 0 & w \\ \hline 0 & (1+aw)^2 & 0 \\ 0 & 0 & 1+aw \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{array}\right].$$

Now, the matrix $\tilde{D}_{EF}(w)$ will be constructed such that $\operatorname{gcddm}\tilde{D}_{EF}(w) = w^2$, which means that the matrices $\tilde{D}_{ij}$ should be appropriately chosen. By (11), we have

$$\tilde{D}_{2\epsilon} := \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \end{bmatrix}, \quad \tilde{D}_{2q} := \begin{bmatrix} \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ \alpha_{22}w^2 + \alpha_{23}w & \alpha_{24}w \end{bmatrix}, \tag{62}$$

where coefficients α_{ij} can freely be chosen. Next, taking into account the relationships (56)–(57) and the fact that $q_1 = 2$, we put $\boldsymbol{k}_\epsilon^\star = \{1\}$, which gives $\deg\det\tilde{D}_{2\epsilon}^{[1]} = \deg\det\alpha_{11} = 0$, i.e. $\alpha_{11} \neq 0$, $\tilde{D}_{2\epsilon}^{[/1]} = \alpha_{21} = 0$, $\tilde{D}_{2q[\ 1]}^{[/1]} = \alpha_{22}w^2 + \alpha_{23}w = 0$. Then, the matrix $\tilde{D}_{EF}(w)$ is found to be of the form

$$\tilde{D}_{EF}(w) = \begin{bmatrix} \alpha_{11} & (1+aw)^2 + \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ 0 & 0 & 1 + aw + \alpha_{24}w \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{bmatrix},$$

where $\alpha_{11} \neq 0$ and the values of all other coefficients α_{ij} are irrelevant.

Let $\tilde{D}_{NF}(w)$ denote a w-analogue of the matrix $D_{NF}(s)$ in (7). This matrix is of the form

$$\tilde{D}_{NF} = \begin{bmatrix} \alpha_{11} & \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ 0 & 0 & \alpha_{24}w \end{bmatrix}, \ \alpha_{11} \neq 0,$$

which gives the state feedback gain $F = \left[\begin{smallmatrix} \alpha_{11} & \alpha_{12}-a\alpha_{13} & \alpha_{13} & \alpha_{14} \\ 0 & 0 & 0 & \alpha_{24} \end{smallmatrix}\right]$ with $\alpha_{11} \neq 0$ (the values of other α_{ij} are irrelevant).

It can be easily verified that for such an F the pencil $sE - A - BF$ has the pole at infinity of order 2,

$$[sE-A-BF] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -\alpha_{11} & -\alpha_{12}+a\alpha_{13} & s-\alpha_{13} & -\alpha_{14} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & s-\alpha_{24} \end{bmatrix} \sim \begin{bmatrix} -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & s \end{bmatrix}.$$

Remark 4.10. For the systems of low dimensions, like the above one, a state feedback gain F can be calculated directly from the forms of dm $\tilde{D}_{EF}(w)$. For example, consider $\tilde{D}_{EF}(w)$ with the matrices $\tilde{D}_{ij}$ defined by (62), i. e.

$$\tilde{D}_{EF}(w) = \begin{bmatrix} \alpha_{11} & (1+aw)^2+\alpha_{12}w^2+\alpha_{13}w & \alpha_{14}w \\ \alpha_{21} & \alpha_{22}w^2+\alpha_{23}w & 1+aw+\alpha_{24}w \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{bmatrix} \cong \begin{bmatrix} \alpha_{11} & 1+(2a+\alpha_{13})w & 0 \\ \alpha_{21} & 0 & 1 \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{bmatrix}.$$

There are four dominant minors of the above matrix, namely

$$\begin{aligned} \det \tilde{D}_{EF}(w)^{[/1]} &= w^3\alpha_{21}, \\ \det \tilde{D}_{EF}(w)^{[/2]} &= w^3\alpha_{11}, \\ \det \tilde{D}_{EF}(w)^{[/3]} &= w\alpha_{21}[1+(2a+\alpha_{13})w], \\ \det \tilde{D}_{EF}(w)^{[/4]} &= w^2\alpha_{11}, \end{aligned}$$

which implies that the maximal number of assignable poles is $d = 2$ (the full column rank hypothesis implies that at least one coefficient $\alpha_{i1},\ i = 1, 2$ is nonzero). The result follows on putting $\alpha_{11} \neq 0,\ \alpha_{21} = 0$.

5. CONCLUSIONS

The problem of pole assignment by state feedback to the non-square implicit systems (1) is considered in the paper. If the condition (a) or (b) of Theorem 3.1 is satisfied, then the problem is well-defined and such systems are called weakly (row or column) regularizable - the weak regularizability can be viewed as an analog of the regularizability known [11] in the case of square systems (1). The main result of the paper is stated in Theorem 4.2 and Theorem 4.6 that give necessary conditions of solvability. The conditions are stated in terms of the greatest common divisor of the dominant minors of $\left[w\tilde{E} - \tilde{A} - \tilde{B}(w)F\right]$, which is an analog of the determinant of the square pencils. Just sufficient conditions for particular cases are also given in Theorem 4.4 and Theorem 4.7. In a special case, when the maximal number of zeros (both finite and infinite) is to be assigned to the pencil $sE - A - BF$, Theorem 4.8 gives necessary and sufficient conditions.

By investigating the results of the section 4, the concluding remarks are given below. We begin with the row regularizable systems ($k_\epsilon \geq k_q$ and $k_\eta = 0$). In the case when $k_\epsilon = k_q$ (regularizable system), the conditions of Theorem 4.2 coincide with those of Theorem 4.4 (a set $\{\epsilon'_i\}_{i=1}^{k_q}$ is the whole set $\{\epsilon_i\}_{i=1}^{k_\epsilon}$ now). In other words, the conditions of these theorems are necessary and sufficient. And in fact, these theorems reduce to Proposition 2.3. The same holds in the case of column regularizable systems.

The blocks causing the non-regularizability of the system, but not breaking the weak regularizability, deserve a special attention. Particularly, the extension of the regularizability to its weak analogue is due to these blocks, which will hereafter be called as NS blocks. In the case of row regularizable systems they generate $k_\epsilon - k_q$ nonproper indices, while the column regularizable systems may possess $k_q - k_\epsilon$ NS q-blocks. In addition, all the η-blocks belong to the NS blocks, too. Their influence upon the dynamics of the system (1) is described below.

First, it should be noted that the maximal number of assignable poles (with multiplicities included) of a weakly regularizable system cannot be increased by its NS blocks.

Second, the number of poles that can freely be assigned to weakly regularizable systems can be different from the similar number in regularizable systems, see Remark 2.4. In the row regularizable systems this number is described by the term $\tilde{\psi}'(w)$ in (25). Since $\deg \tilde{\psi}'(w) \leq \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i$, it follows that NS ϵ-blocks may lead to the cancellation of all such poles (in the case when $\deg \tilde{\psi}'(w) = 0$). The same inequality, inspired by (48), holds in the column regularizable systems, caused by NS q-blocks.

The last remark concerns the quantities $\alpha_i(s), q_i, p_i$ in the column regularizable systems. Particularly, the conditions (36), (37) imply that only $k_l - k_q + k_\epsilon$ smallest $\alpha_i(s)$ and $k_\epsilon + k_p$ smallest indices of $\{q_i\}_{i=1}^{k_q} \cup \{p_i\}_{i=1}^{k_p}$ are not changed by state feedback (2), which is different from the analogue conclusions for regularizable systems (see Remark 2.4) and row regularizable systems (see Theorem 4.2).

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REFERENCES

[1] R. Bacos, J. E. R. Cury, and J. J. Loiseau: Control of a class of constrained implicit systems. In: Proc. IFAC Symposium on System Structure and Control, Ancona 2010.

[2] J. Y. Ishihara and M. H. Terra: Impulse controllability and observability of rectangular descriptor systems. IEEE Trans. Automat. Control *46* (2001), 6, 991–994.

[3] T. Kaczorek: Infinite eigenvalue assignment by an output feedback for singular systems. Internat. J. Appl. Math. Comput. Sci. *14* (2004), 1, 19–23.

[4] T. Kailath: Linear systems. Englewood Cliffs, Prentice Hall, NJ 1980.

[5] T. Korotka, J. J. Loiseau, and P. Zagalak: A note on pole assignment in linear singular systems. In: Proc. 11th International Ph.D. Workshop on Systems and Control, Veszprem 2010.

[6] V. Kučera and P. Zagalak: Fundamental theorem of state feedback for singular systems. Automatica *24* (1988), 5, 653–658.

[7] L. Libeau: Sur l'utilisation des divoïdes pour la commande des systèmes à événements discrets. Ph.D. Thesis, Ecole Centrale de Nantes 1996.

[8] G. P. Liu and R. J. Patton: Eigenstructure Assignment for Control System Design. John Wiley and Sons, New York 1998.

[9] J. J. Loiseau, K. Özçaldiran, M. Malabre, and N. Karcanias: Feedback canonical forms of singular systems. Kybernetika *27* (1991), 4, 289–305.

[10] J. J. Loiseau and P. Zagalak: On pole structure assignment in linear systems. Internat. J. Control *82* (2009), 7, 1179–1192.

[11] K. Özçaldiran and F. L. Lewis: On the regularizability of singular systems. IEEE Trans. Automat. Control *35* (1990), 10, 1156–1160.

[12] R. R. Rosenbrock: State–space and Multivariable Theory. Nelson, London 1970.

[13] A. I. G. Vardulakis, D. N. J. Limebeer, N. Karcanias: Structure and Smith–McMillan form of a rational matrix at infinity. Internat. J. Control *35* (1982), 4, 701–725.

[14] I. Zaballa: Interlacing inequalities and control theory. Lin. Algebra Appl. *101* (1988), 9–31.

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