VYSOKÁ ŠKOLA STROJNÍ A TEXTILNÍ V LIBERCI
PEDAGOGICKÁ FAKULTA

Miroslav Brzezina

# KAPACITY, HARMONICKÉ MORFIZMY A VĚTY WIENEROVA TYPU V TEORII POTENCIÁLU

(Habilitační práce)

Liberec, 1993

Předkládaná habilitační práce je souborem 10 prací věnovaných některým problémům z teorie potenciálu (viz [9], [10], [12] – [19] v seznamu literatury). Práce vznikly v letech 1988–1993, převážně během výzkumného pobytu na matematickém ústavu Friedrich-Alexandrovy univerzity v Erlangen, na katedře Prof. Dr. Dr.h.c. Heinze Bauera. O většině výsledků předkládané práce jsem přednesl referáty na seminářích prof. Bauera v Erlangen a prof. Krále v Praze a na ICPT 91 v Holandsku.

Za financování pobytu ve Spolkové republice Německo patří poděkování Gottlieb Daimlerově a Karl Benzově nadaci a přírodovědecké fakultě Ostravské univerzity. Prof. Dr. Dr.h.c. Heinzovi Bauerovi a Priv. Doz. Dr. Nielsovi Jacobovi děkuji za cenné rady a náměty k práci a za vytvoření vynikajících pracovních podmínek.

Za všestrannou pomoc a podporu, rady, náměty a připomínky, které provázely vznik většiny mých prací, děkuji Prof. RNDr. Ivanu Netukovi, DrSc.

Liberec, listopad 1993

# Obsah

# Úvod

Teorie parciálních diferenciálních rovnic má široké užití nejen ve fyzice, ale i v technické praxi. Teorie potenciálu představuje v dnešní době účinný matematický aparát ke studiu parciálních diferenciálních operátorů především eliptického a parabolického typu.

Pro začátek uvažujme Laplaceův operátor v $\mathbb{R}^n$, $n \geq 3$, tj. diferenciální operátor tvaru

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Klasickou teorii potenciálu lze chápat jako teorii Laplaceova operátoru.

Předpokládejme, že $U \subset \mathbb{R}^n$ je otevřená, relativně kompaktní množina a $f$ spojitá funkce na hranici $\partial U$ ($f \in \mathcal{C}(\partial U)$) množiny U. *Klasická Dirichletova úloha*, nazývaná často klasická první okrajová úloha, spočívá v nalezení funkce $u$ spojité na uzávěru množiny $U$ ($u \in \mathcal{C}(\overline{U})$) takové, že na $U$ platí $\Delta u = 0$ a restrikce $u|_{\partial U}$ funkce $u$ na $\partial U$ splývá s $f$.

Je-li $U$ otevřená jednotková koule v $\mathbb{R}^n$ bez středu, nelze očekávat, že má klasická Dirichletova úloha řešení pro každou spojitou funkci na $\partial U$. Podobně pro tzv. Lebesgueův hrot, viz např. [24], s. 175, nemá tato úloha řešení pro každou spojitou hraniční funkci .

Vzniká tedy potřeba vhodným způsobem zobecnit formulaci a pojem řešení Dirichletovy úlohy. Pro otevřenou relativně kompaktní množinu $U \subset \mathbb{R}^n$ označme

$$\mathcal{H}(U) := \{h \in \mathcal{C}^2(U);\ \Delta h = 0 \ \text{ na } \ U\};$$

prvky $\mathcal{H}(U)$ nazýváme harmonickými funkcemi na $U$. *Zobecněná Dirichletova úloha* pro $U$ a $f \in \mathcal{C}(\partial U)$ spočívá v nalezení funkce $H_f^U$, harmonické na $U$, přičemž přiřazení $f \mapsto H_f^U$ je nezáporné a lineární a přitom $H_f^U$ splývá s řešením klasické Dirichletovy úlohy, pokud toto řešení pro $f$ existuje. Takové přiřazení se zpravidla považuje za *řešení zobecněné Dirichletovy úlohy.*

Nejvíce propracovaná je tzv. PWB (Perron, Wiener, Brelot) metoda založená na pojmu hyperharmonické funkce. Zdola polospojitá funkce $v > -\infty$ na otevřené množině $U \subset \mathbb{R}^n$ se nazývá *hyperharmonická*, jestliže pro každou uzavřenou kouli $K_r(z)$ o středu $z$ a poloměru $r > 0$, $K_r(z) \subset U$, platí tzv. podmínka nadprůměru, tj.

$$\frac{1}{\omega_n r^{n-1}} \int_{\partial K_r(z)} u(y)\, \sigma(dy) \leq u(z);$$

zde $\omega_n$ označuje povrch jednotkové koule v $\mathbb{R}^n$ a $\sigma$ povrchovou míru na $\partial K_r(z)$.

Pro relativně kompaktní otevřenou množinu $U \subset \mathbb{R}^n$ a $f \in \mathcal{C}(\partial U)$ označme $\mathcal{U}_f^U$ množinu všech zdola omezených hyperharmonických funkcí na $U$, jejichž liminf na hranici $\partial U$ majorizuje $f$. Funkce $H_f^U := \inf \mathcal{U}_f^U$ se nazývá *PWB-řešení* zobecněné Dirichletovy úlohy. Je známo, viz např. [24], s. 157, že $f \mapsto H_f^U$ je nezáporné lineární zobrazení vektorového prostoru $\mathcal{C}(\partial U)$ do vektorového prostoru $\mathcal{H}(U)$. Dále je známo, že pro funkci $g \in \mathcal{C}(\partial U)$, pro kterou existuje klasické řešení Dirichletovy úlohy, je $H_g^U(x) \to g(z)$ pro $x \to z$, kdykoliv $z \in \partial U$. Je tedy PWB-řešení řešením zobecněné Dirichletovy úlohy.

Hraniční chování funkce $H_f^U$ je obecně složité. Je-li $z \in \partial U$ a pro každou funkci $f \in \mathcal{C}(\partial U)$ je $H_f^U(x) \to f(z)$ pro $x \to z$, nazývá se $z$ *regulární bod*. Množina regulárních bodů se značí $U_r$ a často se nazývá *regulární hranicí* $U$.

Nechť $\tau$ je nejhrubší ze všech topologií v $\mathbb{R}^n$, pro kterou jsou všechny hyperharmonické funkce na $\mathbb{R}^n$ spojité. Pro $A \subset \mathbb{R}^n$ označme symbolem $b(A)$ množinu všech bodů v $\mathbb{R}^n$, které jsou $\tau$-hromadnými body pro $A$. Množina $b(A)$ se nazývá *bází množiny* $A$ a topologie $\tau$ *jemnou topologií*. Jemná topologie úzce souvisí s regulární hranicí. Platí totiž následující rovnost, srov. [6]:

$$U_r = b(\complement U) \cap \overline{U},$$

($\complement$ je symbol pro doplněk množiny). Lze tedy bázi množiny chápat jako zobecnění pojmu regulární hranice na libovolné množiny.

Pro další potřeby označme pro relativně kompaktní, otevřenou množinu $U \subset \mathbb{R}^n$

$$H(U) := \{h \in \mathcal{C}(\overline{U});\ h|_U \in \mathcal{H}(U)\}.$$

Pro $z \in \overline{U}$ označme symbolem $\mathcal{M}_z$ množinu všech nezáporných Radonových měr $\mu$ na $\overline{U}$, pro něž je $\mu(f) = f(z)$ pro všechny funkce $f \in H(U)$. Zřejmě je *Diracova míra* $\varepsilon_z$ soustředěná v bodě $z$ prvkem $\mathcal{M}_z$. Množina

$$Ch_{H(U)}\overline{U} := \{z \in \overline{U};\ \mathcal{M}_z = \{\varepsilon_z\}\}$$

se nazývá *Choquetova hranice* $\overline{U}$ vzhledem k $H(U)$. Choquetova hranice má úzký vztah k regulární hranici, platí totiž rovnost $Ch_{H(U)}\overline{U} = U_r$.

Problémem charakterizace množiny regulárních bodů se zabýval N.Wiener. Než uvedeme *Wienerovo kritérium regularity*, budeme se zabývat pojmem Newtonovy kapacity. Pro $A \subset \mathbb{R}^n$ nechť $\mathcal{M}^+(A)$ označuje množinu všech nezáporných Radonových měr s kompaktním nosičem v $A$. *Newtonovo jádro* $N$ je definováno rovností

$(n \geq 3)$:

$$N : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty] : N(x,y) := \begin{cases} \infty & , x = y, \\ \|x-y\|^{2-n} & , x \neq y; \end{cases}$$

zde $\|\cdot\|$ označuje eukleidovskou normu v $\mathbb{R}^n$. *Newtonova kapacita* kompaktní množiny $L \subset \mathbb{R}^n$ je definována rovností:

$$N\text{-cap}(L) := \sup\{\mu(L);\ \mu \in \mathcal{M}^+(L),\ N_\mu(x) := \int_{\mathbb{R}^n} N(x,y)\,\mu(dy) \leq 1 \text{ pro } x \in \mathbb{R}^n\}. \tag{1}$$

Není těžké ukázat, že pro kompaktní množiny $L_k \subset \mathbb{R}^n$, $k \in \mathbb{N}$, platí:

(i) $0 \leq N\text{-cap}(L)$;

(ii) $N\text{-cap}(L_1) \leq N\text{-cap}(L_2)$, je-li $L_1 \subset L_2$;

(iii) $N\text{-cap}(L_1 \cup L_2) + N\text{-cap}(L_1 \cap L_2) \leq N\text{-cap}(L_1) + N\text{-cap}(L_2)$;

(iii') $N\text{-cap}(L_1 \cup L_2) \leq N\text{-cap}(L_1) + N\text{-cap}(L_2)$;

(iv) je-li $L_{k+1} \subset L_k$ pro $k \in \mathbb{N}$, a $L = \bigcap_{k=1}^{\infty} L_k$, potom $N\text{-cap}(L_k) \to N\text{-cap}(L)$ pro $k \to \infty$.

(Vlastnosti (ii) se říká *monotonie*, (iii) *silná subaditivita*, (iii') *subaditivita* a vlastnost (iv) se zpravidla nazývá *spojitost shora*.)

Množinová funkce *cap* definovaná na kompaktních podmnožinách lokálně kompaktního topologického prostoru $X$ se spočetnou bází se nazývá *Choquetova kapacita na* $X$, jsou-li splněny pro *cap* předcházející čtyři podmínky. Je tedy $N$-cap Choquetova kapacita na $\mathbb{R}^n$.

V roce 1924 dokázal N.Wiener v [37] následující charakterizaci regulárních bodů:

$$z \in U_r \text{ právě tehdy, je-li } \sum_{k=1}^{\infty} 2^{k(n-2)} N\text{-cap}(\complement U \cap K_{2^{-k}}(z)) = \infty.$$

M.Brelot podal v roce 1944 v [6] charakterizaci bodů báze $b(A)$ množiny $A \subset \mathbb{R}^n$ ve tvaru Wienerovy řady:

$$z \in b(A) \text{ právě tehdy, je-li } \sum_{k=1}^{\infty} 2^{k(n-2)} N\text{-cap}^*(A \cap K_{2^{-k}}(z)) = \infty;$$

zde $N$-cap* označuje vnější Newtonovu kapacitu.

V třicátých letech jsou v pracích Petrovského a Sternberga metody známé z klasické teorie potenciálu aplikovány také na jiné parciální diferenciální operátory. Jedná se především o tepelný operátor v $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, tj. operátor tvaru

$$H := \Delta - \frac{\partial}{\partial t};$$

zde $\Delta$ označuje Laplaceův operátor v $\mathbb{R}^n$, $n \in \mathbb{N}$, body v $\mathbb{R}^{n+1}$ píšeme ve tvaru $(x, t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Především z potřeby důkladného studia PWB metody řešení Dirichletovy úlohy a jejího užití v teorii parciálních diferenciálních rovnic vzniká na konci padesátých let axiomatická teorie potenciálu. Podrobně se lze s touto teorií seznámit např. v [3] a [20]. Zde uvedeme jen několik základních definic a pojmů, které budeme dále potřebovat.

Předpokládejme, že $X$ je lokálně kompaktní topologický prostor se spočetnou bází a $\mathcal{H}$ svazek vektorových prostorů spojitých (reálných) funkcí. Každé otevřené podmnožině $V \neq \emptyset$ prostoru $X$ tedy zobrazení $\mathcal{H}$ přiřazuje vektorový prostor $\mathcal{H}(V) \subset \mathcal{C}(V)$ tak, že platí tyto podmínky:

- je-li $f \in \mathcal{H}(V)$ a $W \neq \emptyset$ otevřená podmnožina $V$, potom $f|_W \in \mathcal{H}(W)$;
- je-li $\{V_j;\ j \in I\}$ systém neprázdných otevřených množin, $V$ jeho sjednocení a $f$ je funkce na $V$, pak $f \in \mathcal{H}(V)$, pokud je $f|_{V_j} \in \mathcal{H}(V_j)$ pro všechna $j \in I$.

Prvkům prostoru $\mathcal{H}(V)$ budeme říkat *harmonické funkce na* $V$. Relativně kompaktní otevřená množina $V \subset X$ se nazývá *regulární*, je-li $\partial V \neq \emptyset$ a pro každou funkci $f \in \mathcal{C}(\partial V)$ existuje funkce $H_f^V \in \mathcal{H}(V)$ tak, že pro každé $z \in \partial V$ platí: $H_f^V(x) \to f(z)$ pro $x \to z$; dále $H_f^V \geq 0$, pokud $f \geq 0$. Zdola polospojitá funkce $v > -\infty$ na otevřené množině $U$ se nazývá *hyperharmonická na* $U$, jestliže pro každou regulární množinu $V$, $\overline{V} \subset U$, a každou funkci $f \in \mathcal{C}(\partial V)$ platí: je-li $f \leq u$ na $\partial V$, potom $H_f^V \leq u$ na $V$. Množinu všech hyperharmonických funkcí na $U$ označme ${}^*\mathcal{H}(U)$. Svazek $\mathcal{H}$ se nazývá *harmonická struktura* na $X$, platí-li následující axiómy:

- **axióm konvergence:** limita neklesající posloupnosti funkcí harmonických na otevřené množině je harmonická funkce, pokud je konečná na husté podmnožině;

- **axióm báze:** regulární množiny tvoří bázi topologie prostoru $X$;
- **oddělovací axióm:** systém $^*\mathcal{H}(X)$ odděluje lineárně body prostoru $X$, tj., jsou-li $x, y \in X$, $x \neq y$, potom existují funkce $f, g \in {}^*\mathcal{H}(X)$ tak, že $f(x)g(y) \neq f(y)g(x)$; dále pro každou otevřenou, relativně kompaktní množinu $U$ existuje $h \in \mathcal{H}(U)$ tak, že $h > 0$ na $U$.

Prostor $X$ opatřený harmonickou strukturou se nazývá *harmonický prostor*. Je známo, viz např. [30], že široká třída eliptických a parabolických diferenciálních operátorů 2. řádu definuje přirozeným způsobem harmonický prostor. Je-li $L$ takový operátor v $\mathbb{R}^n$, pak se pro otevřenou množinu $V \subset \mathbb{R}^n$ definuje $\mathcal{H}(V) = \{h \in C^2(V);\ Lh = 0 \text{ na } V\}$. Mezi standardní příklady patří Laplaceův a tepelný operátor.

Analogicky jako v případě klasické teorie potenciálu se definuje *PWB-řešení* zobecněné Dirichletovy úlohy (s využitím hyperharmonických funkcí) a pojem regulárního bodu. *Jemná topologie* je i v harmonických prostorech definována jako nejhrubší ze všech topologií, při níž jsou všechny hyperharmonické funkce na $X$ spojité. *Báze množiny* je pak definována podobně jako v klasickém případě.

Abychom mohli formulovat některé další výsledky, uvedeme zde ještě definici podstatné báze a semipolárních množin. Množina $S \subset X$ se nazývá *semipolární*, je-li $S$ spočetným sjednocením množin $T_n$, $n \in \mathbb{N}$, pro které je $b(T_n) = \emptyset$. *Podstatnou bází* množiny $A \subset X$ nazýváme množinu $\beta(A)$ všech bodů $z \in X$ takových, že pro žádné jemné okolí $V$ bodu $z$ není množina $V \cap A$ semipolární.

Mnoho výsledků známých z klasické teorie potenciálu lze snadno přenést do axiomatické teorie potenciálu. Avšak situace zde již není tak jednoduchá jako v případě Laplaceova operátoru, existují dokonce některé podstatné rozdíly. Abychom se vyhnuli zbytečným komplikacím, budeme demonstrovat tyto rozdíly na teorii potenciálu pro tepelný operátor.

Pro relativně kompaktní otevřenou množinu $U \subset \mathbb{R}^{n+1}$ označme $K(U) := \{h \in \mathcal{C}(\overline{U});\ h|_U \in \mathcal{H}(U)\}$; zde $\mathcal{H}(U)$ označuje harmonické funkce vzhledem k tepelnému operátoru, tzv. *kalorické funkce* na $U$. Choquetova hranice $Ch_{K(U)}\overline{U}$ množiny $\overline{U}$ vzhledem ke $K(U)$ je definována podobně jako v případě Laplaceova operátoru. Zatímco v klasické teorii potenciálu je $Ch_{H(U)}\overline{U} = U_r$, v tepelné teorii potenciálu tato rovnost obecně neplatí, viz [29], ale platí pouze inkluze:

$$Ch_{K(U)}\overline{U} \subset U_r.$$

Dále platí (viz [5]):

$$U_r = b(\complement U) \cap \overline{U}, \quad Ch_{K(U)}\overline{U} = \beta(\complement U) \cap \overline{U}.$$

Lze tedy podstatnou bázi chápat jako zobecnění pojmu Choquetovy hranice na libovolné množiny. Poznamenejme, že v klasické teorii potenciálu vždy platí rovnost $b = \beta$, obecně ale platí pouze inkluze ($A \subset X$): $\beta(A) \subset b(A)$. I v tepelné teorii potenciálu bylo po značném úsilí matematiků dokázáno kritérium regularity Wienerova typu, viz [21]. Pro $z = (x,t)$, $z' = (x',t') \in \mathbb{R}^{n+1}$ položme

$$W(x,t;x',t') := \begin{cases} 0 & \text{, je-li } t \leq t', \\ \frac{1}{(4\pi(t-t'))^{\frac{n}{2}}} \exp(-\frac{\|x-x'\|^2}{4(t-t')}) & \text{, je-li } t > t'; \end{cases}$$

tzv. *Weierstrassovo (tepelné) jádro*. Příslušnou $W$-kapacitu označme ${}^h\text{cap}$. *Tepelná koule* o středu $z = (x,t) \in \mathbb{R}^{n+1}$ a poloměru $r > 0$ je definována rovností:

$$B_r(z) := \overline{\{(x',t') \in \mathbb{R}^{n+1};\, W(x,t;x',t') \geq (4\pi r)^{-\frac{n}{2}}\}}.$$

(Poznamenejme, že střed tepelné koule leží na její hranici.) Wienerův test má v tomto případě tvar:

$$z \in U_r \text{ právě tehdy, je-li } \sum_{k=1}^{\infty} 2^{\frac{kn}{2}}\, {}^h\text{cap}(\complement U \cap B_{2^{-k}}(z)) = \infty.$$

Přirozené jsou otázky, zda existuje i v tomto případě analogie Wienerova kritéria pro bázi (viz zmíněný Brelotův výsledek z roku 1944) a zda lze nalézt test Wienerova typu pro podstatnou bázi a tedy i pro Choquetovu hranici. V pracích [10] a [18] jsou tyto otázky řešeny v kontextu výmetových prostorů, které (zhruba řečeno) zobecňují pojem harmonického prostoru a umožňují metodami teorie potenciálu zkoumat i některé nelokální (pseudodiferenciální) operátory. (Definice je uvedena na s. 11, k podrobnému studiu odkazujeme na monografii [5].) V souvislosti s Wienerovými testy pro podstatnou bázi je zaveden pojem $\alpha$-kapacity a jsou studovány její vlastnosti a vztah k tzv. spojité kapacitě, kterou pro tepelný operátor zavedl v roce 1967 G.Anger v [1].

V roce 1930 dokázal F.Riesz následující tvrzení:

> Je-li $\mu$ Radonova míra na $\mathbb{R}^n$, $n \geq 3$, a není-li $N_\mu$ identicky rovna $+\infty$, potom je $N_\mu$ hyperharmonická funkce na $\mathbb{R}^n$ a největší harmonická minoranta pro $N_\mu$ je nulová funkce.

Platí také obrácení tohoto tvrzení, totiž:

Je-li $p$ nezáporná hyperharmonická funkce na $\mathbb{R}^n$, $n \geq 3$, která není identicky rovna $+\infty$ a jejíž největší harmonická minoranta je nulová funkce, potom existuje Radonova míra $\mu$ na $\mathbb{R}^n$ tak, že $p = N_\mu$.

Tato tvrzení byla v axiomatické teorii potenciálu vzata za základ definice potenciálu. Nezáporná hyperharmonická funkce, která je konečná na husté podmnožině $X$ a jejíž největší harmonická minoranta je nulová funkce, se nazývá *potenciál* na $X$. Rieszova věta tedy říká, že každý "abstraktní" potenciál $p$ lze v klasické teorii potenciálu vyjádřit ve tvaru $p = N_\mu$ pro vhodnou Radonovu míru $\mu$. Analogie Rieszovy věty platí i v tepelné teorii potenciálu, viz [34]. Obdobné tvrzení pro *Kolmogorovův operátor*, tj. operátor

$$K := \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} \quad v \quad \mathbb{R}^3,$$

dokázáno v [11].

V dalších odstavcích uvedeme některé hlavní výsledky habilitační práce.

Protože je většina výsledků formulována v kontextu výmetového prostoru, zavedeme nejprve tento pojem.

Nechť $X$ je lokálně kompaktní (Hausdorffův) topologický prostor se spočetnou bází. Označme $\mathcal{B}(X)$ systém všech borelovských numerických funkcí na $X$ a $\mathcal{C}(X)$ prostor spojitých funkcí na $X$. Nechť $\mathcal{F} \subset \mathcal{B}(X)$. Označme

$$\mathcal{S}(\mathcal{F}) := \{\sup_{n\in\mathbb{N}} f_n;\ (f_n)_{n=1}^{\infty} \text{ neklesající posloupnost v } \mathcal{F}\}.$$

Říkáme, že $\mathcal{F}$ je *$\sigma$-stabilní*, je-li $\mathcal{S}(\mathcal{F}) = \mathcal{F}$.

Systém $\mathcal{F}$ se nazývá *lineárně oddělující*, jestliže pro každou dvojici bodů $x$, $y \in X$, $x \neq y$ a $\lambda \in \mathbb{R}$, $\lambda \geq 0$, existuje $f \in \mathcal{F}$ tak, že $f(x) \neq \lambda f(y)$.

Konvexní kužel $\mathcal{S} \subset \mathcal{C}(X)$ se nazývá *kužel funkcí*, jestliže $\mathcal{S}$ splňuje následující podmínky:

(i) existuje $s_0 \in \mathcal{S}$ tak, že $s_0 > 0$ na $X$;

(ii) nezáporné funkce z $\mathcal{S}$ tvoří lineárně oddělující systém;

(iii) pro každou $f \in \mathcal{S}$ existuje nezáporná funkce $g \in \mathcal{S}$ taková, že pro libovolné $\varepsilon > 0$ existuje kompaktní množina $K \subset X$ tak, že $|f(x)| \leq \varepsilon g(x)$ pro $x \in \complement K$.

Nechť $\mathcal{W}$ je konvexní kužel nezáporných, zdola polospojitých funkcí na $X$. Nejhrubší topologii na $X$, která je jemnější než původní topologie a pro kterou jsou všechny funkce z $\mathcal{W}$ spojité, nazýváme *jemnou topologií*. Topologické pojmy, které se vztahují k jemné topologii, budeme označovat přívlastkem "'jemný". Jemný uzávěr množiny $A \subset X$ budeme označovat symbolem $\overline{A}^f$.

Pro numerickou funkci $f : Y \to [-\infty, \infty]$ na topologickém prostoru $Y$ označuje $\hat{f}$ největší zdola polospojitou minorantu $f$, tj. funkci, definovanou rovností

$$\hat{f}(x) = \liminf_{y \to x} f(y), \quad x \in X.$$

Systém všech nezáporných Radonových měr na prostoru $X$ budeme označovat $\mathcal{M}^+$.

Nechť $X$ je lokálně kompaktní (Hausdorffův) topologický prostor se spočetnou bází a $\mathcal{W}$ konvexní kužel zdola polospojitých funkcí na $X$. Dvojice $(X, \mathcal{W})$ se nazývá **výmetový prostor**, jestliže platí:

(i) kužel $\mathcal{W}$ je $\sigma$-stabilní;

(ii) je-li $\mathcal{V} \subset \mathcal{W}$, potom $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$; zde $\widehat{\ }^f$ označuje zdola polospojitou regularizaci vzhledem k jemné topologii;

(iii) pro $u, v_1, v_2 \in \mathcal{W}$, $u \leq v_1 + v_2$, existují funkce $u_1, u_2 \in \mathcal{W}$ tak, že $u = u_1 + u_2$, $u_1 \leq v_1$ a $u_2 \leq v_2$;

(iv) existuje kužel funkcí $\mathcal{P}$ tvořený nezápornými spojitými funkcemi takový, že $\mathcal{W} = \mathcal{S}(\mathcal{P})$.

Jestliže $X$ je harmonický prostor, v němž kužel $\mathcal{W}$ nezáporných hyperharmonických funkcí lineárně odděluje body, potom $(X, \mathcal{W})$ je výmetový prostor. Za kužel $\mathcal{P}$ z bodu (iv) lze vzít systém spojitých potenciálů na $X$. (Poznamenejme, že teorie výmetových prostorů umožňuje také např. studium excesivních funkcí příslušných semigrupě jader či zahrnuje diskrétní teorii potenciálu.)

Připomeňme, že množina $P \subset X$ se nazývá *polární*, existuje-li $u \in \mathcal{W}$ tak, že $u$ je konečná na husté podmnožině $X$ a pro všechna $x \in P$ je $u(x) = \infty$.

V práci [12] jsou zkonstruovány některé nové příklady výmetových prostorů generovaných pseudodiferenciálními operátory, speciálně je zde ukázáno, že tzv. *tepelný operátor řádu* $\alpha$, $\alpha \in$, tj.

$$\frac{\partial}{\partial t} - (-\Delta)^\alpha,$$

generuje výmetový prostor (zde $(-\Delta)^\alpha$ označuje $\alpha$-zlomkovou mocninu $-\Delta$ v $\mathbb{R}^n$.

## Harmonické morfizmy

Uvažujme nyní klasickou teorii potenciálu v $\mathbb{R}^n$, $n \geq 3$. Nechť $R$ je ortogonální matice typu $n \times n$, $k \geq 0$, $c \in \mathbb{R}^n$. Je-li $u : \mathbb{R}^n \to \mathbb{R}$ harmonická funkce, je zřejmě i funkce

$$(Su)(x) := u(kR \cdot x + c) \tag{2}$$

harmonická na $\mathbb{R}^n$.

Nechť dále $y \in \mathbb{R}^n$ a $\rho > 0$. Transformaci

$$(Ku)(x) := \frac{\rho^{n-2}}{\|x-y\|^{n-2}} u\left(y + \frac{\rho^2}{\|x-y\|^2}(x-y)\right), \quad x \in \mathbb{R}^n \setminus \{y\}, \tag{3}$$

budeme nazývat *Kelvinovou transformací s pólem v bodě y*. V [24], s. 36, je ukázáno, že Kelvinova transformace zachovává harmonicitu, tj., je-li $u$ harmonická na $\mathbb{R}^n \setminus \{y\}$, je funkce $Ku$ harmonická na téže množině. Složením transformací typu (2) a (3) dostaneme opět transformaci, která převádí harmonické funkce na jisté otevřené množině na harmonické funkce na jiné otevřené množině. Všechny tyto transformace jsou tvaru ($U$, $V \subset \mathbb{R}^n$ otevřené, $\varphi : U \to \mathbb{R}$, $\varphi > 0$, $\Psi : U \to V$ bijekce $U$ na $V$ a $u : V \to \mathbb{R}$):

$$(Tu)(x) := \varphi(x)u(\Psi(x)). \tag{4}$$

Přirozená je otázka, zda existují transformace tvaru (4), které převádějí harmonické funkce na harmonické a které nevzniknou složením transformací typu (2) a (3). Odpověď na tento problém je uveden v knize O.D.Kelloga, viz [28], s. 235. Je zde uvedeno, že složením transformací typu (2) a (3) obdržíme všechny transformace tvaru (4).

V případě rovnice vedení tepla je situace poněkud komplikovanější. Je-li $u$ kalorická funkce, je zřejmě i funkce ($\lambda$, $\mu \in \mathbb{R}$, $\lambda > 0$, $R$ ortogonální matice typu $n \times n$, $c \in \mathbb{R}^n$)

$$(Su)(x,t) := u(\lambda R + c, \lambda^2 t + \mu) \tag{5}$$

kalorická. Roli Kelvinovy transformace zde zaujímá Appellova transformace, tj. transformace tvaru

$$(Au)(x,t) := |t|^{-n/2} \exp\left(\frac{\|x\|^2}{4t}\right) u\left(\frac{x}{t}, \frac{1}{t}\right) \tag{6}$$

definovaná na $\mathbb{R}^{n+1}$ mimo rovinu $t = 0$; srov. [2].

H.Leutwiler v [31] zavádí pojem kalorického morfizmu. Uveďme zde jeho definici. Nechť $U$, $V \subset \mathbb{R}^{n+1}$, $n \geq 1$ jsou otevřené množiny, $\varphi : U \to \mathbb{R}$, $\varphi > 0$, a nechť $\Psi : U \to V$ je bijekce $U$ na $V$. Nechť dále $u : V \to \mathbb{R}$. Zobrazení ($(x,t) \in U$)

$$(Tu)(x,t) := \varphi(x,t)u(\Psi(x,t)) \tag{7}$$

se nazývá *kalorický morfizmus na $U$*, jestliže platí: je-li funkce $u$ kalorická na $V$, je funkce $Tu$ kalorická na $U$.

Leutwilerův výsledek lze nyní snadno formulovat: *každý kalorický morfizmus je složením morfizmů typu (5) a (6).*

Práce [17] se zabývá problematikou $L$-harmonických morfizmů pro Kolmogorovův operátor (definice je uvedena v [17], s. 3). Je zde nalezena transformace, která v tomto případě zaujímá roli Appellovy transformace. Dále je zde podána úplná charakterizace $L$-harmonických morfizmů na dané otevřené množině (viz Theorem na s. 3 v [17]).

## Kapacity

Práce [13] a [16] se zabývají problematikou kapacit. V úvodní části jsme zavedli pojmy Newtonova a Choquetova kapacita. Přirozeným zobecněním Newtonovy kapacity docházíme k pojmu **K**-kapacity.

Nechť $X$ je lokálně kompaktní Hausdorffův topologický prostor se spočetnou bází. Zdola polospojitou funkci $\mathbf{K} : X \times X \to [0,\infty]$ nazýváme *jádrem* na $X$. Pro míru $\mu \in \mathcal{M}^+$ definujeme **K**-*potenciál* míry $\mu$ rovností

$$\mathbf{K}_\mu(x) := \int_X \mathbf{K}(x,y)\mu(dy)\,, \quad x \in X.$$

Množinovou funkci $c : \mathcal{K} \to [0,\infty]$ definovanou na systému $\mathcal{K}$ všech kompaktních podmnožin $X$ vztahem

$$c(L) := \sup\{\mu(L);\ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \leq 1 \text{ na } X\}$$

budeme nazývat **K-kapacitou** (podrobněji: **K**-*kapacitou odvozenou od jádra* **K**).

Obecně není **K**-kapacita Choquetovou kapacitou. Příslušný příklad je uveden v [16], s. 91. Snadno lze ukázat, že každá **K**-kapacita je monotónní, subaditivní a shora spojitou funkcí na $\mathcal{K}$, viz např. [8]. Tedy, **K**-kapacita je Choquetovou kapacitou právě tehdy, je-li silně subaditivní.

Pomocí následujících principů lze odpovědět na otázku, za jakých podmínek kladených na jádro $\mathbf{K}$ je $\mathbf{K}$-kapacita Choquetovou kapacitou.

Nechť $\mathbf{K}$ je jádro na $X$. Říkáme, že jádro $\mathbf{K}$ splňuje **spojitý dominační princip**, jestliže platí:

jsou-li míry $\nu$, $\mu \in \mathcal{M}^+$ takové, že $\mathbf{K}_\nu$, $\mathbf{K}_\mu$ jsou spojité a omezené $\mathbf{K}$-potenciály na $X$ a $\mathbf{K}_\nu \leq \mathbf{K}_\mu$ na supp $\nu$, potom $\mathbf{K}_\nu \leq \mathbf{K}_\mu$ na $X$.

Říkáme, že jádro $\mathbf{K}$ splňuje **spojitý princip rovnovážného rozložení**, jestliže platí:

je-li $L \subset X$ kompaktní a $G$ otevřená, relativně kompaktní podmnožina $X$, $L \subset G$, potom existuje míra $\mu \in \mathcal{M}^+(G)$ taková, že $\mathbf{K}_\mu \leq 1$ a spojitý na $X$ a $\mathbf{K}_\mu = 1$ na okolí $L$.

K jádru $\mathbf{K}$ na $X$ definujeme rovností $\tilde{\mathbf{K}}(x, y) := \mathbf{K}(y, x)$, $x, y \in X$ *adjungované jádro* $\tilde{\mathbf{K}}$.

Hlavním výsledkem práce [13] je následující věta (viz Theorem 1, s. 92, v [13]). Při důkazu této věty se podstatným způsobem využívá následující lemma, které umožňuje definovat $\mathbf{K}$-kapacitu jiným způsobem.

**Lemma.** *Nechť jádra* $\mathbf{K}$ *a* $\tilde{\mathbf{K}}$ *splňují spojitý princip rovnovážného rozložení,* $L \subset X$ *je kompaktní množina a* $c$, *resp.* $\tilde{c}$, *je* $\mathbf{K}$*-kapacita, resp.* $\tilde{\mathbf{K}}$*-kapacita, odvozená od jádra* $\mathbf{K}$, *resp.* $\tilde{\mathbf{K}}$, *na* $X$. *Potom platí:*

$$c(L) = \inf\{\mu(X);\ \mu \in \mathcal{M}^+,\ \mathbf{K}_\mu \geq 1 \text{ na okolí } L,\ \mathbf{K}_\mu \text{ je spojitý } \mathbf{K}\text{-potenciál na } X\};$$

$$\tilde{c}(L) = \inf\{\mu(X);\ \mu \in \mathcal{M}^+,\ \tilde{\mathbf{K}}_\mu \geq 1 \text{ na okolí } L,\ \tilde{\mathbf{K}}_\mu \text{ je spojitý } \tilde{\mathbf{K}}\text{-potenciál na } X\};$$

$$c(L) = \tilde{c}(L).$$

**Věta.** *Nechť jádra* $\mathbf{K}$ *a* $\tilde{\mathbf{K}}$ *splňují spojitý princip rovnovážného rozložení a jádro* $\mathbf{K}$ *navíc spojitý dominační princip. Potom je* $\mathbf{K}$*-kapacita Choquetovou kapacitou na* $X$.

Uvedená věta zobecňuje výsledky M. Brelota z [8], které jsou aplikovatelné na jádra (fundamentální řešení) odvozená především od parciálních diferenciálních operátorů 2. řádu eliptického typu, na obecnější jádra, např. na tepelná jádra, pro která principy z [8] neplatí.

Přirozená je otázka, zda vůbec jádra, která se v teorii potenciálu vyskytují, výše uvedené principy splňují. Odpověď lze nalézt v [13], s. 91; zhruba řečeno, jádra uvedené principy splňují, pokud lze abstraktní potenciály reprezentovat pomocí jádra **K** ve smyslu Rieszovy věty z úvodu.

Další z výsledků [13] se zabývají vztahem polárních množin a množin nulové kapacity (viz Theorem 2, s. 97, v [13]). Abychom nezacházeli příliš do podrobností, budeme tyto výsledky demonstrovat na dvou příkladech.

Pro $z = (x, y, t) \in \mathbb{R}^3$ označme

$$\overline{E}(z) := \begin{cases} 0 & \text{, je-li } t \leq 0, \\ \frac{\sqrt{3}}{2\pi t^2} \exp\{-\frac{x^2}{t} + \frac{3x(y+tx)}{t^2} - \frac{3(y+tx)^2}{t^3}\} & \text{, je-li } t > 0. \end{cases}$$

*Kolmogorovovo jádro* $E : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty]$ je definováno rovností ($z = (x, y, t)$, $z_0 = (x_0, y_0, t_0) \in \mathbb{R}^3$):

$$E(z; z_0) := \overline{E}(x - x_0, y - y_0 + x_0(t - t_0), t - t_0).$$

Označme ${}^{k}\mathrm{cap}$, resp. ${}^{\tilde{k}}\mathrm{cap}$, $E$-kapacitu, resp. $\tilde{E}$-kapacitu, na $\mathbb{R}^3$ a nechť $\mathrm{cap}^*$ označuje vnější kapacitu odvozenou od cap, viz s. 20.

**Věta.** *Je-li $P \subset \mathbb{R}^3$, potom jsou následující podmínky ekvivalentní:*

*(i) $P$ je polární pro Kolmogorovův operátor;*

*(ii) ${}^{k}\mathrm{cap}^*(P) = 0$;*

*(iii) $P$ je polární pro adjungovaný Kolmogorovův operátor;*

*(iv) ${}^{\tilde{k}}\mathrm{cap}^*(P) = 0$.*

Jako další příklad uvažujme tepelnou teorii potenciálu řádu $\alpha$. *Tepelné jádro řádu* $\alpha$ je definováno rovností ($(x, t)$, $(y, s) \in \mathbb{R}^{n+1}$):

$$W^{(\alpha)}(x, t; y, s) := \begin{cases} 0 & \text{, je-li } t \leq s, \\ (4\pi)^{-n} \int_{\mathbb{R}^n} \exp\{-(t - s)\|\xi\|^{2\alpha} + i(x - y, \xi)\}\, d\xi & \text{, je-li } t > s; \end{cases}$$

zde $(\cdot, \cdot)$ označuje skalární součin v $\mathbb{R}^n$. Symbolem ${}^{(\alpha)}\mathrm{cap}$, resp. ${}^{(\tilde{\alpha})}\mathrm{cap}$, označme $W^{(\alpha)}$-kapacitu, resp. $\tilde{W}^{(\alpha)}$-kapacitu, na $\mathbb{R}^{n+1}$, tzv. *tepelnou, resp. adjungovanou tepelnou, kapacitu řádu $\alpha$.*

**Věta.** *Je-li $P \subset \mathbb{R}^{n+1}$, potom jsou následující podmínky ekvivalentní:*

*(i) P je polární pro tepelný operátor řádu* $\alpha$;

*(ii)* ${}^{(\alpha)}cap^*(P) = 0$;

*(iii) P je polární pro adjungovaný tepelný operátor řádu* $\alpha$;

*(iv)* ${}^{(\alpha)\bar{}}cap^*(P) = 0$.

Uvažujme nyní **K**-kapacity na $\mathbb{R}$. Pro $\alpha > 0$ označme:

$$K^\alpha(t,s) := \begin{cases} |t-s|^{-\alpha}, & \text{je-li } t \neq s, \\ \infty, & \text{je-li } t = s, \end{cases}$$

tzv. *Rieszovo jádro řádu* $\alpha$. Dále definujeme ${}^+K^\alpha(t,s) := K^\alpha(t,s) \cdot 1_{]0,\infty[}(t-s)$ a ${}^-K^\alpha(t,s) := {}^+K^\alpha(s,t)$, $t$, $s \in \mathbb{R}$. Zřejmě jsou funkce $K^\alpha$, ${}^+K^\alpha$ a ${}^-K^\alpha$ jádra na $\mathbb{R}$. Odpovídající kapacity označme po řadě symboly ${}^\alpha$cap, ${}^\alpha_+$cap a ${}^\alpha_-$cap. Kapacita ${}^\alpha$cap se zpravidla nazývá *Rieszova kapacita řádu* $\alpha$.

R.Kaufman a J.M.Wu v [27] ukazují, že systémy množin nulové kapacity pro všechny tři výše uvedené kapacity splývají v případě $\alpha = \frac{1}{2}$. Jejich důkaz je založen na vztahu polárních a adjungovaných polárních množin pro tepelnou teorii potenciálu. Kaufman a Wu se dále ptají, zda uvedené tvrzení platí i pro jiná $\alpha$. Odpověď na tento problém v případě $\alpha \in [\frac{1}{2}, 1]$ dává následující věta (hlavní výsledek práce [16]):

**Věta.** *Nechť* $\alpha \in [1/2, 1[$ *a* $P \subset \mathbb{R}$. *Potom jsou následující podmínky ekvivalentní:*

*(i) P is* $\alpha$*-polární;*

*(ii)* ${}^\alpha cap^*(P) = 0$;

*(iii)* ${}^\alpha_+ cap^*(P) = 0$;

*(iv)* ${}^\alpha_- cap^*(P) = 0$.

Důkaz tohoto tvrzení je uveden v [16]. Poznamenejme zde, že se v důkaze podstatným způsobem využívá výsledků [13] pro tepelný operátor řádu $\alpha$ (viz věta na straně 15).

## Kapacitní interpretace Fulksovy míry

V této části práce se budeme zabývat kapacitní interpretací Fulksovy míry (definice následuje dále). Připomeňme nejprve jeden výsledek klasické teorie potenciálu, totiž Gaussovu větu:

**Věta.** *Nechť $U \subset \mathbb{R}^d$, $d \geq 2$, je otevřená množina a $u$ je harmonická funkce na $U$. Potom pro každou kouli $B_r(x)$ se středem $x \in \mathbb{R}^d$ a poloměrem $r > 0$, $\overline{B_r(x)} \subset U$, platí*

$$u(x) = \int_{\partial B_r(x)} u \, d\sigma_{x,r}; \tag{8}$$

*zde $\sigma_{x,r}$ označuje normalizovanou povrchovou míru na $\partial B_r(x)$.*

*Obráceně, je-li $u : U \rightarrow \mathbb{R}$ spojitá funkce splňující (8) pro všechna $\overline{B_r(x)} \subset U$, potom je $u$ harmonická na $U$.*

Z potenciálně-teoretického hlediska je normalizovaná povrchová míra $\sigma_{x,r}$ na $\partial B_r(x)$ výmetem Diracovy míry $\varepsilon_x$ v bodě $x$ na komplement $B_r(x)$, tj.

$$\sigma_{x,r} = \varepsilon_x^{\complement B_r(x)};$$

přesnou definici vymetené míry zde nebudeme uvádět, lze ji nalézt např. v [7], s. 138–139, nebo [3], s. 113–115.

V úvodní části jsme zavedli Newtonovu kapacitu. *Kapacitní mírou* pro kompaktní množinu $K \subset \mathbb{R}^d$, $d \geq 3$, rozumíme nezápornou Radonovu míru $\mu_K$, pro kterou je $N_{\mu_K} = \widehat{R}_1^K$; zde $\widehat{R}_1^K$ označuje výmet funkce identicky rovné jedné na kompaktní množinu $K$. Je-li $\mu_K$ kapacitní míra pro $K$, je $N\text{-cap}(K) = \mu_K(K)$.

Pro uzavřenou kouli $\overline{B} = \overline{B_r(x)}$ lze kapacitní míru snadno nalézt, totiž

$$\mu_{\overline{B}}(\overline{B}) = (d-2)\omega_d r^{d-2};$$

zde $\omega_d$ označuje povrch jednotkové koule v $\mathbb{R}^d$.

V případě teorie potenciálu pro rovnici vedení tepla je situace poněkud složitější. Abychom mohli formulovat hlavní výsledek práce [14], musíme zavést následující označení.

Nechť $\Omega(z,c)$ označuje vnitřek tepelné koule o středu $z$ a poloměru $c$ v $\mathbb{R}^{d+1}$ a $B(z,c) := \partial\Omega(z,c)$ *tepelnou sféru.* Označme symbolem $\sigma := \sigma_{B(z,c)}$ povrchovou míru na $B(z,c)$ (tj. $d$–dimensionální Hausdorffovu míru). Pro $z' = (x',t') \in \mathbb{R}^{d+1}$, definujme funkci $Q : (\mathbb{R}^d \times ]0,\infty[) \cup \{0\} \to \mathbb{R}$ rovností

$$Q(x',t') := \begin{cases} \|x'\|^2[4\|x'\|^2t'^2 + (\|x'\|^2 - 2dt')^2]^{-1/2}, & t' > 0, \\ 1, & (x',t') = 0. \end{cases}$$

Dále, pro $z = (x,t)$ definujme funkci $q_z(z') := Q(z - z')$, $z' \in \mathbb{R}^{d+1}$, na $(\mathbb{R}^d \times ]-\infty, t[) \cup \{z\}$ a položme $q_{z,c} := q_z|_{B(z,c)}$. Nezáporná Radonova míra

$$\mu_{z,c} := (4\pi c)^{-d/2} q_{z,c}\sigma_{B(z,c)}$$

na $B(z,c)$ se nazývá *Fulksova míra* pro tepelnou kouli $\Omega(z,c)$; srov. [4], s. 70–71.

Tepelné koule a Fulksova míra hrají pro rovnici vedení tepla tutéž roli jako euklidovské sféry a normalizovaná povrchová míra pro Laplaceovu rovnici. W. Fulks, viz [22], dokázal analogii Gaussovy věty v tomto případě.

**Věta.** *Nechť $U \subset \mathbb{R}^{d+1}$, $d \geq 1$, je otevřená množina a $u : U \to \mathbb{R}$ kalorická funkce na $U$, tj. $\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial t} \in \mathcal{C}(U)$, $i = 1, \ldots, d$, a $Hu = 0$ na $U$. Je-li $z \in U$, $c > 0$ a $\overline{\Omega(z,c)} \subset U$, potom $u$ splňuje na $U$ následující podmínku průměru:*

$$u(z) = \int u \, d\mu_{z,c}. \tag{9}$$

*Obráceně, splňuje-li spojitá funkce $u : U \to \mathbb{R}$ podmínku (9) pro všechny $\overline{\Omega(z,c)} \subset U$, potom je $u$ kalorická na $U$.*

Označme $^*\mu_K$ kapacitní míru kompaktní množiny $K \subset \mathbb{R}^{d+1}$ vzhledem k adjungované tepelné teorii potenciálu (definici lze nalézt v [14], s. 4). Nyní již můžeme přistoupit k formulaci výsledků práce [14]:

**Věta.** *Nechť $z \in \mathbb{R}^{d+1}$, $c > 0$ a $\Omega := \Omega(z,c)$. Potom platí následující rovnosti*

$$\mu_{z,c} = \varepsilon_z^{\complement\Omega} = (4\pi c)^{-d/2}\, {}^*\mu_{\overline{\Omega}}.$$

Poznamenejme, že důkaz první rovnosti lze nalézt v [4], jinými prostředky v [32]. V [14] je ukázáno, jak lze první rovnost dokázat pomocí výsledků N. Watsona z [36]. Důkaz druhé rovnosti lze zobecnit a dokázat v kontextu výmetových prostorů s Greenovou funkcí, splňující jisté vlastnosti (výsledek nebyl publikován).

## Spojité kapacity

K formulaci dalších výsledků potřebujeme zavést některé pojmy.

Nechť $E \subset X$ a $z \in X$. Připomeňme, srov. s. 5 a 8, že množina $E$ *je tenká v bodě* $z$, není-li $z$ jemně hromadný bod pro $E$. *Bází množiny* $E$ nazýváme množinu

$$b(E) := \{z \in X; E \text{ není tenká v } z\}.$$

Množina $S \subset X$ se nazývá *semipolární*, je-li spočetným sjednocením množin $T_n$, $n \in \mathbb{N}$, takových, že $b(T_n) = \emptyset$ pro $n \in \mathbb{N}$. (Tedy, semipolární množiny, báze množiny a tenkost množiny v daném bodě se i v případě výmetových prostorů definují stejně jako v klasické teorii potenciálu či v harmonických prostorech, srov. definice na s. 5 a 8.)

Nechť $E \subset X$ a $z \in X$. Říkáme, že množina $E$ je *semipolární v bodě* $z$, jestliže existuje jemné okolí $V$ bodu $z$ tak, že množina $E \cap V$ je semipolární. *Podstatnou bází množiny* $E$ nazýváme množinu

$$\beta(E) := \{z \in X;\ E \text{ není semipolární v } z\}.$$

Nechť $\gamma$ je Choquetova kapacita na $X$ a $K \subset X$ kompaktní množina. Vztahem

$$\alpha(K) := \gamma(\beta(K))$$

definujeme *$\alpha$-kapacitu* (podrobněji: *$\alpha$-kapacitu odvozenou od kapacity $\gamma$*) množiny $K$.

Množinová funkce $\alpha$ je monotónní a silně subaditivní, obecně však $\alpha$ není Choquetova kapacita na $X$. Následující věta, jejíž důkaz je uveden v [19], dává odpověď na otázku, kdy je $\alpha$-kapacita Choquetova kapacita na $X$.

**Věta.** *Nechť $(X, \mathcal{W})$ je výmetový prostor a $\gamma$ Choquetova kapacita na $X$, pro kterou platí:*

*kompaktní množina $K \subset X$ je polární, právě když $\gamma(K) = 0$.*

*Potom jsou následující podmínky ekvivalentní:*

*(i) $\alpha$ je Choquetova kapacita na $X$;*

*(ii) $\alpha \equiv \gamma$;*

(iii) *výmetový prostor* $(X, \mathcal{W})$ *splňuje axióm polarity, tj., semipolární množiny v* $X$ *jsou polární.*

Nechť $\mathbf{K}$ je jádro na $X$. Množinovou funkci $\sigma : \mathcal{K} \to [0, \infty]$ definovanou rovností $(L \in \mathcal{K})$

$$\sigma(L) := \sup\{\mu(X);\ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \leq 1 \text{ a spojitý } \mathbf{K}\text{-potenciál na } X\}$$

budeme nazývat *spojitou* $\mathbf{K}$*-kapacitou* na $X$.

Vztah mezi spojitou $\mathbf{K}$-kapacitou a $\alpha$-kapacitou odvozenou od jádra $\mathbf{K}$ na $X$ je vyjasněn v [19]. Dále je zde podána charakterizace borelovských semipolárních množin pomocí spojité kapacity.

V [19] jsou také uvedeny odpovědi na některé dosud neřešené problémy z knihy G. Angera [1].

## Wienerovská kritéria pro bázi a podstatnou bázi množin

Pro $z \in X$ a $r \in ]0, 1]$ označme $B^r(z)$ kompaktní podmnožinu v $X$ takovou, že

$B^r(z) \subset B^s(z)$ pro $r < s$;

$\bigcap_{0<r\leq 1} B^r(z) = \{z\}$.

Pro $\lambda \in ]0, 1[$ a $k \in \mathbb{N}$ píšeme $B_k(z, \lambda)$ místo $B^{\lambda^k}(z)$.

Nechť $\gamma$ je Choquetova kapacita na $X$. Pro $A \subset X$ definujeme *vnitřní*, resp. *vnější, Choquetovu kapacitu* rovností:

$$\gamma_*(A) := \sup\{\gamma(K);\ K \subset A,\ K \text{ kompaktní}\},$$

resp.

$$\gamma^*(A) := \inf\{\gamma_*(U);\ U \supset A,\ U \text{ otevřená }\}.$$

Pro $\alpha$-kapacitu $\alpha$ odvozenou od Choquetovy kapacity $\gamma$ na $X$ definujeme *vnitřní* $\alpha$*-kapacitu* $\alpha_*$ rovností $(A \subset X)$:

$$\alpha_*(A) := \sup\{\alpha(K);\ K \subset A,\ K \text{ kompaktní}\}.$$

K hlavním výsledkům prací [10] a [18] patří následující věty:

**Věta.** *Nechť $z \in X$, $E$ je libovolná podmnožina $X$, $\lambda \in ]0,1[$ a nechť $\gamma$ je Choquetova kapacita na $X$. Předpokládejme, že množina $\{z\}$ je tenká v bodě $z$ a že platí následující podmínka (P):*

> *Existuje posloupnost nezáporných čísel $(c_k(z,\lambda))_{k=1}^{\infty}$ tak, že následující podmínky jsou ekvivalentní, kdykoliv $F \subset X$ je kompaktní:*
>
> *$F$ je tenká v $z$;*
> $\sum_{k=1}^{\infty} c_k(z,\lambda)\gamma(F \cap B_k(z,\lambda)) < \infty$.

*Potom $E$ je tenká v bodě $z$ právě tehdy, je-li řada*

$$\sum_{k=1}^{\infty} c_k(z,\lambda)\gamma^*(E \cap B_k(z,\lambda))$$

*konvergentní.*

**Věta.** *Nechť $z \in X$, $B$ je borelovská podmnožina $X$, $\lambda \in ]0,1[$ a nechť $\gamma$ je Choquetova kapacita na $X$, pro kterou platí:*

> *je-li $A$ borelovská, relativně kompaktní množina, potom $\gamma(A) = \gamma(\overline{A}^f)$.*

*Nechť dále $\alpha_*$ označuje vnitřní $\alpha$-kapacitu odvozenou od kapacity $\gamma$. Předpokládejme, že množina $\{z\}$ je tenká v bodě $z$ a že platí podmínka (P) z předcházející věty. Potom $B$ je semipolární v bodě $z$ právě tehdy, je-li řada*

$$\sum_{k=1}^{\infty} c_k(z,\lambda)\alpha_*(B \cap B_k(z,\lambda))$$

*konvergentní.*

V případě tepelné teorie potenciálu a teorie potenciálu pro tepelný operátor řádu $\alpha$ jsou dále v [9] a [15] mimo jiné odvozeny některé nutné podmínky geometrického charakteru pro to, aby bod patřil do báze či podstatné báze množiny. Tyto výsledky zobecňují tzv. "tusk-condition" z [26] a [33].

# Literatura


[1] G.Anger, *Funktionalanalytische Betrachtungen bei Differentialgleichungen unter Verwendung von Methoden der Potentialtheorie I*, Akademie-Verlag, Berlin, 1967.

[2] P.Appell, *Sur l'équation $\partial^2 z/\partial x^2 - \partial z/\partial y = 0$ et la théorie du chaleur*, J. Math. Pures. Appl **(4)8**(1892), 187–216.

[3] H.Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Mathematics 22, Springer-Verlag, Berlin, 1966.

[4] H.Bauer, *Heat balls and Fulks measures*, Ann.Acad.Sci.Fenn.Ser.A.I.Math. **10**-(1985), 62–82.

[5] J.Bliedtner, W.Hansen, *Potential theory, An Analytic and Probabilistic Approach to Balayage*, Springer-Verlag, Berlin, 1986.

[6] M.Brelot, *Sur les ensembles effilés*, Bull. Sci. Math. **68**(1944), 12–36.

[7] M.Brelot, *Élémentes de la théorie classique du potentiel*, $2^e$ ed., Centre de Documentation Universitaire, Paris, 1961.

[8] M.Brelot, *Lectures on potential theory*, Tata institut of fundamental research, Bombay, 1967.

[9] M.Brzezina, *On the base and the essential base in parabolic potential theory*, Czechoslovak Math. J. **40**(1990), 87–103.

[10] M.Brzezina, *Wiener's test of thinness in potential theory*, Comm. Math. Univ. Carolinae **31**(1990), 227–232.

[11] M.Brzezina, *Báze, podstatné báze a Wienerovo kritérium ve výmetových prostorech*, disertace, MFF UK, Praha, 1991.

[12] M.Brzezina, *On a class of translation invariant balayage spaces*, Exposition. Math. **11**(1993), 181–184.

[13] M.Brzezina, *Kernels and Choquet capacities*, Aequationes Math. **45**(1993), 89–99.

[14] M.Brzezina, *Capacitary interpretation of the Fulks measure*, Exposition. Math. **11**(1993), 469–474.

[15] M.Brzezina, *Tusk type conditions and thinness for the parabolic operator of order $\alpha$*, Manuscripta Math. **80**(1993), 21–26.

[16] M.Brzezina, *On capacities related to the Riesz kernel*, zasláno do Aequationes Math.

[17] M.Brzezina, *Appell type transformation for the Kolmogorov equation*, Math. Nachr., v tisku.

[18] M.Brzezina, *Essential base, Choquet's boundary and Wiener's test in potential theory*, preprint.

[19] M.Brzezina, *On continuous capacities*, preprint.

[20] C.Constantinescu, A.Cornea, *Potential theory on harmonic spaces*, Springer-Verlag, Berlin, 1972.

[21] C.L.Evans, F.R.Gariepy, *Wiener's criterion for the heat equation*, Arch. Rational Mech. Anal. **78**(1982), 293–314.

[22] W.Fulks, *A mean value theorem for the heat equation*, Proc.Amer.Math.Soc. **17**(1966), 6–11.

[23] N.Garofalo, F.Segala, *Estimates of the fundamental solution and Wiener's criterion for the heat equation on the Heisenberg group*, Indiana Univ. Math. J. **39**(1990), 1155–1196.

[24] L.L.Helms, *Introduction to potential theory*, Wiley Interscience, New York, 1969.

[25] W.Hoh, N.Jacob, *On some translation invariant balayage spaces*, Comm. Math. Univ. Carolinae **32**(1991), 471–478.

[26] M.Itô, M.Nishio, *Poincaré type conditions of regularity for the parabolic operator of order $\alpha$*, Nagoya Math. J. **115**(1989), 1–22.

[27] R.Kaufman, J.M.Wu, *Parabolic potential theory*, J. Differential Equations **43**-(1982), 204–234.

[28] O.D.Kellogg, *Foundations of potential theory*, Springer-Verlag, Berlin, 1929.

[29] J.Köhn, M.Sieveking, *Reguläre und extremale Randpunkte in der Potentialtheorie*, Rev. Roumaine Math. Pures Appl. **12**(1967), 1489–1502.

[30] P.Kröger, *Harmonic spaces associated with parabolic and elliptic differential operators*, Math. Ann. **285**(1989), 393–403.

[31] H.Leutwiler, *On the Appell transformation*, In: Potential Theory, ed. J.Král et al., Plenum Press, New York, 1987, 215–222.

[32] I.Netuka, *Fine behaviour of solution of the Dirichlet problem near an irregular point*, Bull.Sci.Math. **114**(1990), 1–22.

[33] M.Nishio, *The Wiener criterion of regular points for the parabolic operator of order $\alpha$*, Nagoya Math. J. **116**(1989), 163–179.

[34] N.A.Watson, *Green functions, potentials, and the Dirichlet problem for the heat equation*, Proc. London Math. Soc. **(3)33**(1976), 251–298.

[35] N.A.Watson, *Thinness and boundary behaviour of potentials for the heat equation*, Mathematika **32**(1985), 90–95.

[36] N.A.Watson, *A convexity theorem for local mean values of subtemperatures*, Bull.London Math.Soc. **22**(1990), 245–252.

[37] N.Wiener, *The Dirichlet problem*, J. Math. Phys. **3**(1924), 127–146.

# Přílohy

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

# ON THE BASE AND THE ESSENTIAL BASE IN PARABOLIC POTENTIAL THEORY

MIROSLAV BRZEZINA, Praha

(Received March 3, 1988)

## INTRODUCTION

Several decades ago it was recognized that methods of classical potential theory can be applied to investigation of the heat equation. In particular it is possible to define the PWB-solution of the Dirichlet problem and the notion of the regular point, see [10], [2].

While the necessary and sufficient condition of the regularity of a boundary point in the form of the so called series of Wiener's type has been known in the classical case since 1924, see [29], the way to the analogous criterion in the heat case, in spite of considerable similarity with the classical case, took more than 50 years.

First sufficient conditions of the regularity of a region with continuous or differentiable boundary were proved in [23] and [21]. Necessary or sufficient conditions of the regularity were shown in [22], [17]. In 1982, Evans and Gariepy proved the heat analogue of the Wiener test, see [12]. Results from [12] generalized to parabolic equations with variable coefficients can be found in [13]. Regularity of a boundary point is very closely connected with the notion of thinness of a set at a given point, see e.g. [10]. M. Brelot proved the Wiener test of thinness in the classical potential theory in [6]. One of the main results of this paper is the test of Wiener's type of thinness in the heat potential theory, see Theorem 1.11. In our proof, results of [12] are used in an essential way. The probabilistic approach to the criterion of thinness for analytic sets is given in [25]. The paper [24] deals with a probabilistic interpretation of thinness of a set.

In modern potential theory there is a possibility of formulating a number of fundamental results in terms of the base and the essential base of a set (see Definitions 1.12 and 2.7 below). Theorem 1.13 gives necessary and sufficient conditions for a point to belong to the base of a set.

In the second part, a continuous capacity is introduced using continuous potentials, and its fundamental properties are established. Relations between capacity and continuous capacity are cleared up in Theorem 2.9 using results from [16]. Theorem 2.16 is the test of Wiener type for a point to belong to the essential base of a Borel set.

Papers [14], [11], [18], [27] deal with the so called "tusk condition" for regularity of a boundary point. Corollary 2.21 and Corollary 3.5 are "tusk conditions" for a Borel set to be semipolar at a point (see Definition 2.7) and for a point to belong to the Choquet boundary.

The results of this paper were presented with more details in [8] and announced in [9].

## 0. NOTATION

The set of all positive integers is denoted by $N$. For $n \in N$, the symbol $R^{n+1} = R^n \times R^1$ will stand for the $(n+1)$-dimensional Euclidean space. We will often write a typical point $z \in R^{n+1}$ as $z = (x, t)$, $x \in R^n$, $t \in R^1$. Let the Euclidean norm be denoted by $|\cdot|$, the set difference by $A \setminus B$, and $R^{n+1} \setminus A$ by $A^c$. For $A \subset R^{n+1}$, $\bar{A}$ denotes the closure of $A$ and $\partial A$ the boundary of $A$. For $z = (x, t) \in R^{n+1}$, let us define

$$F(z) = \begin{cases} (4\pi t)^{-n/2} \exp\left[-|x|^2/4t\right] & t > 0 \\ 0 & t \leqq 0 ; \end{cases}$$

this is the fundamental solution of the heat equation. Further, for $z \in R^{n+1}$ and $c > 0$ denote

$$B(z, c) = \{w \in R^{n+1};\ F(z - w) \geqq (4\pi c)^{-n/2}\} \cup \{z\},$$

the so called "heat ball", and for $k \in N$ put

$$B_k(z) = B(z, 2^{-k}), \quad A_k(z) = \mathrm{cl}\,(B_k(z) \setminus B_{k+1}(z)).$$

## 1. THERMAL CAPACITY, THINNESS AND BASE

For a set $E \subset R^{n+1}$, let us denote by $\mathcal{M}^+(E)$ the collection of all nonnegative Radon measures on $R^{n+1}$ with compact support in $E$; the support is denoted by spt. For $\mu \in \mathcal{M}^+(R^{n+1})$ we set

$$P_\mu(z) = \int_{R^{n+1}} F(z - w)\, \mathrm{d}\mu(w), \quad z \in R^{n+1};$$

$P_\mu$ is the heat potential of $\mu$. The heat potentials are lower semicontinuous but not continuous in general. The coarsest topology making every heat potential continuous is called the fine topology for the heat equation. Topological concepts related to the fine topology will be used with the attribute fine.

**1.1. Definition.** Let $K \subset R^{n+1}$ be a compact set. The *capacity* (in detail: the *thermal capacity*) of $K$ is defined as

$$\gamma(K) = \sup\{\mu(R^{n+1});\ \mu \in \mathcal{M}^+(K),\ P_\mu \leqq 1 \text{ in } R^{n+1}\};$$

cf. [17].

Fundamental properties of the termal capacity are summarized in the following lemma.

**1.2. Lemma.** *Let $K, K_j$, $j \in N$, be compact subsets of $R^{n+1}$. For $s > 0$ define $s \odot K = \{(sx, s^2 t); (x, t) \in K\}$. Then*

(1) $\gamma(K) < \infty$;

(2) $\gamma(K_1 \cup K_2) \leqq \gamma(K_1) + \gamma(K_2)$ *(subadditivity of* $\gamma$*)*;

(3) $K_1 \subset K_2$ *implies* $\gamma(K_1) \leqq \gamma(K_2)$ *(monotonicity of* $\gamma$*)*;

(4) $\gamma(\{z\}) = 0$ *for all* $z \in R^{n+1}$;

(5) $\gamma(s \odot K) = s^n \gamma(K)$;

(6) *if* $\{K_j\}_{j=1}^{\infty}$ *is a decreasing sequence of sets with the intersection* $K$, *i.e.* $K_j \searrow K$, *then*

$$\lim_{j \to \infty} \gamma(K_j) = \gamma(K) .$$

Proof. See [17], pp. 85, 89.

**1.3. Definion.** Let $E \subset R^{n+1}$ be an arbitrary set. Then

$$\gamma_*(E) = \sup \{\gamma(K);\ K \subset E, K \text{ compact}\}$$

is called the *inner thermal capacity of* $E$ and

$$\gamma^*(E) = \inf \{\gamma_*(G);\ G \supset E,\ G \text{ open}\}$$

the *outer thermal capacity of* $E$.

**1.4. Lemma.** *Let $E, E_j$, $j \in N$, be arbitrary subsets of $R^{n+1}$. Then*

(1) $0 \leqq \gamma_*(E) \leqq \gamma^*(E)$;

(2) $\gamma^*(E_1 \cup E_2) \leqq \gamma^*(E_1) + \gamma^*(E_2)$;

(3) $E_1 \subset E_2$ *implies* $\gamma_*(E_1) \leqq \gamma_*(E_2)$, $\gamma^*(E_1) \leqq \gamma^*(E_2)$;

(4) *if* $s > 0$, *then* $\gamma^*(s \odot E) = s^n \gamma^*(E)$;

(5) *if* $\{E_j\}_{j=1}^{\infty}$ *is an increasing sequence of sets with the union* $E$, *i.e.* $E_j \nearrow E$, *then*

$$\lim_{j \to \infty} \gamma^*(E_j) = \gamma^*(E) .$$

Proof. See [26], p. 352.

**1.5. Lemma.** *Let $t \in R^1$, let $F$ be a Borel subset of $R^n$, $K = \{(x, t) \in R^{n+1}; x \in F\}$, and let $\lambda_n$ stand for the Lebesgue measure in $R^n$. Then*

$$\gamma(K) = \lambda_n(F) .$$

Proof. See [26], p. 355.

**1.6. Remark.** It follows from the definition of the inner and outer capacities and from Lemma 1.2 (6) that $\gamma(K) = \gamma_*(K) = \gamma^*(K)$ whenever $K$ is a compact subset of $R^{n+1}$. A subset $E$ of $R^{n+1}$ is said to be $\gamma$-capacitable, if $\gamma_*(E) = \gamma^*(E)$. It can be shown, see [26], p. 352, that all Borel subsets of $R^{n+1}$ are $\gamma$-capacitable. We can extend the set function $\gamma$ which is defined for compact sets only to $\gamma$-capacitable sets $E \subset R^{n+1}$ by defining $\gamma(E) = \gamma_*(E)$. In particular, we will write $\gamma(E)$ instead of $\gamma_*(E)$ and $\gamma^*(E)$ whenever $E \subset R^{n+1}$ is capacitable.

**1.7. Definition.** The balayage of the function identically equal to 1 on a subset $E$ of $R^{n+1}$ will be denoted by $\hat{R}_1^E$. For $z = (x, t) \in R^{n+1}$ and $r > 0$, let

$$C(z, r) = \{(\chi, \tau) \in R^{n+1};\ |\chi - x| \leqq r,\ -r^2 \leqq \tau - t \leqq 0\}.$$

We say that a set $E$ is *thin* at a point $z \in R^{n+1}$ if there exists $r > 0$ such that

$$\hat{R}_1^{E \cap C(z,r)}(z) < 1.$$

It can be shown, see [10], p. 158 and p. 141, that this notion of thinness coincides with the notion usually adopted in potential theory (see e.g. [10], p. 149).

**1.7. Remark.** If $E \subset R^{n+1}$ is thin at a point $z \in R^{n+1}$ and $E' \subset E$, then $E'$ is also thin at $z$. If the set $E \subset R^{n+1} \setminus \{z\}$ is thin at a point $z$, then there exists an open set $G \subset R^{n+1}$ such that $E \subset G$ and $G$ is thin at $z$.

**1.9. Lemma.** *Let $K \subset R^{n+1}$ be a compact set. Then*

(1) $\hat{R}_1^K = 1$ *on* int $K$, *the interior of* $K$;

(2) *there exists a unique Radon measure $\tilde{\mu} \in \mathcal{M}^+(K)$ such that*

$$\hat{R}_1^K = P_{\tilde{\mu}} \quad \text{and} \quad \tilde{\mu}(R^{n+1}) = \gamma(K)$$

($\tilde{\mu}$ *is called the equilibrium measure for* $K$);

(3) *if $\tilde{\mu}$ is the equilibrium measure for $K$, then $P_\mu \leqq P_{\tilde{\mu}}$ for all $\mu \in \mathcal{M}^+(K)$ such that $P_\mu \leqq 1$ in $R^{n+1}$.*

Proof. See [17], pp. 86–88.

In [12], Evans and Gariepy proved the criterion of regulatity for the heat equation. If we use the well known relation between regularity and thinness of a set at a given point, see e.g. [17], p. 94, we obtain the following Theorem 1.10 a generalization of which to arbitrary sets is contained in Theorem 1.11.

**1.10. Theorem.** *Let $F \subset R^{n+1}$ be a closed set and $z \in R^{n+1}$. Then $F$ is thin at $z$ if and only if the series*

$$\sum_{k=1}^{\infty} 2^{nk/2}\, \gamma(F \cap A_k(z))$$

*is convergent.*

Proof. See [12], p. 295 and p. 298.

**1.11. Theorem.** *Let $E \subset R^{n+1}$ be an arbitrary set and $z \in R^{n+1}$. Then $E$ is thin at $z$ if and only if the series*

$$\sum_{k=1}^{\infty} 2^{nk/2}\, \gamma^*(E \cap A_k(z)) \tag{1.1}$$

*is convergent.*

Proof. Assume that the series in (1.1) is convergent. Choose $\varepsilon_k > 0$, $k \in N$, such that the series

$$(1.2) \qquad \sum_{k=1}^{\infty} 2^{nk/2} \varepsilon_k$$

is convergent. Let $G_k \subset R^{n+1}$, $k \in N$, be open sets such that $E \cap A_k(z) \subset G_k$ and $\gamma(G_k) \leqq \gamma(E \cap A_k(z)) + \varepsilon_k$. It follows from (1.1) and (1.2) that the series

$$(1.3) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \gamma(G_k)$$

is convergent. Let $G = (\bigcup_{k=1}^{\infty} (G_k \cap A_k(z))) \cap B_1^c(z)$. Clearly $G$ is a Borel set and $E \subset G$. We are going to show that the set $G$ is thin at $z$, which in turn implies that $E$ is thin at $z$. To this end, we shall show that there exists $r_0 > 0$ such that

$$\hat{R}^{G \cap C_{r_0}}(z) < 1 ;$$

here $C_r$ is used instead of $C(z, r)$. Let us choose $0 < \eta < 1$ and show that there exists $r_0 > 0$ such that $\hat{R}_1^{K \cap C_{r_0}}(z) < \eta$ whenever $K$ is a compact subset of $G$. Then $\hat{R}_1^{G \cap C_{r_0}}(z) < 1$ according to [10], p. 132, because $\hat{R}_1^{G \cap C_{r_0}} = \sup \{\hat{R}_1^{K \cap C_{r_0}};\ K \subset G, K \text{ compact}\}$. So let $0 < \eta < 1$, let $K$ be an arbitrary compact subset of $G$ and $r > 0$ arbitrary. Putting $D_k(r) = A_k(z) \cap K \cap C_r$, $k \in N$, $D_0 = \mathrm{cl}\,(K \cap C_r) \setminus \bigcup_{k=1}^{\infty} D_k(r))$, we get

$$(1.4) \qquad K \cap C_r \subset \bigcup_{k=0}^{\infty} D_k(r) \subset C_r \,.$$

Let $\mu \in \mathscr{M}^+(K \cap C_r)$ be the equilibrium measure for $K \cap C_r$ (see Lemma 1.9 (2)), $\nu_k$ the restriction of the measure $\mu$ to the set $D_k(r)$, $k \in N \cup \{0\}$, and $\nu'_k \in \mathscr{M}(D_k(r))$ the equilibrium measure for $D_k(r)$, $k \in N \cup \{0\}$. According to Lemma 1.9 (3), (2) $P_{\nu_k} \leqq \leqq P_{\nu'_k}$, $k \in N \cup \{0\}$, and

$$\hat{R}_1^{K \cap C_r}(w) = P_{\tilde{\mu}}(w) = \int_{K \cap C_r} F(w - v)\, \mathrm{d}\tilde{\mu}(v) \,.$$

Now (1.4) yields

$$(1.5) \qquad \hat{R}_1^{K \cap C_r}(w) \leqq \sum_{k=0}^{\infty} \int_{D_k(r)} F(w - v)\, \mathrm{d}\tilde{\mu}(v) \leqq \sum_{k=0}^{\infty} P_{\nu'_k}(w) \,.$$

Since $D_k(r) \subset A_k(z)$ for $k \in N$, we have

$$F(z - v) \leqq \left(\frac{2^k}{2\pi}\right)^{n/2}$$

for $v \in D_k(r)$. The same inequality holds for $k = 0$ because $D_0(r) \subset (\mathrm{int}\, B_1(z))^c$. Consequently

$$P_{\nu'_k}(z) \leqq \left(\frac{2^k}{2\pi}\right)^{n/2} \nu'_k(D_k(r)) \,, \quad k \in N \cup \{0\} \,,$$

and we obtain from (1.5)

$$\hat{R}_1^{K\cap C_r}(z) \leqq \left(\frac{1}{2\pi}\right)^{n/2} \sum_{k=0}^{\infty} 2^{nk/2}\, v_k'(D_k(r))\,.$$

According to Lemma 1.9 (2) $v_k'(D_k(r)) = \gamma(D_k(r))$, and so

$$\hat{R}_1^{K\cap C_r}(z) \leqq \left(\frac{1}{2\pi}\right)^{n/2} \left(\gamma(C_r) + \sum_{k=1}^{\infty} 2^{nk/2}\, \gamma(K \cap A_k(z) \cap C_r)\right).$$

Since $K \subset G$ and the inclusion

$$G \cap A_k(z) \subset G_{k-1} \cup G_k \cup G_{k+1} \cup \{z\}$$

holds for $k \in N$ (with $G_0 = \emptyset$), it follows from monotonicity and subaditivity of $\gamma$ that

$$\gamma(K \cap A_k(z) \cap C_r) \leqq \gamma(G_{k-1}) + \gamma(G_k) + \gamma(G_{k+1})\,, \quad k \in N\,.$$

Since $\gamma(C_r) \to 0$ for $r \to 0^+$ and the series (1.3) converges, we easily establish the existence of $r_0 > 0$ such that

$$\hat{R}_1^{K\cap C_{r_0}}(z) < \eta < 1$$

whenever $K$ is a compact subset of $G$. The first part of the proof is complete.

Suppose now that the set $E$ is thin at $z$. We can assume that $z \notin E$. Let $G \supset E$ be an open set thin at $z$. Let $\varepsilon_k$ be strictly positive numbers satisfying (1.2). Since $G \cap A_k(z)$ is a $K_\sigma$ – set there exist compact sets $K_k \subset G \cap A_k(z)$ such that

$$\gamma(G \cap A_k(z)) \leqq \gamma(K_k) + \varepsilon_k\,. \tag{1.6}$$

Obviously, the set $K = \bigcup_{k=1}^{\infty} K_k \cup \{z\}$ is compact and $K \subset G \cup \{z\}$. Consequently, the set $K$ is thin at $z$. According to Theorem 1.10, the series

$$\sum_{k=1}^{\infty} 2^{nk/2} \gamma(K \cap A_k(z)) \tag{1.7}$$

is convergent. From $E \subset G$, from the inequality (1.6) and from the monotonicity of $\gamma$ it follows that

$$\gamma^*(E \cap A_k(z)) \leqq \gamma(K_k) + \varepsilon_k \leqq \gamma(K \cap A_k(z)) + \varepsilon_k\,, \quad k \in N\,.$$

The last inequality, the convergence of the series in (1.7) and (1.2) imply that the series in (1.1) is convergent.

The proof of Theorem 1.10 is complete.

**1.12. Definition.** Let $E$ be an arbitrary subset of $R^{n+1}$. The set $b(E)$ of all points $z \in R^{n+1}$ such that $E$ is not thin at $z$ will be called the *base of* $E$.

**1.13. Theorem.** *For an arbitrary set* $E \subset R^{n+1}$, *the following conditions are equivalent:*

(1) $\quad z \in b(E)\,;$

(2) $\qquad \int_0^1 \gamma^*(E \cap B(z, c))/c^{n/2+1}\,\mathrm{d}c = \infty\,;$

(3) $\qquad \sum_{k=1}^{\infty} 2^{nk/2}\,\gamma^*(E \cap B_k(z)) = \infty\,;$

(4) $\qquad \sum_{k=1}^{\infty} 2^{nk/2}\,\gamma^*(E \cap A_k(z)) = \infty\,.$

Proof. (1) eq. (4). This follows from Theorem 1.11 and Definition 1.12.

(4) implies (3). Since $E \cap A_k(z) \subset E \cap B_k(z)$, according to Lemma 1.4 (3) we have $\gamma^*(E \cap A_k(z)) \leqq \gamma^*(E \cap B_k(z))$; this implies (3).

(3) implies (4). Since $B_k(z) \subset A_k(z) \cup B_{k+1}(z)$, according to Lemma 1.4 (2), (3) we have

$$\gamma^*(E \cap B_k(z)) \leqq \gamma^*(E \cap A_k(z)) + \gamma^*(E \cap B_{k+1}(z))\,.$$

Multiplying this inequality by $2^{nk/2}$ and summing from $k = 1$ to $M$ we easily obtain

$$(1 - 2^{-n/2}) \sum_{k=2}^{M} 2^{nk/2}\,\gamma^*(E \cap B_k(z)) + 2^{n/2}\,\gamma^*(E \cap B_1(z)) \leqq$$

$$\leqq \sum_{k=1}^{M} 2^{nk/2}\,\gamma^*(E \cap A_k(z)) + 2^{Mn/2}\,\gamma^*(E \cap B_{M+1}(z))\,.$$

Further, $\gamma^*(E \cap B_1(z)) < \infty$. There is $c$ such that

$$2^{Mn/2}\,\gamma^*(E \cap B_{M+1}(z)) \leqq c\,.$$

This and the previous relations imply (4).

(4) eq. (2). For $k \in N \cup \{0\}$,

$$2^{(k+1)n/2}(1 - 2^{-n/2})\,\gamma^*(E \cap B_{k+1}(z)) \leqq$$

$$\leqq \int_{2^{kn/2}}^{2^{(k+1)n/2}} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t \leqq 2^{nk/2}(2^{n/2} - 1)\,\gamma^*(E \cap B_k(z))\,.$$

Summing from $k = 1$ to $M$ we conclude that the series in (3) is divergent if and only if

(1.8) $\qquad \int_1^{\infty} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t = \infty\,.$

The change of variables $c = t^{-2/n}$ transforms the integral in (1.8) to the integral in (2). Hence the theorem is proved.

**1.14. Corollary.** *Let $E \subset R^{n+1}$ be an arbitrary set and $z \in R^{n+1}$. If $E$ is thin at $z$, then*

$$\lim_{t \to 0+} \frac{\gamma^*(E \cap B(z, t))}{\gamma^*(B(z, t))} = 0\,.$$

Proof. Since $E$ is thin at $z$ we obtain from the proof of Theorem 1.13 that the integral

$$\int_1^{\infty} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t$$

is convergent. Hence

$$\lim_{t \to 0+} t\,.\,\gamma^*(E \cap B(z, t^{-2/n})) = 0\,,$$

i.e.

$$\lim_{c \to 0+} \frac{\gamma^*(E \cap B(z, c))}{c^{n/2}} = 0 .$$

The assertion is then obtained by virtue of Lemma 1.4 (4) and from $\gamma(B(z, 1)) > 0$.

**1.15. Remark.** The so called "tusk condition" from [11], [14] and [18] can be then deduced using Corollary 1.14.

## 2. CONTINUOUS THERMAL CAPACITY AND ESSENTIAL BASE

**2.1. Definition.** Let $K$ be a compact set. The *continuous capacity* (in detail: the *continuous thermal capacity*) of $K$ is defined as

$$\alpha(K) = \sup\,\{\mu(R^{n+1});\ \mu \in \mathcal{M}^+(K),\ P_\mu \leqq 1 \text{ and continuous in } R^{n+1}\}\ ;$$

cf. [1]. Obviously, $\alpha(K) \leqq \gamma(K)$ for every compact subset $K$ of $R^{n+1}$.

For an arbitrary subset $E$ of $R^{n+1}$ the inner continuous capacity $\alpha_*(E)$ and the outer continuous capacity $\alpha^*(E)$, respectively, are defined in a similar way as $\gamma_*(E)$ and $\gamma^*(E)$.

Let $K = [0, 1]^n \times \{0\}$ and $K_j \subset R^{n+1}$, $j \in N$, be compact sets such that $K_{j+1} \subset \subset \operatorname{int} K_j$, $j \in N$, and $K = \bigcap_{j=1}^{\infty} K_j$. Then $\alpha(K) = 0$ and $\alpha(K_j) \geqq 1$; see Theorem 2.9.

Consequently, the continuous thermal capacity is not the Choquet capacity in the sense of [7]. Further, $\alpha_*(K) < \alpha^*(K)$ for $K = [0, 1]^n \times \{0\}$. (This follows immediately from Definition 2.7 and Theorem 2.9.) Hence we do not define the $\alpha$-capacitable sets. Of course, $\alpha_*(K) = \alpha(K)$ whenever $K \subset R^{n+1}$ is a compact set.

Proofs of the following lemmas are left to the reader.

**2.2. Lemma.** *Let $K, K_1, K_2$ be compact subsets of $R^{n+1}$. Then*

(1) $\alpha(K) < \infty$;

(2) $\alpha(K_1 \cup K_2) \leqq \alpha(K_1) + \alpha(K_2)$ *(subadditivity of $\alpha$)*;

(3) $K_1 \subset K_2$ *implies* $\alpha(K_1) \leqq \alpha(K_2)$ *(monotonicity of $\alpha$)*,

(4) $\alpha(\{z\}) = 0$ *for all* $z \in R^{n+1}$;

(5) *if* $s > 0$, *then* $\alpha(s \odot K) = s^n\,\alpha(K)$.

**2.3. Lemma.** *Let $E, E_1, E_2$ be arbitrary subsets of $R^{n+1}$. Then*

(1) $0 \leqq \alpha_*(E) \leqq \alpha^*(E)$;

(2) $\alpha^*(E_1 \cup E_2) \leqq \alpha^*(E_1) + \alpha^*(E_2)$;

(3) $E_1 \subset E_2$ *implies* $\alpha_*(E_1) \leqq \alpha_*(E_2)$, $\alpha^*(E_1) \leqq \alpha^*(E_2)$;

(4) *if* $s > 0$, *then* $\alpha_*(s \odot E) = s^n\,\alpha_*(E)$, $\alpha^*(s \odot E) = s^n\,\alpha^*(E)$.

**2.4. Definition.** A subset $T$ of $R^{n+1}$ is called *totally thin* if it is thin at every point $z \in R^{n+1}$, i.e. if $b(T) = 0$. A subset $S$ of $R^{n+1}$ is called *semipolar* if it is a countable union of totally thin sets.

Obviously, every subset of a semipolar set is semipolar and every countable union of semipolar sets is semipolar.

**2.5. Lemma.** *Let $S$ be a Borel subset of $R^{n+1}$. Then $S$ is semipolar if and only if $\alpha_*(S) = 0$.*

Proof. Let $S$ be a semipolar set and $K \subset S$ an arbitrary compact set. According to [19], p. 121, $\alpha(K) = 0$, consequently $\alpha_*(S) = 0$. If $S$ is not semipolar, then $S$ contains a nonsemipolar compact set $K$; see [16], p. 498. According to [19], p. 121, $\alpha(K) > 0$ and so $\alpha_*(S) > 0$.

**2.6. Remark.** It follows from Definition 2.1 and the proof of Lemma 2.5 that $\alpha_*(S) = 0$ for every semipolar set. The converse implication is not true as shown by the following example. We say that a subset $A$ of $R^1$ is of Bernstein's type if the intersection of both $A$ and $A^c$ with every closed uncountable set is nonempty. Let $T$ be a set of Bernstein's type (see [20] for the existence) and $S_1 = R^n \times T$, $S_2 = R^n \times T^c$. Let $K$ be an arbitrary compact subset of $S_1$. Then $K \subset R^n \times L$ for a suitable countable set $L \subset T$. But the set $R^n \times L$ is semipolar and by Lemma 2.5 and the monotonicity of $\alpha$ we have $\alpha(K) = 0$. Consequently, $\alpha_*(S_1) = 0$. In a similar way, $\alpha_*(S_2) = 0$. Since $S_1 \cup S_2 = R^{n+1}$, at least one of the sets $S_i$, $i = 1, 2$, is not semipolar. Consequently, there exists a nonsemipolar subset $A$ of $R^{n+1}$ such that $\alpha_*(A) = 0$. According to Lemma 2.5, $A$ cannot be a Borel set.

**2.7. Definition.** Let $E$ be an arbitrary subset of $R^{n+1}$ and $z \in R^{n+1}$. Then $E$ is said to be *semipolar* at $z$ if there exists a fine neighborhood $V$ of $z$ such that the set $E \cap V$ is semipolar. Let $\beta(E)$ be the set of all points $z \in R^{n+1}$ such that $E$ is not semipolar at $z$. The set $\beta(E)$ is called the *essential base* of $E$.
(In [16], the essential base is denoted by $\varrho$ and called the quasibase. It can be shown, see [3], p. 184, that $\beta(E)$ is a $G_\delta$-set.)

**2.8. Theorem.** *Let $B$ be a Borel subset of $R^{n+1}$. Then*

$$\hat{R}_1^{\beta(B)} = \sup\{P_\mu;\ \mu \in \mathcal{M}^+(R^{n+1}),\ P_\mu \leqq 1 \text{ and continuous in } R^{n+1},\ \operatorname{spt}\mu \subset B\}.$$

Proof. See [16], p. 502.

**2.9. Theorem.** *Let $B$ be a Borel subset of $R^{n+1}$. Then*

$$\alpha_*(B) = \gamma(\beta(B)).$$

Proof. We will use the following notation. If $E \subset R^{n+1}$, then $^*E = \{(x, t) \in R^{n+1};\ (x, -t) \in E\}$. If $E \subset R^{n+1}$ is a Borel set and $\nu \in \mathcal{M}^+(R^{n+1})$, then we define $^*\nu(E) = \nu(^*E)$. Clearly, $^*(^*\nu) = \nu$, $^*\nu \in \mathcal{M}^+(R^{n+1})$ and

$$\int_{R^{n+1}} F(z - w)\,\mathrm{d}\nu(w) = \int_{R^{n+1}} F(w - z)\,\mathrm{d}(^*\nu)(w). \tag{2.1}$$

We first assume that $B$ is a bounded Borel set. From [10], p. 133 and p. 127, and from [28], p. 279 it follows that there exists $\mu \in \mathcal{M}^+(\operatorname{cl}\beta(B))$ such that $\hat{R}_1^{\beta(B)} = P_\mu$.

Let $L \subset R^{n+1}$ be a compact set such that

(2.2) $\quad \operatorname{cl} \beta(B) \subset \operatorname{int} L .$

According to Lemma 1.9 (1), (2) there exists $\nu \in \mathcal{M}^+(L)$ such that $\hat{R}_1^L = P_\nu$ and $P_\nu = 1$ on int $L$. Consequently,

$$\mu(R^{n+1}) = \int_{R^{n+1}}(\int_{R^{n+1}} F(z - w)\, d\nu(w))\, d\mu(z) .$$

Applying Fubini's theorem and the relation (2.1) we obtain

(2.3) $\quad \mu(R^{n+1}) = \int_{R^{n+1}}(\int_{R^{n+1}} F(w - z)\, d\mu(z))\, d(^*\nu)(w) .$

Since $P_\mu = \hat{R}_1^{\beta(B)}$ and $\{P_{\mu'}; \mu' \in \mathcal{M}^+(R^{n+1}),\ P_{\mu'} \leqq 1$ and continuous in $R^{n+1}$, $\operatorname{spt} \mu' \subset B\}$ is an upper directed family of continuous functions, see e.g. [10], p. 40, then according to Deny-Cartan's lemma, see e.g. [7], and to Theorem 2.8 we have

$$\mu(R^{n+1}) = \sup\{\int_{R^{n+1}} P_{\mu'}\, d(^*\nu);\ \mu' \in \mathcal{M}^+(R^{n+1}),\ P_{\mu'} \leqq 1$$
$$\text{and continuous in } R^{n+1},\ \operatorname{spt} \mu' \subset B\} .$$

This together with (2.1) and (2.2) and an application of Fubini's theorem yields

$$\mu(R^{n+1}) = \sup\{\mu'(R^{n+1});\ \mu' \in \mathcal{M}^+(R^{n+1}),\ P_{\mu'} \leqq 1$$
$$\text{and continuous in } R^{n+1},\ \operatorname{spt} \mu' \subset B\} .$$

Consequently,

(2.4) $\quad \mu(R^{n+1}) = \alpha_*(B) .$

According to [10], p. 132, $\hat{R}_1^{\beta(B)} = \sup\{\hat{R}_1^K;\ K \subset \beta(B),\ K \text{ compact}\}$. Using (2.3) and $\hat{R}_1^{\beta(B)} = P_\mu$ we get

$$\mu(R^{n+1}) = \int_{R^{n+1}} \sup\{\hat{R}_1^K;\ K \subset \beta(B),\ K \text{ compact}\}\, d(^*\nu) .$$

Since $\{\hat{R}_1^K;\ K \subset \beta(B),\ K \text{ compact}\}$ is obviously an upper directed family of lower semicontinuous functions we ave according to Deny-Cartan's lemma, see [7],

$$\mu(R^{n+1}) = \sup\{\int_{R^{n+1}} \hat{R}_1^K\, d(^*\nu);\ K \subset \beta(B),\ K \text{ compact}\} .$$

Lemma 1.9 (2) implies the existence of a Radon measure $\mu_K \in \mathcal{M}^+(K)$ such that $\hat{R}_1^K = P_{\mu_K}$ and

(2.5) $\quad \gamma(K) = \mu_K(R^{n+1}) .$

Fubini's theorem and (2.1) give

$$\mu(R^{n+1}) = \sup\{\int_{R^{n+1}} P_\nu\, d\mu_K;\ K \subset \beta(B),\ K \text{ compact}\} .$$

Hence using the inclusion (2.2), the equality $P_\nu = 1$ on int $L$, the relation (2.5) and the definition of $\gamma_*$, we obtain $\mu(R^{n+1}) = \gamma(\beta(B))$. This together with (2.4) implies the desired equality.

Now let $B$ be an arbitrary Borel set. Let $U_k \subset R^{n+1}$, $k \in N$, be bounded open sets such that $U_k \nearrow R^{n+1}$. Since $\beta(B) \cap U_k \subset \beta(B \cap U_k)$, $k \in N$, the above proved equality and the monotonicity of $\alpha_*$ and $\gamma$ imply

$$\alpha_*(B) \geqq \gamma(\beta(B) \cap U_k), \quad k \in N .$$

Since $\beta(B) \cap U_k \nearrow \beta(B)$, Lemma 1.4 (5) yields

$$\lim_{k \to \infty} \gamma(\beta(B) \cap U_k) = \gamma(\beta(B)),$$

and so

$$\alpha_*(B) \geqq \gamma(\beta(B)). \tag{2.6}$$

Let $K \subset B$ be an arbitrary compact set. Since $\alpha_*(K) = \gamma(\beta(K))$ and $\beta(K) \subset \beta(B)$, the monotonicity of $\gamma$ gives $\alpha(K) \leqq \gamma(\beta(B))$. Consequently,

$$\alpha_*(B) \leqq \gamma(\beta(B)). \tag{2.7}$$

The inequalities (2.6) and (2.7) complete the proof.

**2.10. Remark.** Let $S_1$ and $S_2$ be as in Remark 2.6. Then $\alpha_*(S_1) = \alpha_*(S_2) = 0$ and $R^{n+1} = \beta(R^{n+1}) = \beta(S_1) \cup \beta(S_2)$. Consequently,

$$\infty = \gamma(R^{n+1}) \leqq \gamma(\beta(S_1)) + \gamma(\beta(S_2)).$$

This implies that there exists a set $B \subset R^{n+1}$ such that $\alpha_*(B) \neq \gamma(\beta(B))$. Theorem 2.9 fails for arbitrary sets.

**2.11. Lemma.** *Let $B$ be a Borel subset of $R^{n+1}$. Then there exists a Borel semipolar set $S \subset B$ such that*

$$\alpha_*(B) = \gamma(B \setminus S).$$

Proof. Obviously, $S = B \setminus \beta(B)$ is a Borel semipolar set. Further, $B \setminus S \subset \beta(B)$. From the monotonicity of $\gamma$ and from Theorem 2.9 we obtain $\gamma(B \setminus S) \leqq \alpha_*(B)$. According to [5], p. 296, $\beta(B \setminus S) = \beta(B)$. This, Theorem 2.9 and the relation betweeen $\alpha_*$ and $\gamma_*$ imply $\alpha_*(B) \leqq \gamma(B \setminus S)$.

**2.12. Remark.** Given a compact set $K \subset R^{n+1}$, let us denote $\alpha_1(K) = \gamma(\beta(K))$ and

$$\alpha_2(K) = \inf \{\gamma_*(K \setminus S);\ S \subset R^{n+1},\ S \text{ semipolar}\}. \tag{2.8}$$

Then $\alpha_1(K) = \alpha_2(K) = \alpha(K)$ whenever $K$ is a compact subset of $R^{n+1}$. The equalities can be easily deduced from Theorem 2.9, from Lemma 2.11 and from the fact that every semipolar set is contained in a Borel semipolar set, see e.g. [5], p. 282. Lemma 2.11 says that the right hand side of (2.8) attains the minimum.

**2.13. Lemma.** *Let $U$ be an arbitrary subset of $R^{n+1}$ and let $z \in R^{n+1}$. Then $U$ is a fine neighborhood of $z$ if and only if $z \in U$ and $U^c$ is thin at $z$.*

Proof. See [10], p. 152.

**2.14. Theorem.** *Let $B$ be a Borel set and let $z \in R^{n+1}$. The set $B$ is semipolar at $z$*

*if and only if the series*

$$(2.9) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap A_k(z))$$

*is convergent. If $B$ is an arbitrary set semipolar at $z$, then the series in* (2.9) *is convergent.*

Proof. Assume that $B$ is a Borel set and the series in (2.9) is convergent. For every $k \in N$ let $S_k$ be a Borel semipolar set such that $S_k \subset B \cap A_k(z)$ and

$$(2.10) \qquad \gamma((B \cap A_k(z)) \setminus S_k) = \alpha_*(B \cap A_k(z))\,;$$

see Lemma 2.11. Consequently, the set $S = \{z\} \cup \bigcup_{k=1}^{\infty} S_k$ is semipolar. For $k \in N$ we have $(B \setminus S) \cap A_k(z) \subset (B \cap A_k(z)) \setminus S_k$. Thus (2.10) and the monotonicity of $\gamma$ yield

$$\gamma((B \setminus S) \cap A_k(z)) \leqq \alpha_*(B \cap A_k(z))\,.$$

Since the series in (2.9) is convergent, we obtain from the above relation that the series

$$\sum_{k=1}^{\infty} 2^{nk/2}\, \gamma((B \setminus S) \cap A_k(z))$$

is convergent. According to Theorem 1.11, the set $B \setminus S$ is thin at $z$, hence $V = (B \setminus S)^c$ is a fine neighborhood of the point $z$ (see Lemma 2.13). Since the set $V \cap B$, being a subset of $S$, is semipolar, the set $B$ is semipolar at $z$ by Definition 2.7.

Let $B$ be a Borel set semipolar at $z$. Then there exists a fine neighborhood $V$ of the point $z$ such that the set $V \cap B$ is semipolar. Since $z$ has a fundamental system of fine neighborhoods which are compact in the Euclidean topology we can assume that $V$ is compact. Since $B \cap A_k(z)$ and $(B \setminus V) \cap A_k(z)$ differ for a semipolar set, according to Theorem 2.9 we have

$$(2.11) \qquad \alpha_*(B \cap A_k(z)) = \alpha_*((B \setminus V) \cap A_k(z))\,.$$

As $V$ is a fine neighborhood of the point $z$, the series

$$(2.12) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \gamma(V^c \cap A_k(z))$$

is convergent by Theorem 1.11 and Lemma 2.13. Using the equality (2.11), the relation between $\alpha_*$ and $\gamma_*$, the inclusion $(B \setminus V) \cap A_k(z) \subset V^c \cap A_k(z)$ and the monotonicity of $\gamma_*$ we obtain

$$\alpha_*(B \cap A_k(z)) \leqq \gamma(V^c \cap A_k(z))\,.$$

This and the convergence of the series in (2.12) imply that the series (2.9) is convergent.

Now let $B$ be an arbitrary set semipolar at $z$. From Definition 2.7 and from [5], p. 285, it follows that there exists a Borel set $B'$ such that $B \subset B'$ and $B'$ is semipolar at $z$. Applying the assertion proved above for the Borel set $B'$ and using the monotonicity of $\alpha_*$ we obtain the convergence of the series in (2.9).

The proof of Theorem 2.14 is complete.

**2.15. Remark.** Let $A \subset R^{n+1}$ be a nonsemipolar set such that $\alpha_*(A) = 0$, see Remark 2.6. Since $A$ is nonsemipolar, $\beta(A) \neq \emptyset$ according to [5], p. 296. Consequently, there exists a point $z \in R^{n+1}$ such that $A$ is not semipolar at the point $z$. As $\alpha_*(A) = 0$, the series in (2.9) (for $A$ instead of $B$) is convergent. Consequently, the assumption in Theorem 2.14 that $B$ is a Borel set is essential.

**2.16. Theorem.** *For an arbitrary Borel set $B$, the following conditions are equivalent:*

(1) $z \in \beta(B)$ ;

(2) $\int_0^1 \alpha_*(B \cap B(z, c))/c^{n/2+1}\, \mathrm{d}c = \infty$ ;

(3) $\sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap B_k(z)) = \infty$ ;

(4) $\sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap A_k(z)) = \infty$ .

Proof. The assertions are proved using Theorem 2.14 in an analogous way as in the proof of Theorem 1.13.

**2.17. Corollary.** *Let $B \subset R^{n+1}$ be a set and $z \in R^{n+1}$. If $B$ is semipolar at $z$, then*

$$\lim_{t \to 0+} \frac{\alpha_*(B \cap B(z, t))}{\alpha_*(B(z, t))} = 0 .$$

Proof. In a similar way as in the proof of Corollary 1.14 we obtain from Theorem 2.16 and from Theorem 2.14

$$\lim_{t \to 0+} \frac{\alpha_*(B \cap B(z, t))}{t^{n/2}} = 0 .$$

Since the set $B(z, 1)$ is not semipolar, $\alpha_*(B(z, 1)) > 0$ according to Lemma 2.5. The assertion follows from Lemma 2.3 (4).

**2.18. Definition.** Let $E$ be an arbitrary subset of $R^{n+1}$ and let $z = (x, t) \in R^{n+1}$. We say that *$E$ lies parabolically below $z$* provided there is $b > 0$ such that

$$\tau - t < -b|\chi - x|^2$$

for any $(\chi, \tau) \in E$. For $c > 0$ and $z = (x, t) \in R^{n+1}$ put

$$D(z, c) = \{(\chi, \tau) \in R^{n+1};\ t - c \leqq \tau \leqq t\} .$$

**2.19. Corollary.** *Let $B$ be a subset of $R^{n+1}$ and let $z \in R^{n+1}$. If $B$ lies parabolically* *below $z$ and $B$ is semipolar at $z$, then*

$$\lim_{c \to 0+} \frac{\alpha_*(B \cap D(z, c))}{c^{n/2}} = 0 .$$

Proof. Since $B$ lies parabolically below $z = (x, t)$, there exists $b > 0$ such that $(\tau - t) < -b|\chi - x|^2$ for any $(\chi, \tau) \in B$. We put $k = \exp(1/nb)$. An easy calculation shows that $B \cap D(z, c) \subset B \cap B(z, k \cdot c)$. This and Corollary 2.17 give the assertion.

**2.20. Definition.** Let $T$ be a subset of $R^1$ and let $t \in R^1$. The point $t$ is said to be a *condensation point of* $T$ if the set $]t - \varepsilon, t + \varepsilon[ \cap T$ is uncountable whenever $\varepsilon > 0$.

**2.21. Corollary.** *Let $B_0$ be an arbitrary subset of $R^1$, $T \subset ]0, \infty[$, let 0 be a condensation point of $T$ and*

$$B = \{(tx, -t^2) \in R^{n+1};\ x \in B_0,\ t \in T\} .$$

*If $B$ is semipolar at $(0, 0)$, then $\lambda_n(B_0) = 0$.*

Proof. We first assume that $B_0$ is a bounded set and $B$ is semipolar at $z = (0, 0)$. Since 0 is a condensation point of $T$, there exists a decreasing sequence of positive numbers $\{c_j\}_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} c_j = 0 \tag{2.14}$$

and for every $j \in N$ the set $T \cap ]c_j/2, c_j[$ is uncountable. It follows from Definition 2.7 and from [5], p. 285 that there exists a Borel set $\tilde{B}$ such that $B \subset \tilde{B}$ and $\tilde{B}$ is semipolar at $z$. Since $B_0$ is bounded, we can assume that $\tilde{B}$ lies parabolically below $z$. For every $j \in N$ let $S^j \subset R^{n+1}$ be a semipolar Borel set such that

$$\alpha_*(\tilde{B} \cap D(z, c_j^2)) = \gamma(\tilde{B} \cap D(z, c_j^2) \setminus S^j) ; \tag{2.15}$$

see Lemma 2.9. For $M \subset R^{n+1}$ and $t \in R^1$ we define

$$(M)_t = \{(x, -t) \in R^{n+1};\ (x, -t) \in M\} .$$

Putting $S = \bigcup_{j=1}^{\infty} S^j$, we get a semipolar Borel set and $S = \bigcup \{(S)_r;\ r \in R^1\}$. According to [15] and Lemma 1.5, the set $P = \{p \in R^1;\ \lambda_n((S)_{p^2}) > 0\}$ is countable. We put $T' = T \setminus P$. For every $j \in N$ there exists $d_j \in ]c_j/2, c_j[ \cap T'$ such that

$$\lambda_n((\tilde{B} \setminus S^j)_{d^2_j}) = \lambda_n((\tilde{B})_{d^2_j}) . \tag{2.16}$$

The monotonicity of $\gamma$, the relations (2.15) and (2.16) and Lemma 1.5 imply

$$\alpha_*(\tilde{B} \cap D(z, c_j^2)) \geqq \lambda_n((\tilde{B})_{d^2_j}) .$$

Denoting by $\lambda_n^*$ the outer Lebesgue measure, we have from the inclusions $B \subset \tilde{B}$ that

$$\lambda_n((\tilde{B})_{d^2_j}) \geqq d_j^n\, \lambda_n^*(B_0) .$$

Consequently, $\alpha_*(\tilde{B} \cap D(z, d_j^2)) \geqq d_j^n\, \lambda_n^*(B_0)$. Since $d_j \in ]c_j/2, c_j[$, we obtain

$$\frac{\alpha_*(\tilde{B} \cap D(z, d_j^2))}{d_j^n} \geqq \left(\frac{1}{2}\right)^n \lambda_n^*(B_0) . \tag{2.17}$$

As the set $\tilde{B}$ lies parabolically below $z$ and is semipolar at $z$, Corollary 2.19 and the relations (2.14) and (2.17) imply that $\lambda_n^*(B_0) = 0$ and hence $\lambda_n(B_0) = 0$.

Now let $B_0$ be an arbitrary set. Then $B_0 = \bigcup_{k=1}^{\infty} B_{0k}$, where $B_{0k}$, $k \in N$, are bounded sets. Then by the first part of the proof $\lambda_n(B_{0k}) = 0$ for any $k \in N$. Consequently, $\lambda_n(B_0) = 0$.

**2.22. Remark.** Suppose that $B_0 = R^n$, $T \subset ]0, \infty[$ and 0 is not a condensation point of $T$, i.e. there exists $\varepsilon > 0$ such that $T_\varepsilon = T \cap ]0, \varepsilon[$ is at most countable. The set $B_\varepsilon = \{(tx, -t^2) \in R^{n+1};\ x \in B_0,\ t \in T_\varepsilon\}$ is semipolar. According to [5], p. 296, $\beta(B_\varepsilon) = \emptyset$. Consequently, $B$ is semipolar at the point $(0, 0)$, while $\lambda_n(B_0) = \infty$. The assumption that 0 is a condensation point of $T$ cannot be omitted in Corollary 2.21.

## 3. THE CHOQUET BOUNDARY IN PARABOLIC POTENTIAL THEORY

**3.1. Definition.** Let $X$ be a metrizable compact topological space and let $\mathscr{C}(X)$ be the Banach space of continuous functions on $X$. Suppose that $P$ is a closed linear subspace of $\mathscr{C}(X)$ which separates points of $X$ and contains the constant functions. For every $x \in X$ the symbol $\mathscr{M}_x$ stands for the set of all positive Radon measures $\mu$ on $X$ such that $f(x) = \mu(f)$ whenever $f \in P$. Obviously, the Dirac measure $\varepsilon_x$ concentrated at $x$ belongs to $\mathscr{M}_x$. The set

$$\mathrm{Ch}_P X = \{x \in X;\ \mathscr{M}_x = \{\varepsilon_x\}\}$$

is called the *Choquet boundary of X* (with respect to $P$).

**3.2. Notation and Definition.** Let $U$ be a bounded open subset of $R^{n+1}$. A function $u$ is said to be *caloric* on $U$, if $u$ has continuous second partial derivatives on $U$ and

$$\frac{\partial u}{\partial t}(x, t) - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x, t) = 0 \quad \text{for all} \quad (x, t) \in U .$$

The set of all continuous functions on $\overline{U}$ whose restriction to $U$ is caloric will be denoted by $K(U)$. We will apply Definition 3.1 to the following situation: $X = \overline{U}$ and $P = K(U)$.

**3.3. Theorem.** *Let $U$ be a bounded open subset of $R^{n+1}$. Then*

$$\mathrm{Ch}_{K(U)} \overline{U} = \beta(U^c) \cap \overline{U} .$$

Proof. See [4], pp. 101, 103 and [16], p. 516.

**3.4. Theorem.** *Let $U$ be a bounded open subset of $R^{n+1}$. Then the following conditions are equivalent:*

(1) $\quad z \in \mathrm{Ch}_{K(U)} \overline{U}$ ;

(2) $\quad \int_0^1 \alpha(U^c \cap B(z, c))/c^{n/2+1}\, \mathrm{d}c = \infty$ ;

$$(3) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha(U^c \cap B_k(z)) = \infty\,;$$

$$(4) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha(U^c \cap A_k(z)) = \infty\,.$$

Proof. The assertions follow from Theorems 3.3 and 2.16.

**3.5. Corollary.** *Let $U$ be a bounded open subset of $R^{n+1}$ and let $z = (x, t) \in \partial U$. Let $B_0$ be an arbitrary subset of $R^n$ and $T$ a subset of $]0, \infty[$ such that 0 is a condensation point of $T$. If*

$$\{(x + \tau\chi,\ t - \tau^2) \in R^{n+1};\ \chi \in B_0,\ \tau \in T\} \subset U^c$$

*and $\lambda_n^*(B_0) > 0$, then $z \in \mathrm{Ch}_{K(U)}\overline{U}$.*

Proof. We may assume that $z = (0, 0)$. We shall prove that $z \in \beta(U^c)$. Since

$$B = \{(\tau\chi, \tau^2) \in R^{n+1};\ \chi \in B_0,\ \tau \in T\} \subset U^c$$

and $\lambda_n^*(B_0) > 0$, the set $B$ is not semipolar at $z$ by Corollary 2.21, hence $z \in \beta(B)$. Since $B \subset U^c$, we have $\beta(B) \subset \beta(U^c)$ too. Consequently, $z \in \beta(U^c)$.

The proof of Corollary 3.5 is complete.

*References*


[1] *G. Anger:* "Funktionalanalytische Betrachtungen bei Differentialgleichungen unter Verwendung von Methoden der Potentialtheorie I", Akademie-Verlag, Berlin, 1967.

[2] *H. Bauer:* "Harmonische Räume und ihre Potentialtheorie", Lecture Notes in Mathematics *22*, Springer-Verlag, Berlin, 1966.

[3] *U. Bauermann:* Balayage-Operatoren in der Potentialtheorie, Math. Ann. *231* (1977), 181—186.

[4] *J. Bliedtner, W. Hansen:* Simplicial cones in potential theory, Invent. Math. *29* (1975), 83—110.

[5] *J. Bliedtner, W. Hansen:* "Potential theory, An Analytic and Probabilistic Approach to Balayage", Springer-Verlag, Berlin, 1986.

[6] *M. Brelot:* Sur les ensembles effilés, Bull. Sci. Math. *68* (1944), 12—36.

[7] *M. Brelot:* "Éléments de la theorie classique du potential", 2ᵉ ed. Centre de Documentation Universitaire Paris, 1961.

[8] *M. Brzezina:* "Thinness and essential base for the heat equation", (Thesis in Czech) Charles University, Prague, 1986.

[9] *M. Brzezina:* Base and essential base in parabolic potential theory, Comm. Math. Univ. Carolinae *27* (1986), 631—632.

[10] *C. Constantinescu, A. Cornea:* "Potential theory on harmonic spaces", Springer-Verlag, Berlin, 1972.

[11] *E. G. Effros, L. J. Kazdan:* On the Dirichlet problem for the heat equation, Indiana Univ. Math. J. *20* (1971), 683—693.

[12] *C. L. Evans, F. R. Gariepy:* Wiener's criterion for the heat equation, Arch. Rational Mech. Anal. *78* (1982), 293—314.

[13] *N. Garofalo, E. Lanconelli:* Wiener's criterion for parabolic equation with variable coefficient and its consequences, Trans. Amer. Math. Soc. (to appear).

[14] *W. Hansen:* Fegen und Dünheit mit Anwendungen auf die Laplace- und Wärmeleitungsgleichungen, Ann. Inst. Fourier (Grenoble) *21* (1971), 79—121.
[15] *W. Hansen:* Semi-polar sets are almost negligible, J. reine angew. Math. *314* (1980), 217—220.
[16] *W. Hansen:* Semi-polar sets and quasi-balayage, Math. Ann. *257* (1981), 495—517.
[17] *E. Lanconelli:* Sul problema di Dirichlet per l'equazione del calore, Ann. Math. Pura ed Appl. *97* (1973), 83—113.
[18] *I. Netuka:* Thinness and the heat equation, Časopis Pěst. Mat. *99* (1974), 293—299.
[19] *I. Netuka, J. Veselý:* Harmonic continuation and removable singularities in the axiomatic potential theory, Math. Ann. *234* (1978), 117—123.
[20] *C. J. Oxtoby:* "Measure and category", Springer-Verlag, Berlin, 1971.
[21] *G. I. Petrowsky:* Zur ersten Randwertaufgabe der Wärmeleitungsgleichung, Compositio Math. *1* (1935), 383—419.
[22] *B. Pini:* Sulla regolarita e irregolarita della frontiera per il primo problema di valori al contorno relativo all'equazione del calore, Ann. Math. Pura ed Appl. *40* (1955), 69—88.
[23] *V. Sternberg:* Über die Gleichung der Wärmeleitung, Math. Ann. *101* (1929), 394—398.
[24] *L. Stocia:* On the thinness of a set at a point, Stud. Cerc. Mat. *38* (1986), 382—391.
[25] *K. Uchiyama:* A probabilistic proof and applications of Wiener's test for the heat operator (preprint).
[26] *A. N. Watson:* Thermal capacity, Proc. London Math. Soc. (3) *37* (1978), 342—362.
[27] *A. N. Watson:* Thinness and boundary behaviour of potentials for the heat equation, Mathematika *32* (1985), 90—95.
[28] *A. N. Watson:* Green's functions, potentials, and the Dirichlet problem for the heat equation, Proc. London Math. Soc. (3) *33* (1976), 251—298.
[29] *N. Wiener:* The Dirichlet problem, J. Math. Phys. *3* (1924), 127—146.

*Author's address:* 186 00 Praha 8, Sokolovská 83, Czechoslovakia (Matematicko-fyzikální fakulta UK).

# Wiener's test of thinness in potential theory

MIROSLAV BRZEZINA

*Abstract.* It is proved that Wiener's test of regularity provides a test for thinness of arbitrary sets. The result which is obtained in the context of harmonic spaces can be applied to a wide class of second order partial differential equations of elliptic or parabolic types.

*Keywords:* Wiener's test, regularity, thinness, capacity

*Classification:* 31D05, 35J25, 35K20

## INTRODUCTION

Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic space with countable base such that points of $X$ are polar; for definitions, see e.g. [C–C],[Ba]. Let $\hat{R}_u^E$ stand for the balayage of a hyperharmonic function $u$ on $X$ on a subset $E$ of $X$. A subset $E$ of $X$ is said to be thin at a point $z \in X$ if

$$\hat{R}_p^E(z) < p(z)$$

for some strict potential $p$ on $X$.

In this note we present a Wiener type test of thinness, if a suitable Wiener test for regularity is known.

We shall adopt notations of [B–H2].

## SEMICAPACITY AND THINNESS

Let us denote by $\mathcal{P}(X)$ the collection of all subsets of $X$.

**Definition.** A set function $\gamma : \mathcal{P}(X) \to [0, \infty]$ is called semicapacity on $X$ if the following conditions hold:

(i) $\gamma(A) \leq \gamma(B)$, whenever $A, B \in \mathcal{P}(X)$, $A \subset B$;
(ii) $\gamma(B) = \sup\{\gamma(K); K \subset B, K \text{ compact}\}$, whenever $B$ is a Borel subset of $X$;
(iii) $\gamma(M) = \inf\{\gamma(U); M \subset U, U \text{ open}\}$, whenever $M$ is a subset of $X$.

**Remark.** If $c$ is a Choquet capacity on $X$, then the corresponding outer capacity $c^*$ is a semicapacity on $X$; for definitions, see e.g. [He], [Br1].

**Lemma 1.** *Let $E$ be a subset of $X$, let $(A_n)_{n=1}^\infty$ be a sequence of Borel subsets of $X$ and let $\gamma$ be a semicapacity on $X$. Then there exists a Borel subset $B$ of $X$ such that*

$$\gamma(E \cap A_n) = \gamma(B \cap A_n)$$

*for every $n \in \mathbb{N}$.*

PROOF : Proof follows [Ha]. For $n, k \in \mathbb{N}$ let $U_{n,k} \subset X$ be open sets such that $E \cap A_n \subset U_{n,k}$ and

$$\gamma(E \cap A_n) = \inf\{\gamma(U_{n,k}); k \in \mathbb{N}\}.$$

Let $B_n = \bigcap_{k=1}^{\infty} U_{n,k}$. Then $B_n$ is a Borel set, $E \cap A_n \subset B_n$ and

$$(*) \qquad \gamma(E \cap A_n) = \gamma(B_n).$$

Let $B = \bigcap_{n=1}^{\infty}(B_n \cup (X \setminus A_n))$. Clearly, $B$ is a Borel set and $E \subset B$. Consequently, $B \cap A_n \subset B_n$. Further,

$$\gamma(E \cap A_n) \leq \gamma(B \cap A_n) \leq \gamma(B_n).$$

In view of $(*)$, the assertion follows. ∎

**Lemma 2.** *Let $A$ be a subset of $X$. Then there exists a $G_\delta$ set $A' \supset A$ such that*

$$\hat{R}_u^A = \hat{R}_u^{A'}$$

*for every $u \in \mathcal{H}_+^*(X)$.*

PROOF : See [B–H2], p.250. ∎

**Lemma 3.** *Let $E$ be a subset of $X$, $z \in X$ and let $E$ be thin at the point $z$. Then there exists a Borel subset $B$ of $X$ such that $E \subset B$ and $B$ is thin at $z$.*

PROOF : By Lemma 2, there is a $G_\delta$ set $B \supset E$ such that

$$\hat{R}_u^E = \hat{R}_u^B$$

for all $u \in \mathcal{H}_+^*(X)$, thus also for potentials; now we can apply the assertion from [C–C], p.150.

∎

**Lemma 4.** *For an arbitrary set $E \subset X$ and $z \in X$, the following conditions are equivalent:*

(i) *$A$ is thin at $z$;*
(ii) *$A \setminus \{z\}$ is thin at $z$;*
(iii) *$A \cup \{z\}$ is thin at $z$.*

PROOF : See [C–C], p.152. ∎

**Lemma 5.** *Let $B$ be a Borel set which is not thin at a point $z \in B$. Then there exists a compact subset $K$ of $B$ such that $K$ is not thin at $z$.*

PROOF : This is a special case of Lemma 5.1 from [B–H1]. For the convenience of the reader, we present a direct proof. Let $p$ be a strict potential and let $(V_n)_{n=1}^{\infty}$ be a sequence of relatively compact open sets such that

$$\overline{V}_{n+1} \subset V_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} V_n = \{z\}.$$

Consider $n \in \mathbb{N}$. Then

$$p(z) = \hat{R}_p^B(z) = \hat{R}_p^{B \cap V_n}(z).$$

By [B–H2], p.248, there exists a compact subset $K_n$ of $B \cap V_n$ such that

$$\hat{R}_p^{K_n}(z) > p(z) - \frac{1}{n} \quad .$$

Take $K = \bigcup_{n=1}^{\infty} K_n \cup \{z\}$. Clearly $K$ is a compact subset of $B$ and

$$\hat{R}_p^K(z) = p(z),$$

i.e., $K$ is not thin at the point $z$.

■

**Notation.** For $z \in X$, $r \in ]0,1]$, let $A^r(z)$ denote a compact set in $X$ such that:
(i) $A^r(z) \subset A^s(z)$ for $r < s$;
(ii) $\bigcap_{o<r\leq 1} A^r(z) = \{z\}$.
For $r = 2^{-n}$ write $A_n$ instead of $A^r$.

**Theorem.** *Let $z \in X$, let $E$ be an arbitrary subset of $X$ and let $\gamma$ be a semicapacity on $X$. Suppose that the following condition holds.*

(P) *There exists a sequence of positive numbers $(c_k(z))_{k=1}^{\infty}$ such that the following statements are equivalent, whenever $F \subset X$ is compact:*
(i) *$F$ is thin at $z$;*
(ii) $\sum_{k=1}^{\infty} c_k(z)\gamma(F \cap A_k(z)) < \infty$.

*Then $E$ is thin at $z$, if and only if the series*

$$\sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z))$$

*is convergent.*

PROOF : Let $E$ be not thin at $z$. By Lemma 4, we can assume that $z \in E$. According to Lemma 1, there exists a Borel set $B \supset E$ such that

$$\gamma(E \cap A_n(z)) = \gamma(B \cap A_n(z))$$

holds for every $n \in \mathbb{N}$. Clearly, $B$ is not thin at $z$. By Lemma 5, there exists a compact set $K \subset B$ such that $K$ is not thin at $z$, so according to the condition (P) the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z))$$

is divergent. Since $K \cap A_k(z) \subset B \cap A_k(z)$, we have

$$\gamma(K \cap A_k(z)) \leq \gamma(B \cap A_k(z)) = \gamma(E \cap A_k(z)),$$

hence

$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z)) \leq \sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z)).$$

Consequently, the series on the right hand side is also divergent.

Let now $E$ be thin at $z$. According to Lemma 3, there exists a Borel set $B \supset E$ such that $B$ is thin at $z$. Choose a sequence of strictly positive numbers $(\varepsilon_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} c_k(z)\varepsilon_k < \infty$. Since the set $B \cap A_k(z)$ is a Borel set, there exists , for every $k \in \mathbb{N}$, a compact set $K_k \subset B \cap A_k(z)$ such that

$$\gamma(B \cap A_k(z)) \leq \gamma(K_k) + \varepsilon_k.$$

Clearly, the set $K = \bigcup_{k=1}^{\infty} K_k \cup \{z\}$ is compact and $K \subset B \cup \{z\}$, i.e., the set $K$ is thin at $z$. By the condition (P), the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z))$$

is convergent. Since $E \subset B$, it follows

$$\gamma(E \cap A_k(z)) \leq \gamma(B \cap A_k(z)) \leq \gamma(K_k) + \varepsilon_k \leq \gamma(K \cap A_k(z)) + \varepsilon_k.$$

Thus

$$\sum_{k=1}^{\infty} c_k(z)\gamma(E \cap A_k(z)) \leq \sum_{k=1}^{\infty} c_k(z)\gamma(K \cap A_k(z)) + \sum_{k=1}^{\infty} c_k(z)\varepsilon_k,$$

and the series on the left hand side is convergent because both series on the right hand side are convergent. ■

## Remarks

The Wiener test for regularity in classical potential theory (i.e. for Laplace operator), was proved in 1924 by N.Wiener, see [**W**]. In 1944, M.Brelot proved the Wiener test of thinness in this case, see [**Br2**]. The way to an analogous criterion in the heat case took more than 50 years. In 1982, a heat analogy of the Wiener test for regularity was established in [**E–G**]. The Wiener test of thinness in the heat case was proved in [**Brz**].

If we apply Theorem proved above we get directly the corresponding criterions of thinness in the classical as well as in the heat case, because in these situations the condition (P) is fulfilled (the condition (P) is, as a matter of fact, a reformulation of the criterion of regularity). Thus immediately we get the corresponding assertions from [**Br2**] and [**Brz**].

Theorem can also be applied to parabolic equations with variable coefficients considered in [**G–L**].

In $\mathbb{R}^{n+1}$, $n \geq 1$, we consider the second order operator

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial u}{\partial x_j}) - \frac{\partial u}{\partial t},$$

where $(a_{i,j}(x,t))_{i,j=1,\dots,n}$ is real symmetric, matrix–valued function on $\mathsf{R}^{n+1}$ with $C^\infty$ entries. We assume that there exists $\nu \in ]0,1]$ such that, for every $\xi \in \mathsf{R}^n$ and every $(x,t) \in \mathsf{R}^{n+1}$,

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \leq \nu^{-1}|\xi|^2.$$

Let $\mathcal{H}^L$ be the sheaf of all continuously differentiable (twice with respect to $x_1, \dots, x_n$ and once with respect to $t$) solutions of the differential equation $Lu = 0$. According to [Ba], p.61, $(\mathsf{R}^{n+1}, \mathcal{H}^L)$ is a $\mathcal{P}$–harmonic space. It is easy to see that the points are polar. The capacity $\mathrm{cap_L}$ is defined in a usual way, see e.g.[G–L], $\mathrm{cap}^*_\mathrm{L}$ denotes the outer capacity deduced from the $\mathrm{cap_L}$. Let $\Gamma(x,t;y,s)$ denote the fundamental solution of $L$. Let us denote (for a given $z = (x,t) \in \mathsf{R}^{n+1}$, $k \in \mathsf{N}$, and $\lambda \in ]0,1[$)

$$A(x,t;\lambda^k) = \{(y,s) \in \mathbb{R}^{n+1};\ (4\pi\lambda^k)^{-n/2} \leq \Gamma(x,t;y,s)\} \cup \{(x,t)\}.$$

The validity of the condition (P) with $c_k(x,t) = \lambda^{-kn/2}$, $\lambda \in ]0,1[$, is proved in [G–L]. We have now:

**Theorem.** *Let $E$ be a subset of $\mathsf{R}^{n+1}$, let $\lambda \in ]0,1[$ and let $z = (x,t) \in \mathsf{R}^{n+1}$. Then $E$ is $L$-thin at $z$, if and only if the series*

$$\sum_{k=1}^{\infty} \lambda^{-kn/2}\, \mathrm{cap}^*_\mathrm{L}(E \cap A(x,t;\lambda^k))$$

is divergent.

Similarly, Theorem can be applied to a wide class of degenerate operators considered in [N–S].

## References


[Ba] H.Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Mathematics 22, Springer–Verlag, Berlin, 1966.

[B–H1] J.Bliedtner, W.Hansen, *Simplicial cones in potential theory*, Inventiones Math. 29 (1975), 83–110.

[B–H2] J.Bliedtner, W.Hansen, *Potential theory, An Analytic and Probabilistic Approach to Balayage*, Springer–Verlag, Berlin, 1986.

[Br1] M.Brelot, *Élémentes de la théorie classique du potentiel*, 2e ed., Centre de Documentation Universitaire, Paris, 1961.

[Br2] M.Brelot, *Sur les ensembles effilés*, Bull.Sci.Math. 68 (1944), 12–36.

[Brz] M.Brzezina, *On the base and the essential base in parabolic potential theory*, Czechoslovak Math.J. (to appear).

[C–C] C.Constantinescu, A.Cornea, *Potential theory on harmonic spaces*, Springer–Verlag, Berlin, 1972.

[E–G] C.L.Evans, F.R.Gariepy, *Wiener's criterion for the heat equation*, Arch.Rational Mech. Anal. 78 (1982), 293–314.

[G–L] N.Garofalo, E.Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans.Amer.Math.Soc. 308 (1988), 811–836.

[Ha] W.Hansen, *Private communication.*

[He] L.L.Helms, *Introduction to potential theory*, Wiley Interscience, New York, 1969.

[N–S] P.Negrini, V Scornazzani, *Wiener criterion for a class of degenerate elliptic operators*, J.Differential Equations 66 (1987), 151–164.

[W] N.Wiener, *The Dirichlet problem*, J.Math.Phys. 3 (1924), 127–146.

Mathematical Institute, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

(*Received* February 12, 1990)

EXPOSITIONES
MATHEMATICAE

Expo. Math. 11 (1993), 181 - 184
© Bibliographisches Institut &
F.A. Brockhaus AG 1993

Mathematical Notes

# On a class of translation invariant balayage spaces

Miroslav Brzezina*

In [1] J. Bliedtner and W. Hansen have introduced the notion of a balayage space. In this context it is possible to use methods well-known in classical potential theory and in the theory of harmonic spaces to more general situations. A comprehensive presentation of the theory of balayage spaces is given in the monograph [2].

In the paper [3] W. Hoh and N. Jacob have shown that certain pseudo differential operators $a(D)$ with constant coefficients also generate a balayage space. The purpose of this note is to point out that the same holds true for the corresponding "heat operator" $\frac{\partial}{\partial t} - a(D)$. We refer to [2] and [3] for the notation we use.

Let $a : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. Then $A : \mathbb{R}^{n+1} \to \mathbb{C} : (\xi, t) \mapsto a(\xi) + i \cdot t$ is also a negative definite function. Let $(Q_t)_{t>0}$ and $(P_t)_{t>0}$ be the convolution semigroups corresponding to $A$ and $a$, respectively. Further let $\mathbb{T} := (T_t)_{t>0}$ denote the translation semigroup on $\mathbb{R}$.

We start with an easy observation. Since $\widehat{P}_t(\xi) = e^{-ta(\xi)}$ and $\widehat{T}_t(\tau) = e^{-it\tau}, t > 0$, $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, (here $\widehat{}$ denotes the Fourier-transform) we find by an obvious calculation

$$\widehat{P_t \otimes T_t}(\xi, \tau) = \widehat{P}_t(\xi) \cdot \widehat{T}_t(\tau) = e^{-t(a(\xi)+i\tau)} = \widehat{Q}_t(\xi, \tau).$$

By the uniqueness of the Fourier-transform it follows

**Proposition 1.** *For all $t > 0$ the equality $Q_t = P_t \otimes T_t$ holds.*

The next result is due to J. Bliedtner and W. Hansen, see [2], p.211, and it plays a key role in our construction.

*Suppose that $(X, E_{\mathbb{P}})$ is a balayage space. Then the following statements are equivalent:*

*(i)* $(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ *is a balayage space;*
*(ii)* $\mathbb{P}$ *is a strong Feller semigroup.*

* Financial support by the Gottlieb Daimler- and Karl Benz-Stiftung is gratefully acknowledged.

0723-0869/93/020181-04$2.30

In potential theory for the heat operator polar and semipolar sets do not coincide. The similar result holds for the above mentioned balayage space. Namely the following proposition holds:

**Proposition 2.** *Let $\mathbb{P}$ be a sub-Markovian semigroup on $X$. Suppose that $(X \times \mathbb{R}, E_{\mathbb{P}\times\mathbb{T}})$ is a balayage space. Then the balayage space $(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ does not satisfy the axiom of polarity, i.e. polar and semipolar sets do not coincide.*

**Proof.** We show first that for $t \in \mathbb{R}$ a set $A = X \times \{t\}$ is totally thin. Let $(y, s)$ be an arbitrary point of $X \times \mathbb{R}$. By [2], p.285, the set $\{t\}$ is totally thin in the balayage space $(\mathbb{R}, E_{\mathbb{T}})$, consequently, there exists a strict potential $p \in E_{\mathbb{T}}$ such that $\widehat{R}_p^{\{t\}}(s) < p(s)$. Since $1 \otimes p \in E_{\mathbb{P}} \otimes E_{\mathbb{T}} \subset E_{\mathbb{P}\otimes\mathbb{T}}$,

$$\widehat{R}^A_{1\otimes p}(y, s) \leq (1 \otimes \widehat{R}_p^{\{t\}})(y, s) < (1 \otimes p)(y, s),$$

i.e. the set $A$ is thin at $(y, s)$ and, consequently, $A$ is totally thin. By definition, see [2], p.285, the set $S = X \times \mathbb{Q}$ is semipolar where $\mathbb{Q}$ denotes the set of rational numbers.

We show that $S$ is not polar. Suppose that $S$ is polar. By [2], p.282, there exists $u \in E_{\mathbb{P}\otimes\mathbb{T}}$ such that $\overline{\{u < \infty\}} = X \times \mathbb{R}$ and $u(x, s) = \infty$ for all $x \in X$ and $s \in \mathbb{Q}$. Take $r \in \mathbb{R}$ and $x \in X$ arbitrarily and $t \in \mathbb{R}$ such that $r - t \in \mathbb{Q}$ and $P_t(x, X) > 0$, see [2], p.63. Then

$$(P_t \otimes T_t)u(x, r) = \int_X u(y, r - t) P_t(x, dy) = \infty .$$

From this it follows that $u(x, r) = \infty$, since $u \in E_{\mathbb{P}\otimes\mathbb{T}}$. Consequently, $u \equiv \infty$ on $X \times \mathbb{R}$, which is impossible. Accordingly, $S$ is a nonpolar semipolar set, i.e. $(X \times \mathbb{R}, E_{\mathbb{P}\otimes\mathbb{T}})$ does not satisfy the axiom of polarity.

Let $a : \mathbb{R}^n \to \mathbb{R}$ denote a continuous negative definite function such that

$$a(0) > 0 \tag{1}$$

and that for some $s \in ]0, 2]$

$$a(\xi) \geq c_s |\xi|^s, \quad |\xi| \geq p \tag{2}$$

holds with some $p \geq 0$ and $c_s > 0$.

Let $\mathbb{P} = (P_t)_{t>0}$ be the semigroup associated with $a$. By [3], Theorem 3.1, the semigroup $\mathbb{P}$ is a strong Feller semigroup and $(\mathbb{R}^n, E_{\mathbb{P}})$ is a balayage space. Hence, Proposition 1, Proposition 2 and above mentioned result from [2] yield

**Corollary 1.** *Let $a : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function satisfying (1) and (2) and denote by $\mathbb{Q} = (Q_t)_{t>0}$ the convolution semigroup corresponding to*

$A(\xi,\tau) = a(\xi) + i\tau$, $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{R}$. *Then* $(\mathbb{R}^{n+1}, E_{\mathbb{Q}})$ *is a balayage space for which the axiom of polarity is not valid.*

**Remark 1.** The convolution semigroup $\mathbb{Q}$ from above corresponds to the pseudo differential operators $\frac{\partial}{\partial t} - a(D)$. In a similar way, for the convolution semigroup $\widetilde{\mathbb{Q}}$ which corresponds to the pseudo differential operator $\frac{\partial}{\partial t} + a(D)$ the assertion of Corollary 1 holds.

**Remark 2.** Since $a$ is a real nonnegative function, $a$ satisfies the condition

$$|\mathscr{I}a| \leq M(1 + \mathscr{R}a)$$

on $X$ with some constant $M > 0$. According to [6], p.36, the balayage space $(\mathbb{R}^n, E_{\mathbb{P}})$ satisfies the axiom of polarity.

Let $\mathbb{P} = (P_t)_{t>0}$ be the Brownian semigroup on $\mathbb{R}^n$, $n \in \mathbb{N}$, and $(\eta_t^\alpha)_{t>0}$ be the one-sided stable semigroup for some $\alpha \in ]0,2]$. The symmetric stable semigroup $\mathbb{P}^\alpha = (P_t^\alpha)_{t>0}$ of index $\alpha$ on $\mathbb{R}^n$ is defined by

$$P_t^\alpha f := \int P_s f \eta_t^\alpha(ds), \quad f \in \mathscr{B}^+(\mathbb{R}^n).$$

By [2], p.176, the Brownian semigroup $\mathbb{P}$ on $\mathbb{R}^n$ is a strong Feller semigroup, hence by [2], p.186, $\mathbb{P}^\alpha$ is a strong Feller semigroup on $\mathbb{R}^n$. Further, by [2], p.9, $\lim_{t\to 0} P_t f = f$ locally uniformly for every $f \in \mathscr{K}(\mathbb{R}^n)$. Consequently $\lim_{t\to 0} P_t^\alpha f = f$ locally uniformly for every $f \in \mathscr{K}(\mathbb{R}^n)$. For every $t > 0$ we define the function $g_t$ on $\mathbb{R}^n$ by

$$g_t(x) := \left(\frac{1}{2\pi t}\right)^{n/2} \exp\left(-\frac{\|x\|^2}{2t}\right).$$

Let $W^{(\alpha)}$, $v$ and $w$ are defined for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ as

$$W^{(\alpha)}(x,t) := \begin{cases} 0, & t \leq 0 \\ \int_0^\infty g_s(x)\eta_t^\alpha(ds), & t > 0, \end{cases}$$

$$v(x,t) := 1 \quad \text{and} \quad w(x,t) := t \cdot 1_{[0,\infty[}(t).$$

Then it is obvious that $W^{(\alpha)}$, $v$, $w \in E_{\mathbb{P}^\alpha \otimes \mathbb{T}}$. Further, $u_0 := \inf(W^{(\alpha)}, w) \in E_{\mathbb{P}^\alpha \otimes \mathbb{T}} \cap \mathscr{C}(\mathbb{R}^{n+1})$ and $u_0 > 0$ on $\mathbb{R}^n \times ]0,\infty[$. A straightforward calculation shows that for the function

$$u(x,t) := \sum_{k=0}^{\infty} 2^{-k} u_0(x, t+k), \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

the following holds: $u \in E_{\mathbb{P}^\alpha \otimes \mathbb{T}} \cap \mathscr{C}(\mathbb{R}^{n+1})$, $u > 0$ on $\mathbb{R}^{n+1}$ and $\frac{u}{v} \in \mathscr{C}_0(\mathbb{R}^{n+1})$. By Proposition 5.6 from [2], p.210, the above remarks and Proposition 2 we have

**Corollary 2.** *Let $\mathbb{P}^\alpha$ be the symmetric stable semigroup of index $\alpha \in ]0,2]$ on $\mathbb{R}^n$, $n \in \mathbb{N}$. Then $(\mathbb{R}^{n+1}, E_{\mathbb{P}^\alpha \otimes \mathbb{T}})$ is a balayage space which does not satisfy the axiom of polarity.*

**Remark 3.** The semigroup $\mathbb{P}^\alpha \otimes \mathbb{T}$ is called the heat semigroup of index $\alpha$ on $\mathbb{R}^n \times \mathbb{R}$. In the articles [4], [5] it is proved that the Poincaré condition for regular points and the Wiener test remain valid in this case.

## References


[1] J. Bliedtner, W. Hansen, Markov processes and harmonic spaces, Z. Wahrscheinlichkeitstheorie verw. Gebiete **42** (1978), 309–325.

[2] J. Bliedtner, W. Hansen, Potential theory, An Analytic and Probabilistic Aproach to Balayage, Springer-Verlag, Berlin, 1986.

[3] W. Hoh, N. Jacob, On some translation invariant balayage spaces, Comm. Math. Univ. Carolinae **32** (1991), 471–478.

[4] M. Itô, M. Nishio, Poincaré type conditions of regularity for the parabolic operator of order $\alpha$, Nagoya Math. J. **115** (1989), 1–22.

[5] M. Nishio, The Wiener criterion of regular points for the parabolic operator of order $\alpha$, Nagoya Math. J. **116** (1989), 163–179.

[6] M. Rao, On a result of M. Kanda, Z. Wahrscheinlichkeitstheorie verw. Gebiete **41** (1977), 35–37.


Received: 09.12.91

Mathematical Institute of Charles University
Sokolovská 83
186 00 Praha 8
Czechoslovakia

Mathematical Institute of University of Erlangen-Nürnberg
Bismarckstrasse 1a
D–8520 Erlangen

Aequationes Mathematicae 45 (1993) 89–99
University of Waterloo

0001-9054/93/010089-11 $1.50 + 0.20/0


# Kernels and Choquet capacities

Miroslav Brzezina

*Summary.* M. Brelot showed that the capacity corresponding to a function-kernel is a Choquet capacity, provided that the kernel satisfies the principle of equilibrium, the weak domination principle and the adjoint kernel satisfies the weak principle of equilibrium. This result is not applicable for a series of important kernels in potential theory (e.g. the fundamental solution of the heat equation, or the Kolmogorov equation), since the above principles no longer hold in this situation. New principles for function kernels guaranteeing that the capacity is a Choquet capacity are introduced and applied in the framework of balayage spaces. In particular, polar and adjoint polar sets are shown to coincide in this context.

In [2] M. Brelot proved that the **K**-capacity is a Choquet capacity (see the definitions below), whenever the kernel **K** satisfies the principle of equilibrium, the weak domination principle and the adjoint kernel $\tilde{\mathbf{K}}$ (see the definition below) satisfies the weak principle of equilibrium, see [2], pp. 46–47, for definitions. But many kernels in potential theory do not satisfy the above principles, for example the heat kernel. In this note we introduce two new principles for kernels **K** and $\tilde{\mathbf{K}}$ which guarantee that the **K**-capacity is a Choquet capacity. The heat, the Riesz, the Kolmogorov and many other kernels in potential theory satisfy our new principles.

In the second part of the paper we give some connections between polar sets and sets of outer **K**-capacity zero in the context of balayage spaces.

In the following let $X$ be a locally compact Hausdorff space with a countable base and let $\mathcal{M}^+$ stand for the set of all nonnegative Radon measures on $X$. For a set $E \subset X$, let us denote by $\mathcal{M}^+(E)$ the collection of all nonnegative Radon measures on $X$ with *compact* support in $E$. The support of a measure is denoted by supp. A lower semicontinuous function $\mathbf{K}: X \times X \to [0, \infty]$ is called a **kernel on** $X$.

---

AMS (1991) subject classification: Primary 31B15, 31B35, 31D05, 31C15, Secondary, 35K05.

*Manuscript received February 21, 1992, and in final form, June 4, 1992.*

The **K-potential** of a measure $\mu \in \mathcal{M}^+$ is defined as

$$\mathbf{K}_\mu(x) := \int_X \mathbf{K}(x, y)\mu(dy), \qquad x \in X.$$

For a compact set $L \subset X$, the **K-capacity** (corresponding to the kernel **K**) is defined by

$$\mathbf{K}\text{-cap}(L) := \sup\{\mu(X);\ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \leqslant 1 \text{ on } X\};$$

cf. [2], p. 43.

Fundamental properties of the **K**-capacities are summarized in the following

LEMMA 1. *Let $c$ be the* **K**-*capacity corresponding to a kernel* **K** *on $X$ and let $L$, $L_n$, $n \in \mathbb{N}$, be compact subsets of $X$. Then*

(*i*) $0 \leqslant c(L) \leqslant \infty$;

(*ii*) $L_1 \subset L_2$ *implies* $c(L_1) \leqslant c(L_2)$ (*monotonicity of* $c$);

(*iii*) $c(L_1 \cup L_2) \leqslant c(L_1) + c(L_2)$ (*subadditivity of* $c$);

(*iv*) *if* $(L_n)_{n=1}^\infty$ *is a decreasing sequence of compact sets with the intersection $L$, i.e., $L_n \downarrow L$, then*

$$\lim_{n\to\infty} c(L_n) = c(L)$$

(*right continuity of $c$ on compact sets*);

(*v*) *if* $c(L) < \infty$, *there exists a measure* $\mu \in \mathcal{M}^+(L)$ *such that* $\mathbf{K}_\mu \leqslant 1$ *on $X$ and* $c(L) = \mu(L)$.

*Proof.* Assertions (i)–(iii) follow easily from the definition, for (iv) and (v) see e.g. [2], pp. 43–44.

A set function $c$ defined on the class of all compact subsets of $X$ will be called the **Choquet capacity on** $X$ if it satisfies the conditions (i), (ii), (iv) of Lemma 1 and

(iii′) $c(L_1 \cup L_2) + c(L_1 \cap L_2) \leqslant c(L_1) + c(L_2)$ whenever $L_1$, $L_2$ are compact subsets of $X$ (strong subbadditivity of $c$);

cf. [2].

As the following example shows, the **K**-capacity is in general not a Choquet capacity.

EXAMPLE 1. Let $X := \{1, 2, 3\}$ and let the kernel $\mathbf{K}: X \times X \to [0, \infty]$ be defined by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let $\gamma$ denote the corresponding **K**-capacity. An easy calculation shows that $\gamma(\{\varnothing\}) = 0$, $\gamma(\{1\}) = \gamma(\{2\}) = \gamma(\{3\}) = \gamma(\{1, 3\}) = \gamma(\{2, 3\}) = 1$, $\gamma(\{1, 2\}) = \gamma(\{1, 2, 3\}) = 2$. Consequently,

$$\gamma(\{1, 3\} \cup \{2, 3\}) + \gamma(\{1, 3\} \cap \{2, 3\}) > \gamma(\{2, 3\}) + \gamma(\{2, 3\}),$$

i.e., the **K**-capacity $\gamma$ is not a Choquet capacity on $X$.

In the following we will investigate when the **K**-capacity is a Choquet capacity on $X$. Using the notions from [1], let us note the following

REMARK 1. Let $(X, \mathscr{W})$ be a balayage space and let $1 \in \mathscr{W}$. We show that for any relatively compact open set $G \subset X$ and for any compact set $L \subset G$ there exists a continuous potential $p \in \mathscr{P}$ such that

$$p \leqslant 1 \text{ on } X,\ p = 1 \text{ in a neighborhood of } L \text{ and } C(p) \subset G;$$

here $C(p)$ denotes the carrier of $p$.

Choose auxiliary open sets $U_1, U_2 \subset X$ such that $L \subset U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset G$. Let $f \in \mathscr{C}(X)$, $f|_{\bar{U}_1} = 1$, $f|_{\mathbf{C}U_2} = 0$ and $f(X) = [0, 1]$. Let further $q \in \mathscr{P}$ be a strict potential on $X$ and $\delta > 0$ such that $\delta \cdot q \geqslant 1$ on $\bar{U}_2$. Then it is obvious that $0 \leqslant f \leqslant \delta \cdot q$. According to [1], p. 100, $R_f \in \mathscr{H}^+(\mathbf{C}\mathrm{supp}(f))$, and consequently $C(R_f) \subset \bar{U}_2 \subset G$. Further by [1], p. 66, we have $R_f \in \mathscr{P}$. Since $1 \in \mathscr{W}$ we have $R_f \leqslant 1$ on $X$ and $R_f = 1$ on $U_1 \supset L$. Consequently $R_f$ is a potential with desired properties.

Let $p, q \in \mathscr{P}$ be continuous potentials on $X$. If $p \leqslant q$ on $C(p)$, then $p \leqslant q$ on $X$. Indeed, let $U := \mathbf{C}C(p)$, $r := q - p$. Then $r \geqslant 0$ on $\mathbf{C}U = C(p)$, $r \geqslant -p$ on $X$ and $r \in {}^*\mathscr{H}(U)$, since $q \in \mathscr{P}$ and $p \in \mathscr{H}(U)$. For $z \in \partial U \subset C(p)$ it follows that

$$\liminf_{x \to z,\, x \in U} r(x) = q(z) - p(z) \geqslant 0.$$

The desired inequality follows from the minimum principle, see [1], p. 116.

DEFINITION. Let K be a kernel on $X$. We say that the kernel K satisfies the **continuous domination principle** (briefly: CDP) if, for any two measures $\mu$ and $\nu$ on $X$ with continuous and bounded K-potentials, the condition $K_\nu \leqslant K_\mu$ on supp $\nu$ implies $K_\nu \leqslant K_\mu$ everywhere.

We say that the kernel K satisfies the **continuous principle of equilibrium** (briefly CPE) if, for any relatively compact open set $G \subset X$ and any compact set $L \subset G$, there exists a measure $\mu \in \mathcal{M}^+(G)$ such that $K_\mu \leqslant 1$, $K_\mu$ is continuous on $X$ and $K_\mu = 1$ on neighborhood of $L$.

The **adjoint kernel** $\tilde{K}$ of a kernel K is defined by $\tilde{K}(x, y) := K(y, x)$, $x, y \in X$.

Now we are able to formulate the main result.

THEOREM 1. *Let* K *be a kernel on* $X$. *Let* K *and the adjoint kernel* $\tilde{K}$ *satisfy the continuous principle of equilibrium and further let* K *satisfy the continuous domination principle. Then the* K*-capacity and the* $\tilde{K}$*-capacity are Choquet capacities on* $X$. *Further, for any compact set* $L \subset X$, *we have*

$$\text{K-cap}(L) = \tilde{\text{K}}\text{-cap}(L).$$

The proof of this theorem is based on the following lemma which is similar to a theorem of F. Y. Maeda, see [8], p. 233.

LEMMA 2. *Let* K *be a kernel on* $X$. *Let* K *and the adjoint kernel* $\tilde{K}$ *satisfy the continuous principle of equilibrium. If* $L$ *is a compact subset of* $X$, *then*

$$\text{K-cap}(L) = \inf\{\mu(X);\ \mu \in \mathcal{M}^+,\ K_\mu \geqslant 1 \textit{ in a neighborhood of } L,\ K_\mu \textit{ is a continuous K-potential on } X\};$$

$$\tilde{\text{K}}\text{-cap}(L) = \inf\{\mu(X);\ \mu \in \mathcal{M}^+,\ \tilde{K}_\mu \geqslant 1 \textit{ in a neighborhood of } L,\ \tilde{K}_\mu \textit{ is a continuous } \tilde{\text{K}}\textit{-potential on } X\};$$

$$\text{K-cap}(L) = \tilde{\text{K}}\text{-cap}(L).$$

*Proof.* Denote by $d(L)$ and $\tilde{d}(L)$ the number on the right hand side in the first and the second equality, respectively.

The inequality $\text{K-cap}(L) \leqslant \tilde{d}(L)$ follows easily from Fubini's theorem.

Now we show that $d(L) \leqslant \text{K-cap}(L)$. We may assume that $\text{K-cap}(L) < \infty$. From the right continuity of K-cap on compact sets it follows that there exists a relatively compact open set $G_1 \subset X$ and a positive constant $c$ such that K-cap$(L') \leqslant c$ whenever $L'$ is a compact subset of $G_1$. Further, we can find a sequence $(G_n)_{n=2}^{\infty}$ of open subsets of $G_1$ such that $G_n \downarrow \bigcap_{n=1}^{\infty} G_n = L$. According to the CPE there exists, for every $n \in \mathbb{N}$, a measure $\mu_n \in \mathcal{M}^+(G_n)$ such that

$$\mathbf{K}_{\mu_n} \leqslant 1, \quad \mathbf{K}_{\mu_n} \text{ is continuous on } X \text{ and } \mathbf{K}_{\mu_n} = 1 \text{ in a neighborhood of } L,$$

i.e., for all $n \in \mathbb{N}$ the following holds:

$$d(L) \leqslant \mu_n(X).$$

Since $\mu_n \in \mathcal{M}^+(G_1)$, $\mathbf{K}_{\mu_n} \leqslant 1$ on $X$ for all $n \in \mathbb{N}$, it follows that for every $n \in \mathbb{N}$ the inequality $\mu_n(X) \leqslant c$ holds. We can assume that the sequence $(\mu_n)_{n=1}^{\infty}$ converges vaguely to a measure $\mu \in \mathcal{M}^+$. From the lower semicontinuity of $\mathbf{K}$ it follows that $\mu \in \mathcal{M}^+(L)$, $\mathbf{K}_\mu \leqslant 1$ on $X$ and $\lim_{n\to\infty} \mu_n(X) = \mu(X)$. Consequently, $d(L) \leqslant$ K-cap$(L)$.

From the equality $\tilde{\tilde{\mathbf{K}}} = \mathbf{K}$ the inequality $\tilde{d}(L) \leqslant \tilde{\mathbf{K}}$-cap$(L) \leqslant d(L)$ follows and hence the assertion.

*Proof of Theorem* 1. By Lemma 1 and Lemma 2 it remains to verify that the condition

$$\text{K-cap}(L_1 \cup L_2) + \text{K-cap}(L_1 \cap L_2) \leqslant \text{K-cap}(L_1) + \text{K-cap}(L_2)$$

is satisfied whenever $L_1$ and $L_2$ are two compact subsets of $X$. We can assume that K-cap$(L_1) < \infty$ and K-cap$(L_2) < \infty$. Choose $\varepsilon > 0$. By Lemma 2 it follows that there exist measures $\mu_i \in M^+$, $i = 1, 2$, such that $\mathbf{K}_{\mu_i} \geqslant 1$ in a relatively compact open set $U_i \supset L_i$, $\mathbf{K}_{\mu_i}$ is continuous on $X$ and $\mu_i(X) \leqslant$ K-cap$(L_i) + \varepsilon$, $i = 1, 2$. Let $v_i$, $i = 1, 2$, be measures on $X$, $v_1 \in \mathcal{M}^+(U_1 \cup U_2)$, $v_2 \in \mathcal{M}^+(U_1 \cap U_2)$ such that $\mathbf{K}_{v_i} \leqslant 1$ and $\mathbf{K}_{v_i}$ are continuous on $X$, $i = 1, 2$, and $\mathbf{K}_{v_1} = 1$ in a neighborhood of $L_1 \cup L_2$, $\mathbf{K}_{v_2} = 1$ in a neighborhood of $L_1 \cap L_2$. (The existence of such measures $v_i$, $i = 1, 2$, follows from the CEP for $\mathbf{K}$.) Let $G$ be a relatively compact open set such that $G \supset \overline{U_1 \cup U_2}$. From the CEP for $\tilde{\mathbf{K}}$ it follows that there exists a measure $\kappa \in \mathcal{M}^+(G)$ such that

$$\tilde{\mathbf{K}}_\kappa \leqslant 1 \text{ on } X \text{ and } \tilde{\mathbf{K}}_\kappa = 1 \text{ in a neighborhood of } \overline{U_1 \cup U_2}.$$

Since $\mathbf{K}_{v_2} \leqslant \mathbf{K}_{\mu_2}$ on $U_2 \supset \operatorname{supp} v_2$, it follows by the CDP that $\mathbf{K}_{v_2} \leqslant \mathbf{K}_{\mu_2}$ on $X$. Further, $\mathbf{K}_{v_1} \leqslant \mathbf{K}_{\mu_1}$ on $U_1$, hence

$$\mathbf{K}_{v_1} + \mathbf{K}_{v_2} \leqslant \mathbf{K}_{\mu_1} + \mathbf{K}_{\mu_2} \tag{1}$$

on $U_1$. Similarly, $\mathbf{K}_{v_2} \leqslant \mathbf{K}_{\mu_1}$ on $U_1 \supset \operatorname{supp} v_2$ and by the CDP $\mathbf{K}_{v_2} \leqslant \mathbf{K}_{\mu_1}$ on $X$.

Further, $\mathbf{K}_{\nu_1} \leqslant \mathbf{K}_{\mu_2}$ on $U_2$ and consequently (1) holds also on $U_2$. The inequality (1) holds on $U_1 \cup U_2 \supset \operatorname{supp}(\nu_1 + \nu_2)$ and from the CDP the validity of the inequality (1) on $X$ follows. Now, using Fubini's theorem, Lemma 2 and the inequality (1), we obtain

$$\begin{aligned}
\mathbf{K}\text{-cap}(L_1 \cup L_2) + \mathbf{K}\text{-cap}(L_1 \cap L_2) &\leqslant \int_{U_1 \cup U_2} 1\, d\nu_1 + \int_{U_1 \cap U_2} 1\, d\nu_2 \\
&\leqslant \int_X \tilde{\mathbf{K}}_\kappa\, d\nu_1 + \int_X \tilde{\mathbf{K}}_\kappa\, d\nu_2 = \int_X (\mathbf{K}_{\nu_1} + \mathbf{K}_{\nu_2})\, d\kappa \\
&\leqslant \int_X (\mathbf{K}_{\mu_1} + \mathbf{K}_{\mu_2})\, d\kappa \leqslant \mu_1(X) + \mu_2(X) \\
&\leqslant \mathbf{K}\text{-cap}(L_1) + \mathbf{K}\text{-cap}(L_2) + 2\varepsilon.
\end{aligned}$$

Since this inequality is valid for any $\varepsilon > 0$, the strong subadditivity of $\mathbf{K}$-cap follows. □

In the second part of this note we will investigate some connections between $\mathbf{K}$-capacities and polar sets in the context of balayage spaces. For the definition of the balayage space, some notions and notations we refer to the monograph of J. Bliedtner and W. Hansen [1]. As a special case it is possible to consider a $\mathscr{P}$-harmonic space with countable base, in the sense of C. Constantinescu and A. Cornea, see [4]. We need the following

DEFINITION. Let $(X, \mathscr{W})$ be a balayage space and let $\mathbf{K}$ be a kernel on $X$. Then $(X, \mathscr{W})$ is said to have the property A (with respect to $\mathbf{K}$) if there exists a balayage space $(X, \tilde{\mathscr{W}})$ such that:

- $1 \in \mathscr{W} \cap \tilde{\mathscr{W}}$;
- for every $p \in \mathscr{P}(X)$ there exists exactly one measure $\mu \in \mathscr{M}^+$ such that $\mathbf{K}_\mu = p$ and $\operatorname{supp} \mu = C(p)$;
- if $\mu \in \mathscr{M}^+$ and $\overline{\{\mathbf{K}_\mu < \infty\}} = X$, then $\mathbf{K}_\mu \in \mathscr{P}(X)$;
- for every $\tilde{p} \in \tilde{\mathscr{P}}(X)$ there exists exactly one measure $\mu \in \mathscr{M}^+$ such that $\tilde{\mathbf{K}}_\mu = \tilde{p}$ and $\operatorname{supp} \mu = C(\tilde{p})$, ($\tilde{\mathbf{K}}$ is the adjoint kernel of the kernel $\mathbf{K}$);
- if $\mu \in \mathscr{M}^+$ and $\overline{\{\tilde{\mathbf{K}}_\mu < \infty\}} = X$, then $\tilde{\mathbf{K}}_\mu \in \tilde{\mathscr{P}}(X)$.

(Here and in the following $\mathscr{P}(X)$ and $\tilde{\mathscr{P}}(X)$ stand for the set of all potentials (with respect to $\mathscr{W}$ and $\tilde{\mathscr{W}}$, respectively) on $X$; $C(u)$ and $C(\tilde{u})$ denote the carrier of $u \in \mathscr{W}$ and $\tilde{u} \in \tilde{\mathscr{W}}$, respectively.)

REMARK 2. Let $(X, \mathscr{W})$ be a $\mathscr{P}$-harmonic space, with a countable base, in the sense of [4]. Assume that $(X, \mathscr{W})$ admits a Green function $\mathbf{K}$ with the following properties:

- $\mathbf{K}: X \times X \to [0, \infty]$ is l.s.c. and finite continuous on $(X \times X) \setminus \{(x, x) \in X \pm X; x \in X\}$;
- for every $x \in X$ the function $\mathbf{K}_{\varepsilon_x}$ is a potential on $X$ and $C(\mathbf{K}_{\varepsilon_x}) = \{x\}$; here $\varepsilon_x$ denotes the Dirac measure concentrated in $x$;
- for every $p \in \mathscr{P}(X)$ with compact $C(p)$ there exists exactly one measure $\mu \in \mathscr{M}^+$ such that $p = \mathbf{K}_\mu$.

From the results by T. Ikegami, see [7], it follows that $(X, \mathscr{W})$ has the property A with respect to the kernel $\mathbf{K}$.

EXAMPLE 2. **(a) Riesz potentials.** Let $n \in \mathbb{N}$. For $\alpha \in ]0, 2]$, $\alpha < n$, the **Riesz kernel** (of order $\alpha$) $N_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ is defined by

$$N_\alpha(x, y) := \begin{cases} \infty, & \text{if } x = y, \\ \|x - y\|^{\alpha - n}, & \text{if } x \neq y. \end{cases}$$

From [1], p. 198 and p. 200, it follows that the balayage space $(\mathbb{R}^n, E_{\mathbb{P}\alpha})$ has the property A with respect to the kernel $N_\alpha$. (Here $E_{\mathbb{P}\alpha}$ denotes the set of all excessive functions with respect to the symmetric stable semigroup of index $\alpha$ on $\mathbb{R}^n$; see [1], pp. 50, 191–207.)

**(b) Potential theory of the heat operator.** Let $n \in \mathbb{N}$. The **heat (Weierstrass) kernel** $W: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to [0, \infty]$ is defined by $((x, t), (y, s) \in \mathbb{R}^{n+1})$

$$W(x, t; y, s) := \begin{cases} 0, & \text{if } t \leqslant s, \\ (4\pi(t - s))^{-n/2} \exp(-\|x - y\|^2/4(t - s)), & \text{if } t > s. \end{cases}$$

By [9], pp. 273–280, it follows that the harmonic space corresponding to the heat operator, i.e. $\partial/\partial t - \sum_{i=1}^n \partial^2/\partial x_i^2$, has the property A with respect to the kernel $W$.

**(c) Potential theory of the Kolmogorov operator.** For $z = (x, y, t) \in \mathbb{R}^3$ we denote

$$\bar{E}(z) := \begin{cases} 0, & \text{if } t \leqslant 0, \\ \dfrac{\sqrt{3}}{2\pi t^2} \exp\left\{ -\dfrac{x^2}{t} + \dfrac{3x(y + tx)}{t^2} - \dfrac{3(y + tx)^2}{t^3} \right\}, & \text{if } t > 0. \end{cases}$$

The **Kolmogorov kernel** $E: \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty]$ is defined by $(z = (x, y, t)$, $z_0 = (x_0, y_0, t_0) \in \mathbb{R}^3)$

$$E(z; z_0) := \bar{E}(x - x_0, y - y_0 + x_0(t - t_0), t - t_0).$$

From [3], pp. 34–36, it follows that the harmonic space corresponding to the Kolmogorov operator, i.e. $\partial^2/\partial x^2 + x(\partial/\partial y) - \partial/\partial t$, has the property A with respect to the kernel $E$.

By Remark 1, it follows that the Riesz kernel of order $\alpha$, the heat kernel, the Kolmogorov kernel and the corresponding adjoint kernels satisfy the continuous domination principle and the continuous principle of equilibrium.

COROLLARY. *Let $(X, \mathscr{W})$ be a balayage space with the property A with respect to the kernel* $\mathbf{K}$. *Then the* $\mathbf{K}$*-capacity on $X$ is a Choquet capacity on $X$.*

*Proof.* By Remark 1, the kernels $\mathbf{K}$ and $\tilde{\mathbf{K}}$ satisfy the continuous domination principle and the continuous principle of equilibrium. The assertion follows now from Theorem 1.

LEMMA 3. Let $(X, \mathscr{W})$ *be a balayage space,* $1 \in \mathscr{W}$ *and let $L$ be a compact subset of $X$. If $C(\hat{R}_1^L) \subset L$, then*

$$\hat{R}_1^L = \sup\{p \in \mathscr{P}(X);\, p \leqslant 1 \text{ on } X \text{ and } C(p) \subset L\}. \tag{2}$$

*Proof.* Denote by $\mathscr{F}$ the set on the right hand side in (2). Since $L$ is compact, $\hat{R}_1^L \in \mathscr{P}(X)$. By assumption, $C(\hat{R}_1^L) \subset L$, consequently, $\hat{R}_1^L \in \mathscr{F}$, i.e.,

$$\hat{R}_1^L \leqslant \sup \mathscr{F}.$$

Let $p \in \mathscr{F}$ and let $u \in \mathscr{W}$, $u \geqslant 1$ on $L$. Put $U := \mathbf{C}L$. From the l.s.c. of $u$ on $X$ it follows that for $z \in \partial U < L$ the inequality

$$\liminf_{x \to z,\, x \in U} u(x) \geqslant 1$$

holds. Consequently,

$$\liminf_{x \to z,\, x \in U} (u(x) - p(x)) \geqslant 0.$$

Since $u - p \geqslant 0$ on $\mathbf{C}U$, $u - p$ is hyperharmonic on $U$ and $u - p \geqslant -p$, it follows from the minimum principle (see [1], p. 116) that $u \geqslant p$ on $X$. Consequently, $\sup \mathscr{F} \leqslant R_1^L$. But the $\sup \mathscr{F}$ is l.s.c. on $X$, i.e., $\sup \mathscr{F} \leqslant \hat{R}_1^L$.

PROPOSITION. *Let $(X, \mathscr{W})$ be a balayage space with the property A with respect to the kernel* $\mathbf{K}$ *and let $L$ be a compact subset of $X$. If $C(\hat{R}_1^L) \subset L$, then there exists exactly one measure $\mu_L \in \mathscr{M}^+(L)$ such that*

(*i*) $\hat{R}_1^L = \mathbf{K}_{\mu_L}$;

(*ii*) $\mathbf{K}\text{-cap}(L) = \mu_L(L)$;

(*iii*) *if $\mu \in \mathscr{M}^+(L)$ and if $\mathbf{K}_\mu \leqslant 1$ on $X$, then $\mathbf{K}_\mu \leqslant \mathbf{K}_{\mu_L}$.*

*Proof.* Since $(X, \mathscr{W})$ has the property A with respect to the kernel $\mathbf{K}$ and since $\hat{R}_1^L \in \mathscr{P}(X)$, there exists exactly one measure $\mu_L \in \mathscr{M}^+(L)$ such that $\mathbf{K}_{\mu_L} = \hat{R}_1^L$.

The property in (iii) follows from Lemma 3.

Since $\mu_L \in \mathscr{M}^+(L)$ and $\mathbf{K}_{\mu_L} \leqslant 1$ on $X$, $\mu_L(L) \leqslant \mathbf{K}\text{-cap}(L)$. Let $\mu \in \mathscr{M}^+(L)$, $\mathbf{K}_\mu \leqslant 1$ on $X$. By (iii), $\mathbf{K}_\mu \leqslant \mathbf{K}_{\mu_L}$. Let $U$ be a relatively compact open set, $L \subset U$. From the CEP for $\tilde{\mathbf{K}}$ the existence of a measure $\nu \in \mathscr{M}^+(U)$, with $\tilde{\mathbf{K}}_\nu = 1$ on a neighborhood of $L$, follows. By Fubini's theorem, we have

$$\mu(L) = \int_X \tilde{\mathbf{K}}_\nu \, d\mu = \int_X \mathbf{K}_\mu \, d\nu \leqslant \int_X \mathbf{K}_{\mu_L} \, d\nu = \int_X \tilde{\mathbf{K}}_\nu \, d\mu_L = \mu_L(L).$$

Consequently, $\mathbf{K}\text{-cap}(L) \leqslant \mu_L(L)$.

REMARK 3. The uniquely determined measure $\mu_L$ from the Proposition is called the equilibrium measure of $L$ and $\mathbf{K}_{\mu_L}$ the equilibrium potential of $L$.

This Proposition gives some generalizations of results known from classical potential theory, see e.g. [5], p. 138, and the potential theory for the heat operator, see [6], p. 88.

Recall that a subset $P$ of a balayage space $(X, \mathscr{W})$ is called **polar** (in detail, $\mathscr{W}$-polar) if there exists a function $v \in \mathscr{W}$ such that $v = \infty$ on $P$, but $\{v < \infty\}$ is dense in $X$.

In the potential theory for the heat operator, the polar sets, the polar sets with respect to the adjoint heat operator, the sets of outer thermal capacity zero and the sets of outer adjoint thermal capacity zero do coincide, see [10], p. 353. The same holds also in our situation, namely we have the following

THEOREM 2. *Let $(X, \mathscr{W})$ be a balayage space with the property A with respect to the kernel* $\mathbf{K}$. *Assume that the following condition holds*:

$$\text{if } L \subset X \text{ is compact, then } C(\hat{R}_1^L) \subset L \text{ and } C({}^\sim\hat{R}_1^L) \subset L. \tag{3}$$

*Let further* K-cap* *and* $\tilde{\mathrm{K}}$-cap* *stand for the usual outer* K-*capacity and the outer* $\tilde{\mathrm{K}}$-*capacity, respectively. If* $P \subset X$, *then the following conditions are equivalent*:

(*i*) *P is* $\mathcal{W}$-*polar*;

(*ii*) K-cap*$(P) = 0$;

(*iii*) *P is* $\tilde{\mathcal{W}}$-*polar*;

(*iv*) $\tilde{\mathrm{K}}$-cap*$(P) = 0$.

REMARK 4. In Theorem 2 $^{\sim}\hat{R}_1^L$ stands for the balayage of 1 on $L$ with respect to $(X, \mathcal{W})$.

If $(X, \mathcal{W})$ is a $\mathcal{P}$-harmonic space in the sense of [4], then the condition (3) is always fulfilled, see e.g. [4], p. 127.

*Proof of Theorem* 2. Let $P$ be $\mathcal{W}$-polar. By [1], p. 250 and p. 282, there exists a $\mathcal{W}$-polar $G_\delta$-set $P' \supset P$ such that $\hat{R}_1^{P'} = 0$. Let $L \subset P'$ be compact. Then $\hat{R}_1^L = 0$ and by Proposition $\mu_L(L) = \mathrm{K}\text{-cap}(L) = 0$. Since the set $P'$ is capacitable (see e.g. [1], pp. 27–31), K-cap*$(P') = 0$. Consequently, K-cap*$(P) = 0$.

Let K-cap*$(P) = 0$ and let $G_n$, $n \in \mathbb{N}$, be an open set such that $P \subset G_n$ and K-cap$_*(G_n) < 1/n$, $n \in \mathbb{N}$. (K-cap$_*$ denote the usual inner K-capacity.) Put $P' = \bigcap_{n=1}^{\infty}$. Then $P \subset P'$ and K-cap$_*(P') = 0$. Let $L \subset P'$ be compact. Then K-cap$(L) = 0$ and, by Proposition, $\hat{R}_1^L = 0$. According to [1], p. 282, $L$ is $\mathcal{W}$-polar. From [1], p. 284, it follows that $P'$ is $\mathcal{W}$-polar. Consequently, $P$ is $\mathcal{W}$-polar.

In view of the symmetry of assumptions it is possible to prove the equivalence of (iii) and (iv) in an analogical way.

The equivalence of (ii) and (iv) follows from Theorem 1.

REMARK 4. By Example 2, the assumptions of Theorem 2 are fulfilled specially in the case of potential theory for the Kolmogorov operator. Consequently, we obtain by Theorem 2 that the polar sets for the Kolmogorov operator, the polar sets for the adjoint Kolmogorov operator, i.e. for the operator $\partial^2/\partial x^2 - x(\partial/\partial y) + \partial/\partial t$, the sets of outer Kolmogorov capacity zero and the sets of outer adjoint Kolmogorov capacity zero coincide.

REFERENCES

[1] BLIEDTNER, J. and HANSEN, W., *Potential theory, an analytic and probabilistic approach to balayage*. Springer-Verlag, Berlin, 1986.

[2] BRELOT, M., *Lectures on potential theory*. Tata institute of fundamental research, Bombay, 1967.

[3] BRZEZINA, M., *Bases, essential bases and Wiener's criterion in balayage spaces*, (Thesis in Czech). Charles University, Praha, 1991.

[4] CONSTANTINESCU, C. and CORNEA, A., *Potential theory on harmonic spaces*. Springer-Verlag, Berlin, 1972.

[5] HELMS, L. L., *Introduction to potential theory*. Wiley Interscience, New York, 1969.

[6] Lanconelli, E., *Sul problema di Dirichlet per l'equazione del calore*. Ann. Math. Pura Appl. *97* (1973), 83–114.
[7] Ikegami, T., *Duality on harmonic spaces*. Osaka J. Math. *28* (1991), 93–116.
[8] Maeda, F. Y., *Capacities on harmonic spaces with adjoint structure*. In: *Potential theory* (Proceedings of a conference in Praha). Plenum Press, New York, 1988.
[9] Watson, N. A., *Green functions, potentials, and the Dirichlet problem for the heat equation*. Proc. London Math. Soc. (3)*33* (1976), 251–298.
[10] Watson, N. A., *Thermal capacity*. Proc. London Math. Soc. (3)*37* (1978), 342–362.

*Department of Mathematics,*
*University of Ostrava,*
*Bráfova 7,*
*ČS-701 03 Ostrava 1, Czechoslovakia*

*and*

*Mathematical Institute of University of Erlangen-Nürnberg,*
*Bismarckstrasse 1a,*
*D-8520 Erlangen, Germany*

EXPOSITIONES
MATHEMATICAE

Expo. Math. 11 (1993), 469-474
© Bibliographisches Institut &
F.A. Brockhaus AG 1993

Mathematical Notes

# Capacitary interpretation of the Fulks measure

Miroslav Brzezina*

**Abstract.** In this note we show that, with respect to the adjoint heat equation, the Fulks measure is the equilibrium measure for the heat ball.

Euclidean spheres in $\mathbb{R}^d$ and the surface measure play an important role in classical potential theory. Namely, the following well known mean value property holds:

*Let $U \subset \mathbb{R}^d$, $d \geq 2$, be an open set and $u$ be a harmonic function on $U$,* i.e. $u \in \mathscr{C}^2(U)$ *and $\Delta u := \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = 0$ in $U$. Then for any Euclidean ball $B_r(x)$ with center $x \in \mathbb{R}^d$ and radius $r > 0$ such that $\overline{B_r(x)} \subset U$*

$$u(x) = \int\limits_{\partial B_r(x)} u \, d\sigma_{x,r} \tag{1}$$

*holds, where $\sigma_{x,r}$ denotes the normalized surface measure on $\partial B_r(x)$.*
*Conversely, if a continuous function $u : U \to \mathbb{R}$ satisfies* (1) *for all $\overline{B_r(x)} \subset U$, then $u$ is harmonic in $U$.*

Speaking in potential-theoretic terms, the normalized surface measure $\sigma_{x,r}$ on $\partial B_r(x)$ is the balayage of the Dirac measure $\varepsilon_x$ at the point $x$ on the complement of $B_r(x)$, i.e.

$$\sigma_{x,r} = \varepsilon_x^{\complement B_r(x)} ; \tag{2}$$

for the notion of balayage of measures, see [4, pp. 138–139] or [1, pp. 113–115].

For $d \geq 3$, let $N : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ denote the *Newtonian kernel*, i.e. the function defined for $x, y \in \mathbb{R}^d$ by

$$N(x,y) := \begin{cases} [(d-2)\omega_d]^{-1} \cdot \|x-y\|^{2-d}, & x \neq y, \\ \infty & , \quad x = y; \end{cases}$$

* Financial support by the Gottlieb Daimler- and Karl Benz-Stiftung is gratefully acknowledged.

0723-0869/93/050469-06$2.70

here $\omega_d$ denotes the surface measure of $\partial B_1(0)$ and $\|\cdot\|$ Euclidean norm in $\mathbb{R}^d$. For a positive Radon measure $\mu \in \mathbb{R}^d$, the Newtonian potential of $\mu$ is defined by

$$N^\mu(x) := \int_{\mathbb{R}^d} N(x,y)\mu(dy), \quad x \in \mathbb{R}^d.$$

It follows from [9, p. 138] that for every compact set $K \subset \mathbb{R}^d$ there exists a uniquely determined Radon measure $\mu_K$ such that

$$N^{\mu_K} = \widehat{R}_1^K ;$$

here $\widehat{R}_1^K$ denotes the balayage of 1 on $K$ (cf. [9, p. 135]). The measure $\mu_K$ is called the *equilibrium measure* for $K$ and the number $\mu_K(K)$ the *Newtonian capacity* of $K$, cf. [3, p. 205]. An easy calculation shows that for $B := B_r(x)$,

$$\mu_{\overline{B}}(\overline{B}) = (d-2)\omega_d r^{d-2}.$$

This leads to another characterization of the normalized surface measure:

*Let $x \in \mathbb{R}^d$, $d \geq 3$, and $r > 0$. Denote $B := B_r(x)$. Then*

$$\sigma_{x,r} = \frac{r^{2-d}}{(d-2)\omega_d}\,\mu_{\overline{B}}\,.$$

The proof of this assertion can be found e.g. in [3, p. 205].

In the case of potential theory for the heat operator $H := \frac{\partial}{\partial_t} - \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^{d+1}$, $d \geq 1$, analogous assertion can be obtained. The situation in this case is, however, more complicated.

Denote by $W$ the *heat kernel* in $\mathbb{R}^{d+1}$, i.e. the function defined for $(x,t)$, $(y,s) \in \mathbb{R}^{d+1}$ by

$$W(x,t;y,s) := \begin{cases} (4\pi(t-s))^{-d/2}\exp\{-\|x-y\|^2/4(t-s)\}\,, & t > s\,, \\ 0 & , \quad t \leq s\,. \end{cases}$$

For $z = (x,t) \in \mathbb{R}^{d+1}$ and $c > 0$ we denote by $\Omega(z,c)$ the set of all points $(y,s) \in \mathbb{R}^{d+1}$ satisfying

$$W(x,t;y,s) > (4\pi c)^{-d/2}.$$

The set $\Omega(z,c)$ is called the *heat ball* of the center $z$ and radius $c$ and $B(z,c) := \partial\Omega(z,c)$ the *heat sphere*. We denote by $\sigma := \sigma_{B(z,c)}$ the surface measure on

$B(z,c)$ (i.e. the $d$-dimensional Hausdorff measure). For $z' = (x',t') \in \mathbb{R}^{d+1}$, define $Q : (\mathbb{R}^d \times ]0,\infty[) \cup \{0\} \to \mathbb{R}$ by

$$Q(x',t') := \begin{cases} \|x'\|^2[4\|x'\|^2 t'^2 + (\|x'\|^2 - 2dt')^2]^{-1/2}, & t' > 0, \\ 1 & , \quad (x',t') = 0. \end{cases}$$

Furthermore, for $z = (x,t)$ define the function $q_z(z') := Q(z - z')$, $z' \in \mathbb{R}^{d+1}$, on $(\mathbb{R}^d \times ]-\infty, t[) \cup \{z\}$ and put $q_{z,c} := q_z\big|_{B(z,c)}$. The positive Radon measure

$$\mu_{(z,c)} := (4\pi c)^{-d/2} q_{z,c} \sigma_{B(z,c)}$$

on $B(z,c)$ is called the *Fulks measure* for the heat ball $\Omega(z,c)$; cf. [2, pp. 70–71].

The heat spheres and the Fulks measures play for the heat equation the same role as the Euclidean spheres and normalized surface measures for the Laplace equation, respectively. W. Fulks, see [8], proved that the following form of the mean value property holds:

*Let $U \subset \mathbb{R}^{d+1}$, $d \geq 1$, be an open set and $u : U \to \mathbb{R}$ a caloric function on $U$,* i.e. $\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial u}{\partial t} \in \mathscr{C}(U)$, *$i = 1, \ldots, d$, and $Hu = 0$ in $U$. Then $u$ satisfies in $U$ the following mean value property:*

$$h(z) = \int h \, d\mu_{z,c} \tag{3}$$

*whenever $z \in U$, $c > 0$ and $\overline{\Omega(z,c)} \subset U$.*
*Conversely, if a continuous function $u : U \to \mathbb{R}$ satisfies* (3) *for all $\overline{\Omega(z,c)} \subset U$, then $u$ is caloric on $U$.*

In [2], H. Bauer proved the following theorem which in spirit corresponds to (2):

*For all $z \in \mathbb{R}^{d+1}$ and $c > 0$, the Fulks measure $\mu_{z,c}$ is the balayage of the Dirac measure $\varepsilon_z$ on the complement of $\Omega := \Omega(z,c)$,* i.e.

$$\mu_{z,c} = \varepsilon_z^{\complement\Omega}.$$

In the following, all notion concerning the adjoint heat potential theory, will be denoted by $^*$, e.g. $^*$thin, $^*$polar, $^*$supercaloric, etc. The $^*$*heat potential* (adjoint heat potential) of a positive Radon measure is defined by ($p \in \mathbb{R}^{d+1}$)

$$^*W^\mu(p) := \int_{\mathbb{R}^{d+1}} W(q;p)\, \mu(dq).$$

From [12, p. 348] it follows that for every compact set $K \subset \mathbb{R}^{d+1}$ there exists a uniquely determined Radon measure ${}^{\bullet}\mu_K$ (called the ${}^{\bullet}$*equilibrium measure* for $K$) such that

$$
{}^{\bullet}W^{{}^{\bullet}\mu_K} = {}^{\bullet}\widehat{R}_1^K \,;
$$

here, of course, ${}^{\bullet}\widehat{R}_1^K$ denotes the balayage of 1 on $K$ with respect to the adjoint heat equation. The number ${}^{\bullet}\mu_K(K)$ is called the ${}^{\bullet}$*capacity* of $K$.

Using the same reasoning as in [11, p. 51] and [5, p. 92], we get for $\Omega := \Omega(z,c)$

$$
{}^{\bullet}\mu_{\overline{\Omega}}(\overline{\Omega}) = (4\pi c)^{d/2}.
$$

In what follows, fix $z \in \mathbb{R}^{d+1}$ and $c > 0$ and consider the function

$$
w(p) := \min\{(4\pi c)^{-d/2},\ W(z\,;p)\}\,, \quad p \in \mathbb{R}^{d+1}.
$$

**Proposition 1.** *Let $z \in \mathbb{R}^{d+1}$ and $c > 0$. Denote $\Omega := \Omega(z,c)$. Then*

$$
{}^{\bullet}W^{\varepsilon_z^{\complement\Omega}} = w\,.
$$

**Proof.** Fix $p \in \mathbb{R}^{d+1}$. We have ${}^{\bullet}W^{\varepsilon_z^{\complement\Omega}}(p) = \int_{\mathbb{R}^{d+1}} W(\cdot\,;p)\, d\varepsilon_z^{\complement\Omega}$. Since $W_p(\cdot) := W(\cdot\,;p)$ is supercaloric on $\mathbb{R}^{d+1}$, we get from [3, p. 250] ${}^{\bullet}W^{\varepsilon_z^{\complement\Omega}}(p) = \widehat{R}_{W_p}^{\complement\Omega}(z)$. Denote ${}^{\bullet}W_z(\cdot) := W(z\,;\cdot)$. By [6, p. 342] we have

$$
{}^{\bullet}W^{\varepsilon_z^{\complement\Omega}}(p) = {}^{\bullet}\widehat{R}_{{}^{\bullet}W_z}^{\complement\Omega}(p)\,. \tag{4}
$$

If $\complement\Omega$ is not ${}^{\bullet}$thin at $p$, then

$$
{}^{\bullet}\widehat{R}_{{}^{\bullet}W_z}^{\complement\Omega}(p) = {}^{\bullet}W_z(p)\,. \tag{5}
$$

It follows from [7, p. 688] that (5) holds in particular at every $p \in \complement\Omega \setminus \{z^c\}$; here for $z = (x,t)$, $z^c = (x, t-c)$. Clearly, ${}^{\bullet}W_z(p) = (4\pi c)^{-d/2}$ for every $p \in \partial\Omega \setminus \{z\}$. Recall that ${}^{\bullet}\widehat{R}_{{}^{\bullet}W_z}^{\complement\Omega}$ coincides on $\Omega$ with the solution of the Dirichlet problem (with respect to adjoint theory) for ${}^{\bullet}W_z\big|_{\partial\Omega}$. Since constant functions are ${}^{\bullet}$caloric and $\{z\}$ is ${}^{\bullet}$polar, ${}^{\bullet}\widehat{R}_{{}^{\bullet}W_z}^{\complement\Omega}$ is equal to $(4\pi c)^{-d/2}$ on $\Omega$. We conclude that

$$
{}^{\bullet}\widehat{R}_{{}^{\bullet}W_z}^{\complement\Omega} = \min\{(4\pi c)^{-d/2}, {}^{\bullet}W_z\} \tag{6}
$$

on $\mathbb{R}^{d+1} \setminus \{z^c\}$. (Observe that ${}^{\bullet}W_z(z) = 0$, so the equality (6) is obvious at $z$.) Since $\{z^c\}$ is ${}^{\bullet}$polar, and both functions appearing at (6) are ${}^{\bullet}$supercaloric, (6) holds at $z^c$ as well. Now, from (4) and (6) the assertion follows.

**Lemma.** *Let $z \in \mathbb{R}^{d+1}$ and $c > 0$. Denote $\Omega := \Omega(z,c)$. Then*

$$^\bullet W^{{}^\bullet\mu_{\overline{\Omega}}} = (4\pi c)^{d/2} w\,.$$

**Proof.** We know that $^\bullet W^{{}^\bullet\mu_{\overline{\Omega}}} = {}^\bullet\widehat{R}_1^{\overline{\Omega}}$. Since $(4\pi c)^{d/2} w = 1$ on $\overline{\Omega} \setminus \{z\}$, $^\bullet\widehat{R}_1^{\overline{\Omega}} = {}^\bullet\widehat{R}_1^{\overline{\Omega}\setminus\{z\}} \leq (4\pi c)^{d/2} w$ on $\mathbb{R}^{d+1}$. The set $\overline{\Omega}$ is not $^\bullet$thin at any $p \in \partial\Omega\setminus\{z\}$, consequently,

$$\liminf_{q\to p,\, q\in\complement\overline{\Omega}} ({}^\bullet\widehat{R}_1^{\overline{\Omega}}(q) - (4\pi c)^{d/2} w(q)) \geq 0$$

for all $p \in \partial\Omega$. By minimum principle, see [3, p. 116], $(4\pi c)^{d/2} \leq {}^\bullet\widehat{R}_1^{\overline{\Omega}}$ on $\mathbb{R}^{d+1}$.

The following result has been obtained by N.A. Watson, [13, p. 249].

**Proposition 2.** *Let $z \in \mathbb{R}^{d+1}$ and $c > 0$. Denote $\Omega := \Omega(z,c)$. Then*

$$^\bullet W^{\mu_{z,c}} = w\,.$$

Now, we are in position to prove our main result.

**Theorem.** *Let $z \in \mathbb{R}^{d+1}$ and $c > 0$. Denote $\Omega := \Omega(z,c)$. Then*

$$\mu_{z,c} = \varepsilon_z^{\complement\Omega} = (4\pi c)^{-d/2}\, {}^\bullet\mu_{\overline{\Omega}}\,.$$

**Proof.** By Proposition 1, Proposition 2 and Lemma, we have

$$^\bullet W^{\mu_{z,c}} = {}^\bullet W^{\varepsilon_z^{\complement\Omega}} = (4\pi c)^{-d/2}\, {}^\bullet W^{{}^\bullet\mu_{\overline{\Omega}}}.$$

Now, from uniqueness of representation of $^\bullet$heat potentials, see [6, p. 305], the desired equalities follow.

**Remarks.** The equality $\mu_{z,c} = \varepsilon_z^{\complement\Omega}$ from Theorem was proved by H. Bauer in [2]. An alternative proof of this equality (based on a simpliciality argument) has been given by I. Netuka in [10]. Our considerations provide a new proof of the above using the result by N.A. Watson stated in Proposition 2. On the other hand, Bauer's results and our Proposition 1 combined give an alternative proof of Proposition 2.

## References

[1] Bauer, H., Harmonische Räume und ihre Potentialtheorie, Lecture Notes in Mathematics, 22, Springer Verlag, Berlin, 1966.

[2] Bauer, H., Heat balls and Fulks measures, Ann. Acad. Sci. Fenn. Ser. A.I Math. *10* (1985), 62–82.

[3] Bliedtner, J., Hansen, W., Potential theory. An Analytic and Probalistic Approach to Balayage, Springer Verlag, Berlin, 1986.
[4] Brelot, M., Éléments de la théorie classique du potentiel, 4$^{e}$ ed., Centre de Documentation Universitaire, Paris, 1969.
[5] Brzezina, M., Kernels and Choquet capacities, Aequationes Math. *45* (1992), 89–99.
[6] Doob, L.J., Classsical potential theory and its probalistic counterpart, Springer Verlag, Berlin, 1984.
[7] Effros, G.E., Kazdan, L.J., On the Dirichlet problem for heat equation, Indiana Univ. Math. J. *20* (1971), 683–693.
[8] Fulks, W., A mean value theorem for the heat equation, Proc. Amer. Math. Soc. *17* (1966), 6–11.
[9] Helms, L.L., Introduction to potential theory, Wiley Interscience, New York, 1969.
[10] Netuka, I., Fine behaviour of solution of the Dirichlet problem near an irregular point, Bull. Sci. Math. *114* (1990), 1–22.
[11] Scornazzani, V., Sul problema di Dirichlet per l'operatore di Kolmogorov, Bollettino U.M.I. *18-C* (1981), 43–62.
[12] Watson, N.A., Thermal capacity, Proc. London Math. Soc. *(3)37* (1978), 342–362.
[13] Watson, N.A., A convexity theorem for local mean values of subtemperatures, Bull. London Math. Soc. *22* (1990), 245–252.

Received: 07.09.92

Permanent address:
Department of Mathematics
University of Ostrava
Bráfova 7
701 03 Ostrava 1
Czech Republic

Mathematical Institute
University of Erlangen-Nürnberg
Bismarckstrasse 1 a
D-8520 Erlangen

manuscripta math. 80, 21 - 26 (1993)

manuscripta
mathematica


# Tusk type conditions and thinness for the parabolic operator of order $\alpha$

Miroslav Brzezina [1]

M.Nishio proved in [12] the Wiener criterion of the regularity for the parabolic operator of order $\alpha \in ]0,1[$ in Euclidean space $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, i.e. for the operator

$$L^{(\alpha)} := \partial/\partial t + (-\Delta)^{\alpha};$$

here $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ is the Laplace operator and $(-\Delta)^{\alpha}$ is the $\alpha$-fractional power of $-\Delta$ on $\mathbb{R}^n$. In this note, using methods from [3], the Wiener criterion of thinness and a generalization of the Poincaré type condition of regularity from [8] and [12] are given.

Let $n \in \mathbb{N}$, $\alpha \in ]0,1[$ and let $P = (P_t)_{t>0}$ be the *Brownian semigroup* on $\mathbb{R}^n$, i.e. $P_t$ is the kernel having the density

$$(4\pi t)^{-n/2} \exp(-\|x\|^2/4t)$$

with respect to the Lebesgue measure in $\mathbb{R}^n$ where $x \in \mathbb{R}^n$, $t > 0$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Further, let $(\sigma_t^{\alpha})_{t>0}$ be the *one-sided stable semigroup of order* $\alpha$ on $]0,\infty[$, i.e. a convolution semigroup on $]0,\infty[$ which is uniquely determined by the condition

$$\mathcal{L}\sigma_t^{\alpha}(s) = \exp(-ts^{\alpha})$$

for every $t$, $s > 0$; here $\mathcal{L}$ denotes the Laplace transform. The *symmetric stable semigroup* $P^{\alpha} = (P_t^{\alpha})_{t>0}$ of index $\alpha$ on $\mathbb{R}^n$ is defined for every positive Borel-measurable function $f$ on $\mathbb{R}^n$ by

$$P_t^{\alpha} f := \int_{\mathbb{R}} P_s f \, \sigma_t^{\alpha}(ds).$$

Denote the tensor-product of $P^{\alpha}$ and the translation semigroup $T$ on $\mathbb{R}$ by $P^{\alpha} \otimes T$ (the so called *heat semigroup of order* $\alpha$) and the set of all excessive

[1] Financial support by the Gottlieb Daimler und Karl Benz- Stiftung is gratefully acknowledged.

functions with respect to the semigroup $P^\alpha \otimes T$ by $E_{P^\alpha \otimes T}$ . For the notions used above and for further notions and notations concerning balayage spaces we refer to the monograph by J.Bliedtner and W.Hansen [1]. The following result was proved in [5].

Proposition. *Let $\alpha \in ]0,1[$. Then $(\mathbb{R}^{n+1}, E_{P^\alpha \otimes T})$ is a balayage space which does not satisfy the axiom of polarity, i.e. the systems of polar and semipolar sets with respect to $(\mathbb{R}^{n+1}, E_{P^\alpha \otimes T})$ do not coincide.*

Let $W^{(\alpha)}$ be the fundamental solution of the operator $L^{(\alpha)}$, i.e. for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$

$$W^{(\alpha)}(x,t) := \begin{cases} (4\pi)^{-n} \int_{\mathbb{R}^n} \exp\{-t\|\xi\|^{2\alpha} + ix\cdot\xi\}\, d\xi, & t > 0, \\ 0 & , t \le 0, \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$.

The *$(\alpha)$-parabolic capacity* of a compact set $K \subset \mathbb{R}^{n+1}$ is defined by

$${}^{(\alpha)}\mathrm{cap}(K) := \sup\{\mu(X);\, \mu \in \mathcal{M}^+(K),\, W^{(\alpha)} * \mu \le 1 \text{ on } \mathbb{R}^{n+1}\};$$

here $\mathcal{M}^+(K)$ denotes the set of all positive Radon measures on $\mathbb{R}^{n+1}$ with support in $K$ and $W^{(\alpha)} * \mu$ denotes the convolution of $W^{(\alpha)}$ with $\mu$. Let $E \subset \mathbb{R}^{n+1}$ be an arbitrary set. Then

$${}^{(\alpha)}\mathrm{cap}_*(E) := \sup\{{}^{(\alpha)}\mathrm{cap}(K);\, K \subset E,\, K \text{ compact }\}$$

is called the *inner $(\alpha)$-parabolic capacity* of $E$ and

$${}^{(\alpha)}\mathrm{cap}^* := \inf\{{}^{(\alpha)}\mathrm{cap}_*(G);\, G \supset E,\, G \text{ open }\}$$

the *outer $(\alpha)$-parabolic capacity* of $E$.

Lemma 1. *The $(\alpha)$-parabolic capacity is a Choquet capacity on $\mathbb{R}^{n+1}$, i.e. ${}^{(\alpha)}cap$ is an increasing, strongly subadditive and right continuous positive set function defined on all compact subsets of $\mathbb{R}^{n+1}$.*

*Proof.* See [12], p.171. □

For any $\lambda > 0$, the *$(\alpha)$-parabolic dilation* ${}^{(\alpha)}\tau_\lambda$ is defined by

$${}^{(\alpha)}\tau_\lambda : (x,t) \mapsto (\lambda x, \lambda^{2\alpha} t), \quad x \in \mathbb{R}^n,\, t \in \mathbb{R}.$$

Lemma 2. *(i) Let $K$ be a compact subset of $\mathbb{R}^{n+1}$ and $\lambda > 0$. Then*

$${}^{(\alpha)}cap({}^{(\alpha)}\tau_\lambda(K)) = \lambda^n \cdot {}^{(\alpha)}cap(K).$$

*(ii) Let $L$ be a compact subset of $\mathbb{R}^n$, $t \in \mathbb{R}$ and let $m_n$ stand for the n-dimensional Lebesgue measure. Then*

$${}^{(\alpha)}cap(L \times \{t\}) = m_n(L).$$

*Proof.* See [12], p.172. □

For $r \in ]0,1]$ and $z = (x,t) \in \mathbb{R}^{n+1}$ we set

$$B^r(z) := \overline{\{(y,s) \in \mathbb{R}^{n+1};\ W^{(\alpha)}(x-y,t-s) \geq r^{-n/2\alpha}\}},$$

the so called *heat ball of order* $\alpha$. Especially, for $\lambda \in ]0,1[$ and $k \in \mathbb{N}$ we write $B_k(z,\lambda)$ instead of $B^{\lambda^k}(z)$. Further, put $A_k(z,\lambda) := \overline{B_k(z,\lambda) \setminus B_{k+1}(z,\lambda)}$.

We recall that a set $E \subset \mathbb{R}^{n+1}$ is called $(\alpha)$*-thin at a point* $z \in \mathbb{R}^{n+1}$, if ${}^{(\alpha)}\varepsilon_z^E$, the balayage of Dirac measure $\varepsilon_z$ on $E$ (with respect to the balayage space $(\mathbb{R}^{n+1}, E_{P^\alpha \otimes T}))0$ is different from $\varepsilon_z$.

Let $U$ be an open subset of $\mathbb{R}^{n+1}$. A boundary point $z \in \partial U$ is called $(\alpha)$*-regular*, if ${}^{(\alpha)}\varepsilon_z^{CU} = \varepsilon_z$; here $CU := \mathbb{R}^{n+1} \setminus U$.

Remark 1. In [12], another definition of $(\alpha)$-regular points is used. It follows from [12], p.167 and [1], p.348, that both definitions are actually equivalent.

As shown in [3] in the context of $\mathcal{P}$-harmonic spaces, the Wiener test for regularity provides a Wiener test for thinness. Using methods from [3] and Nishio's result from [12], p.178, we obtain in this situation the following result (for details, see [4]).

Theorem. *Let $E$ be an arbitrary subset of $\mathbb{R}^{n+1}$, $z \in \mathbb{R}^{n+1}$, and $\alpha, \lambda \in ]0,1[$. Then the following conditions are equivalent:*

(*i*) $E$ *is* $(\alpha)$*-thin at* $z$;

(*ii*) $$\sum_{k=1}^{\infty} \lambda^{-kn/2\alpha}\, {}^{(\alpha)}cap^*(E \cap B_k(z,\lambda)) < \infty;$$

(*iii*) $$\sum_{k=1}^{\infty} \lambda^{-kn/2\alpha}\, {}^{(\alpha)}cap^*(E \cap A_k(z,\lambda)) < \infty;$$

(*iv*) $$\int_0^1 {}^{(\alpha)}cap^*(E \cap B^t(z))/t^{n/2\alpha+1}\, dt < \infty.$$

Corollary 1. *Let $E$ be an arbitrary subset of $\mathbb{R}^{n+1}$, $z \in \mathbb{R}^{n+1}$ and $\alpha \in ]0,1[$. If $E$ is $(\alpha)$-thin at the point $z$, then*

$$\lim_{t\to 0+} {}^{(\alpha)}cap^*(E \cap B^t(z))/t^{n/2\alpha} = 0.$$

*Proof.* By our Theorem, the integral in (iv) is convergent. Choose $\varepsilon > 0$. Then there exists $t_0 \in ]0,1[$ such that

$$\int_{t/2}^{t} {}^{(\alpha)}\mathrm{cap}^*(E \cap B^t(z))/t^{n/2\alpha+1}\, dt < \varepsilon,$$

whenever $t \in ]0,t_0[$. Consequently,

$$^{(\alpha)}\mathrm{cap}^*(E \cap B^t(z))/t^{n/2\alpha} < 2\varepsilon$$

for all $t \in ]0, t_0[$. □

For $r > 0$ and $z = (x, t) \in \mathbb{R}^{n+1}$, we set

$$D(z, r) := \{(y, s) \in \mathbb{R}^{n+1}; t - r \leq s \leq t\}.$$

Let $E$ be an arbitrary subset of $\mathbb{R}^{n+1}$, $z = (x, t) \in \mathbb{R}^{n+1}$ and let $\alpha \in ]0, 1]$. The set $E$ is said to *lie $(\alpha)$-parabolically below $z$* provided there exists $c > 0$ such that

$$s - t \leq -c||y - x||^{2\alpha}$$

for all $(y, s) \in E$.

The following corollary is analogous to Corollary 1. Heat balls of order $\alpha$ are replaced by the strips $D(z, r)$.

Corollary 2. *Let $z \in \mathbb{R}^{n+1}$, $\alpha \in ]0, 1[$ and let $E \subset \mathbb{R}^{n+1}$ lie $(\alpha)$-parabolically below $z$. If $E$ is $(\alpha)$-thin at the point $z$, then*

$$\lim_{r \to 0+} {}^{(\alpha)}cap^*(E \cap D(z, r))/r^{n/2\alpha} = 0.$$

*Proof.* We may assume that $z = 0$, i.e. there exists $c > 0$ such that $s < -c||y||^{2\alpha}$ for all $(y, s) \in E$. If $(y, s) \in \mathbb{R}^{n+1}$, $y \neq 0$, $s = -c||y||^{2\alpha}$, then

$$W^{(\alpha)}(-y, -s) = c^{-n/2\alpha} \cdot \tilde{c}^{-n/2\alpha} \cdot ||y||^{-n},$$

with $\tilde{c} = W^{(\alpha)}(-c^{-1/2\alpha} \cdot e, 1)$; here $e = (1, 0, \ldots, 0)$. Consequently,

$$E \cap D(0, r) \subset E \cap B^{\tilde{c} \cdot r}(0).$$

Now the assertion follows from Corollary 1. □

Corollary 3. *Let $B$ be a subset of $\mathbb{R}^n$, $z = (x, t) \in \mathbb{R}^{n+1}$, $\alpha \in ]0, 1[$ and $T \subset ]0, \infty[$. If $\inf T = 0$ and the set*

$$E := (x, t) + \{{}^{(\alpha)}\tau_r(y - x, -1); r \in T, y \in B\}$$

*is $(\alpha)$-thin at the point $z$, then $m_n(B) = 0$.*

*Proof.* We can assume that $B$ is a bounded set, hence $E$ lies $(\alpha)$-parabolically below $z$.

Let $\inf T = 0$. Then there exists a decreasing sequence $(r_j)_{j=1}^{\infty}$ such that $r_j \in T$, $j \in \mathbb{N}$, and $\lim_{j \to \infty} r_j = 0$. Consequently,

$${}^{(\alpha)}\tau_{r_j}(B \times \{-1\}) \subset E \cap D(z, r_j^{2\alpha}).$$

By Lemma 1 and Lemma 2(i), we get for all $j \in \mathbb{N}$

$$r_j^n \cdot {}^{(\alpha)}\mathrm{cap}^*(B \times \{-1\}) = {}^{(\alpha)}\mathrm{cap}^*({}^{(\alpha)}\tau_{r_j}(B \times \{-1\})) \leq {}^{(\alpha)}\mathrm{cap}^*(E \cap D(z, r_j^{2\alpha})).$$

Since $E$ lies $(\alpha)$-parabolically below $z$, we obtain from Corollary 2 that ${}^{(\alpha)}\mathrm{cap}^*(B \times \{-1\}) = 0$. Now, it follows from Lemma 2(ii) that $m_n^*(B) = {}^{(\alpha)}\mathrm{cap}^*(B \times \{-1\}) = 0$ (here $m_n^*$ denotes the outer n-dimensional Lebesgue measure) and, consequently, $m_n(B) = 0$. $\square$

**Remark 2.** From Corollary 3 we obtain easily the following generalization of the tusk condition of regularity from [12]:
*Let $U$ be an open subset of $\mathbb{R}^{n+1}$, $\alpha \in ]0,1[$ and let $z = (x,t) \in \partial U$. Further, let $B$ be a subset of $\mathbb{R}^n$ and $T \subset ]0,\infty[$. If $\inf T = 0$, $m_n^*(B) > 0$ and*

$$(x,t) + \{{}^{(\alpha)}\tau_r(y-x,-1);\, r \in T,\, y \in B\} \subset \mathbb{R}^{n+1} \setminus U,$$

*then $z$ is an $(\alpha)$-regular point of $U$.*

**Remark 3.** We recall that a set $E \subset \mathbb{R}^{n+1}$ is called *$(\alpha)$-semipolar at a point* $z \in \mathbb{R}^{n+1}$ if there exists an $(\alpha)$-semipolar set $S$ (i.e. semipolar with respect to the balayage space $(\mathbb{R}^{n+1}, E_{P^\alpha \otimes T})$) such that the set $E \setminus S$ is $(\alpha)$-thin at the point $z$.
Now, we formulate a result analogous to Corollary 3 concerning sets $(\alpha)$-semipolar at a point.
*Let $B$ be a subset of $\mathbb{R}^n$, $z = (x,t) \in \mathbb{R}^{n+1}$, $\alpha \in ]0,1[$ and $T \subset ]0,\infty[$. If $]0,\varepsilon[ \cap T$ is uncountable for every $\varepsilon > 0$ and the set*

$$E := (x,t) + \{{}^{(\alpha)}\tau_r(y-x,-1);\, r \in T,\, y \in B\}$$

*is $(\alpha)$-semipolar at the point $z$, then $m_n(B) = 0$.*
The proof of this assertion, a Wiener test for sets $(\alpha)$-semipolar at a point and a relation to the Choquet boundary, can be found in [4].

The above mentioned tusk conditions are valid also for potential theory of the heat operator, i.e. in the limit case $\alpha = 1$; see [2].

In this case, taking $B$ a ball and $T = ]0,\infty[$ in the assertion of Remark 2, we obtain the tusk condition from [6]. If the set $T$ in Corollary 2 is countable and shrinkable to 0 (i.e. $\lambda T \subset T$ for arbitrarily small $\lambda > 0$), then we get the assertion from [7]. Corollary 2 was proved in [11]. For another proof, see [2], similar results can be found in [9], [10].

Consider again potential theory of the heat operator of order $\alpha$. Choosing $B$ non–empty open and $T = ]0,p[$, $p > 0$, in the assertion of Remark 2, we obtain the Theorem from [8], p.18. Taking $B$ a Borel set with positive Lebesgue measure and $T = ]0,p[$, $p > 0$, in the assertion of Remark 2, we get Corollary 4.4 from [12], p.178.

## References

1. Bliedtner, J., Hansen, W.: Potential theory. An Analytic and Probabilistic Approach to Balayage. Springer-Verlag, Berlin, 1986
2. Brzezina, M.: On the base and the essential base in parabolic potential theory. Czechoslovak Math. J. **40**, 87-103 (1990)
3. Brzezina, M: Wiener's test of thinness in potential theory. Comm. Math. Univ. Carolinae **31**, 227-232 (1990)
4. Brzezina, M.: Bases, essential bases and Wiener's criterion in balayage spaces, (Thesis in Czech). Faculty of Mathematics and Physics, Charles University, Prag, 1991
5. Brzezina, M.: On a class of translation invariant balayage spaces. Exposition. Math. **11**, 181-184 (1993)
6. Effros, G.E, Kazdan, L.J.: On the Dirichlet problem for heat equation. Indiana Univ. Math. J. **20**, 683-693 (1971)
7. Hansen, W.: Fegen und Dünnheit mit Anwendungen auf die Laplace- und Wärmeleitungsgleichung. Ann. Inst. Fourier (Grenoble) **21**, 79-121 (1971)
8. Itô, M, Nishio, M.: Poincaré type conditions of regularity for the parabolic operator of order $\alpha$. Nagoya Math. J. **115**, 1-22 (1989)
9. Korányi, A., Taylor, J.C.: Fine convergence and parabolic convergence for the Helmholtz equation and the heat equation. Illinois J. Math. **27**, 77-93 (1983)
10. Lanconelli, E.: Sul problema di Dirichlet per l'equatione del calore. Ann. Mat. Pura Appl., IV.Ser. **97**, 83-114 (1973)
11. Netuka, I.: Thinness and the heat equation. Časopis Pěst. Mat. **99**, 293-299 (1974)
12. Nishio, M.: The Wiener criterion of regular points for the parabolic operator of order $\alpha$. Nagoya Math. J. **116**, 163-179 (1989)

Miroslav Brzezina
Department of Mathematics
University of Ostrava
Bráfova 7,
701 03 Ostrava 1
Czech Republic
and
Mathematical Institute
University of Erlangen–Nürnberg,
Bismarckstrasse 1a
D 8520 Erlangen
Federal Republic of Germany

This article was processed by the author
using the Springer-Verlag TEX mamath macro package 1990.

(Received October 29, 1992)

# On capacities related to the Riesz kernel

Miroslav Brzezina *

**Abstract.** For $\alpha > 0$, let $K^\alpha$ be the Riesz kernel of order $\alpha$, ${}^+K^\alpha(t,s) := K^\alpha(t,s) \cdot 1_{]0,\infty[}(t-s)$ and ${}^-K^\alpha(t,s) := {}^+K^\alpha(s,t)$, $t, s \in \mathbb{R}$. Denote by ${}^\alpha\text{cap}$, ${}^\alpha_+\text{cap}$ and ${}^\alpha_-\text{cap}$, the capacities deduced from the kernels $K^\alpha$, ${}^+K^\alpha$ and ${}^-K^\alpha$, respectively.
For $\alpha \in [1/2, 1[$, it is shown that the sets of zero capacity for ${}^\alpha\text{cap}$, ${}^\alpha_+\text{cap}$ and ${}^\alpha_-\text{cap}$ are the same. This give a positive answer to a question by R. Kaufman and J. M. Wu from 1982.

For a set $E \subset \mathbb{R}^n$, $n \in \mathbb{N}$, let us denote by $\mathcal{M}^+(E)$ the collection of all nonnegative Radon measures on $\mathbb{R}^n$ with support in $E$. The support of such a measure $\mu$ is denoted by supp $\mu$. A lower semicontinuous function $\mathrm{K} : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ is called a **kernel** on $\mathbb{R}^n$. The K-potential of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ is defined as

$$\mathrm{K}_\mu(x) := \int_{\mathbb{R}^n} \mathrm{K}(x,y)\,\mu(dy), \quad x \in \mathbb{R}^n.$$

For a compact set $L \subset \mathbb{R}^n$, the **K-capacity** is defined by

$$\text{K-cap}(L) := \sup\{\mu(L);\ \mu \in \mathcal{M}^+(L),\ \mathrm{K}_\mu \le 1 \text{ on } \mathbb{R}^n\}.$$

For $\alpha > 0$, denote by

$$K^\alpha(t,s) := \begin{cases} |t-s|^{-\alpha}, & \text{if } t \ne s, \\ \infty, & \text{if } t = s, \end{cases}$$

the **Riesz kernel of order** $\alpha$. Further define ${}^+K^\alpha(t,s) := K^\alpha(t,s) \cdot 1_{]0,\infty[}(t-s)$ and ${}^-K^\alpha(t,s) := {}^+K^\alpha(s,t)$, $t, s \in \mathbb{R}$. Obviously, the functions $K^\alpha$, ${}^+K^\alpha$ and ${}^-K^\alpha$ are kernels on $\mathbb{R}$. The corresponding capacities will be denoted by ${}^\alpha\text{cap}$, ${}^\alpha_+\text{cap}$ and ${}^\alpha_-\text{cap}$. As usual we call ${}^\alpha\text{cap}$ the **Riesz capacity of order** $\alpha$.

In [6], pp.213–214, R. Kaufman and J. M. Wu asked whether sets of zero capacity with respect to each of these three capacities under consideration do coincide. They proved that this holds if $\alpha = 1/2$. Their proof depends very much on series of results of the potential theory of the heat operator. Further, the authors remark: " It seems interesting to investigate their relationship in the case $\alpha \ne 1/2$, when ${}^+K^\alpha$ and the fundamental solution of the heat equation are unrelated. However the authors have no conjecture on this point."

* Financial support by the Gottlieb Daimler- und Karl Benz- Stiftung is gratefully acknowledged.

For $\alpha \in [1/2, 1[$, we show in this note that the answer to Kaufman's and Wu's question is positive. Our proof is essentially based on the potential theory for the heat operator of order $\gamma$, $\gamma \in ]0,1[$, i.e. for the operator

$$L^{(\gamma)} := \frac{\partial}{\partial t} + (-\Delta)^{\gamma}$$

in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$; here $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^n$ and $(-\Delta)^{\gamma}$ is the $\gamma$-fractional power of $-\Delta$ on $\mathbb{R}^n$.

Let $W^{(\gamma)}$ be the fundamental kernel of the operator $L^{(\gamma)}$, i.e. for $x, y \in \mathbb{R}^n$, $t, s \in \mathbb{R}$,

$$W^{(\gamma)}(x,t;y,s) := \begin{cases} 0, & \text{if } t-s \leq 0, \\ (4\pi)^{-n} \int_{\mathbb{R}^n} \exp(-(t-s)\|\xi\|^{2\gamma} + i(x-y)\cdot\xi)\, d\xi, & \text{if } t-s > 0, \end{cases}$$

where $(x-y)\cdot\xi$ denotes the inner product on $\mathbb{R}^n$ and $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$. We see easily that the fundamental kernel ${}^{*}W^{(\gamma)}$ of the adjoint operator ${}^{*}L^{(\gamma)}$, i.e. ${}^{*}L^{(\gamma)} := -\frac{\partial}{\partial t} + (-\Delta)^{\gamma}$, is given by ${}^{*}W^{(\gamma)}(x,t;y,s) = W^{(\gamma)}(y,s;x,t)$, $x, y \in \mathbb{R}^n$, $t, s \in \mathbb{R}$. Obviously, $W^{(\gamma)}$ and ${}^{*}W^{(\gamma)}$ are kernels on $\mathbb{R}^{n+1}$. The corresponding $W^{(\gamma)}$-capacity and ${}^{*}W^{(\gamma)}$-capacity will be denoted by ${}^{(\gamma)}\mathrm{cap}$ and ${}^{*(\gamma)}\mathrm{cap}$ and called the **heat capacity of order** $\gamma$ and the **adjoint heat capacity of order** $\gamma$, respectively.

In a natural way, the operators $L^{(\gamma)}$ and ${}^{*}L^{(\gamma)}$ generate balayage spaces $(\mathbb{R}^{n+1}, \mathcal{W}^{(\gamma)})$ and $(\mathbb{R}^{n+1}, {}^{*}\mathcal{W}^{(\gamma)})$, respectively; see e.g. [2], [4]. (For the definition of a balayage space, we recommend the monograph by J. Bliedtner and W. Hansen [1].)

By M. Nishio, see [7], p.170, we have the following

**Lemma 1.** *Let $n \in \mathbb{N}$ and $\gamma \in ]0,1[$. Further, let $L$ be a compact set in $\mathbb{R}^{n+1}$. Then there exists exactly one measure $\mu_L \in \mathcal{M}^{+}(L)$ such that*

$${}^{(\gamma)}\widehat{R}_1^L = W_{\mu_L}^{(\gamma)}\mu_L(L) = {}^{(\gamma)}cap(L).$$

*Further,*

$$\mu_L(L) = {}^{(\gamma)}cap(L) \quad and \quad {}^{(\gamma)}cap(L) = {}^{*(\gamma)}cap(L).$$

*Here ${}^{(\gamma)}\widehat{R}_1^L$ is the balayage of 1 on $L$ with respect to $(\mathbb{R}^{n+1}, \mathcal{W}^{(\gamma)})$.*

We recall that a subset $P$ of a balayage space $(X, \mathcal{W})$ is called **polar** (more precisely: $\mathcal{W}$-polar) if there exists a function $v \in \mathcal{W}$ such that $v = \infty$ on $P$ but $\{v < \infty\}$ is dense in X. Now, we can formulate the following

**Proposition 1.** *Let $n \in \mathbb{N}$ and $\gamma \in ]0,1[$. Let further ${}^{(\gamma)}cap^{*}$ and ${}^{*(\gamma)}cap^{*}$ stand for the usual outer $W^{(\gamma)}$-capacity and the outer ${}^{*}W^{(\gamma)}$-capacity, respectively. If $P \subset \mathbb{R}^{n+1}$, then the following conditions are equivalent:*

2

*(i)* $P$ *is* $\mathcal{W}^{(\gamma)}$*-polar;*

*(ii)* ${}^{(\gamma)}cap^{*}(P) = 0$*;*

*(iii)* $P$ *is* ${}^{*}\mathcal{W}^{(\gamma)}$*-polar;*

*(iv)* ${}^{*(\gamma)}cap^{*}(P) = 0$.

PROOF. (Cf. [3], Theorem 2.)
Let $P$ be $\mathcal{W}^{(\gamma)}$-polar. By [1], p.250 and p.282, there exists a $\mathcal{W}^{(\gamma)}$-polar $G_\delta$-set $P' \supset P$ such that ${}^{(\gamma)}\widehat{R}_1^{P'} = 0$. Let $L \subset P'$ be compact. Then ${}^{(\gamma)}\widehat{R}_1^{L} = 0$ and by Lemma 1 $\mu_L(L) = {}^{(\gamma)}\mathrm{cap}(L) = 0$. Since the set $P'$ is capacitable (see e.g. [1], pp.27–31) and ${}^{(\gamma)}\mathrm{cap}$ is a Choquet capacity (see [7], p. 171), ${}^{(\gamma)}\mathrm{cap}^{*}(P') = 0$. Consequently, ${}^{(\gamma)}\mathrm{cap}^{*}(P) = 0$.
Let ${}^{(\gamma)}\mathrm{cap}^{*}(P) = 0$ and let $G_k$, $k \in \mathbb{N}$, be an open set such that $P \subset G_k$ and ${}^{(\gamma)}\mathrm{cap}_{*}(G_k) < \frac{1}{k}$ (here ${}^{(\gamma)}\mathrm{cap}_{*}$ denote the usual inner $W^{(\gamma)}$-capacity). Put $P' = \bigcap_{k=1}^{\infty} G_k$. Then $P \subset P'$ and ${}^{(\gamma)}\mathrm{cap}_{*}(P') = 0$. Let $L \subset P'$ be compact. Then ${}^{(\gamma)}\mathrm{cap}(L) = 0$ and, by Lemma 1, ${}^{(\gamma)}\widehat{R}_1^{L} = 0$. According to [1], p.282, $L$ is $\mathcal{W}^{(\gamma)}$-polar. From [1], p.284, it follows that $P'$ is $\mathcal{W}^{(\gamma)}$-polar. Consequently, $P$ is $\mathcal{W}^{(\gamma)}$-polar.
The equivalence of (iii) and (iv) can be proved in an analogous way using the adjoint version of Lemma 1.
The equivalence of (ii) and (iv) follows from Lemma 1. □

**Remark 1.** An alternative proof of Proposition 1 follows from [3], Theorem 2 and results of Ikegami [4].

In what follows we assume that $n = 1$. For $\gamma \in ]0,1[$, we denote

$$c(\gamma) := \frac{1}{4\pi} \int_{\mathbb{R}} \exp(-|\eta|^{2\gamma})\, d\eta.$$

**Lemma 2.** *Let* $\gamma \in ]1/2, 1[$ *and* $L$ *be a compact subset of* $\mathbb{R}$*. Put* $\alpha := \frac{1}{2\gamma}$*. Then*

$$c(\gamma) \cdot {}^{(\gamma)}cap(\{0\} \times L) = {}^{\alpha}_{+}cap(L) = {}^{\alpha}_{-}cap(L).$$

PROOF. For $t, s \in \mathbb{R}$, we have

$$W^{(\gamma)}(0,t;0,s) = c(\gamma) \cdot {}^{+}K^{\alpha}(t,s).$$

Put $K := \{0\} \times L$. By Lemma 1, there exists a measure $\mu_K \in \mathcal{M}^{+}(K)$ such that $\mu_K(K) = {}^{(\gamma)}\mathrm{cap}(K)$ and $W^{(\gamma)}_{\mu_K} \le 1$ on $\mathbb{R} \times \mathbb{R}$. Since supp $\mu_K \subset \{0\} \times L$, we can assume

3

that $\mu_K = \varepsilon_0 \otimes \mu$ for some $\mu \in \mathcal{M}^+(L)$. (Here $\varepsilon_0$ denotes the Dirac measure at the point 0.) Further,

$$c(\gamma) \cdot {}^+K^\alpha_\mu(t) = W^{(\gamma)}_{\mu_K}(0,t) \le 1$$

for all $t \in \mathbb{R}$. Consequently,

$$c(\gamma) \cdot {}^{(\gamma)}\mathrm{cap}(\{0\} \times L) \le {}^\alpha\mathrm{cap}(L).$$

Let $\mu \in \mathcal{M}^+(L)$ such that ${}^+K^\alpha_\mu \le 1$. If $\nu = \varepsilon_0 \otimes \mu$, then

$$\frac{1}{c(\gamma)} W^{(\gamma)}_\nu(0,t) \le 1, \quad t \in \mathbb{R}.$$

Since $W^{(\gamma)}_\nu(x,t) \le W^{(\gamma)}_\nu(0,t)$ for all $x, t \in \mathbb{R}$ (this follows from [5], p.2), we have

$$\frac{\nu(\{0\} \times L)}{c(\gamma)} \le {}^{(\gamma)}\mathrm{cap}(\{0\} \times L).$$

Consequently, ${}^\alpha_+\mathrm{cap}(L) \le c(\gamma) \cdot {}^{(\gamma)}\mathrm{cap}(\{0\} \times L)$.
In an analogous way, we can prove the equality

$$c(\gamma) \cdot {}^{*(\gamma)}\mathrm{cap}(\{0\} \times L) = {}^\alpha_-\mathrm{cap}(L).$$

From this and from Lemma 1 the rest of the assertion follows. □

For $\alpha \in ]0,1[$, let $(\mathbb{R}, \mathcal{W}^\alpha)$ denote the balayage space corresponding to the Riesz potentials of order $\alpha$, see e.g. [1], p.186. The $\mathcal{W}^\alpha$-polar sets will be called $\alpha$-**polar**.

**Theorem.** *Let $\alpha \in [1/2, 1[$ and $P \subset \mathbb{R}$. Then the following assertions are equivalent:*

*(i) $P$ is $\alpha$-polar;*

*(ii) ${}^\alpha cap^*(P) = 0$;*

*(iii) ${}^\alpha_+cap^*(P) = 0$;*

*(iv) ${}^\alpha_-cap^*(P) = 0$.*

*Here ${}^\alpha cap^*$, ${}^\alpha_+cap^*$ and ${}^\alpha_-cap^*$ denote the usual outer capacity deduced from ${}^\alpha cap$, ${}^\alpha_+cap$ and ${}^\alpha_-cap$, respectively.*

PROOF. We can assume that $\alpha \in ]1/2, 1[$. (For $\alpha = 1/2$, the assertion follows from [6], p.214.)
The equivalence of (i) and (ii) follows from [1], p.291.

Since ${}^{+}K^{\alpha} \leq K^{\alpha}$, we have ${}^{\alpha}\mathrm{cap}^{*}(P) \leq {}^{\alpha}_{+}\mathrm{cap}^{*}(P)$ and hence the implication from (iii) to (ii).

The equivalence of (iii) and (iv) follows from Lemma 2.

Let ${}^{\alpha}\mathrm{cap}^{*}(P) = 0$. By [1], p.200 and 282, there exists a measure $\mu \in \mathcal{M}^{+}(\mathbb{R})$ such that $K^{\alpha}_{\mu} = \infty$ on $P$ and $\overline{\{K^{\alpha}_{\mu} < \infty\}} = \mathbb{R}$. Put $\nu := \varepsilon_0 \otimes \mu$ and $\gamma := \frac{1}{2\alpha}$. Then for all $x, t \in \mathbb{R}$, it holds

$$W^{(\gamma)}_{\nu}(x,t) \leq W^{(\gamma)}_{\nu}(0,t) = {}^{+}K^{\alpha}_{\mu}(t) \leq K^{\alpha}_{\mu}(t),$$
$${}^{*}W^{(\gamma)}_{\nu}(x,t) \leq {}^{*}W^{(\gamma)}_{\nu}(0,t) = {}^{-}K^{\alpha}_{\mu}(t) \leq K^{\alpha}_{\mu}(t).$$

Put $F := \{W^{(\gamma)}_{\nu} = \infty\}$, ${}^{*}F = \{{}^{*}W^{(\gamma)}_{\nu} = \infty\}$ and $S := \{(0,t);\, \mu(\{t\}) > 0\}$. We observe that $S$ is countable, $F$ is $\mathcal{W}^{(\gamma)}$-polar and ${}^{*}F$ is ${}^{*}\mathcal{W}^{(\gamma)}$-polar. Further

$$\{0\} \times P \subset F \cup {}^{*}F \cup S.$$

By Proposition 1, ${}^{(\gamma)}\mathrm{cap}^{*}(\{0\} \times P) = 0$. Consequently, ${}^{\alpha}\mathrm{cap}^{*}(P) = 0$. □

**Remark 2.** The case $\alpha \in ]0, 1/2[$ remains still open.

# References


[1] J.Bliedtner, W.Hansen, *Potential theory. An Analytic and Probabilistic Approach to Balayage*, Springer-Verlag, Berlin, 1986.

[2] M.Brzezina, *On a class of translation invariant balayage spaces*, Exposition. Math., (in press).

[3] M.Brzezina, *Kernels and Choquet capacities*, Aequationes Math., (in press).

[4] T.Ikegami, *Duality on balayage spaces*, preprint, 1992.

[5] M.Itô, M.Nishio, *Poincaré type conditions of regularity for the parabolic operator of order $\alpha$*, Nagoya Math. J. **115** (1989), 1–22.

[6] R.Kaufman, J.M.Wu, *Parabolic potential theory*, J. Differential Equations **43** (1982), 204–234.

[7] M.Nishio, *The Wiener criterion of regular points for the parabolic operator of order $\alpha$*, Nagoya Math. J. **116** (1989), 163–179.



Department of Mathematics, University of Ostrava, Bráfova 7, 701 03 Ostrava 1, Czech Republic

Mathematical Institute, University of Erlangen–Nürnberg, Bismarckstrasse 1a,
D 8520 Erlangen, Germany

# Appell type transformation for the Kolmogorov operator

Miroslav Brzezina *

**Abstract.** For the Kolmogorov operator $L =: \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}$ we describe all transformations mapping solutions of the equation $Lu = 0$ into solutions.

**Keywords:** potential theory, Kolmogorov equation, harmonic function

**Mathematical subject classification:** 31B35, 35H05, 35M99

## 0. Introduction.

The Laplace operator and the heat operator are well known examples for differentials operator generating harmonic spaces in the sense of H.Bauer, see e.g. [2]. The Kolmogorov operator generates also a harmonic space. In [5], [7] and [9], potential theory of the Kolmogorov operator was investigated. In particular a mean value theorem was proved and its potential theoretical interpretation was discussed; furthermore a maximum principle and the Wiener type test for regularity was established. Harnack inequality and a mean value theorem are given in [3] for operators of Kolmogorov type.

In this note, we investigate $L$-harmonic morphism in the sense of the definition given below.

It is well known that the Kelvin transformation for Laplace operator in $\mathbb{R}^3$ (up to compositions with similarities) is the only one which maps the class of harmonic functions into itself; see [6], cf. [4] for $\mathbb{R}^n$.

For the heat operator $H = \Delta - \frac{\partial}{\partial t}$ in $\mathbb{R}^{n+1}$, the Appell transformation plays the same rôle as the Kelvin transformation for the Laplace operator, see e.g. [1], [8].

In [8], H. Leutwiler has shown that the Appell transformation is essentially the only transformation mapping caloric functions (i.e. the solutions of the equation $Hu = 0$) into another. A similar result holds also for the Kolmogorov operator as will be proved in this note.

## 1. Appell type transformation.

We consider the Kolmogorov operator

$$L := \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial t}$$

*Financial support by the Volkswagen Stiftung is gratefully acknowledged.

in $\mathbb{R}^3$. Let $U \subset \mathbb{R}^3$ be an open set and let $C^{2,1,1}(U) := \{u : U \to \mathbb{R};\ \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t} \in C(U)\}$. The function $u \in C^{2,1,1}(U)$ is said to be *L-harmonic* on $U$, if $Lu = 0$ on $U$. For any $(x, y, t) \in \mathbb{R}^3$, $t \neq 0$, let

$$E(x,y,t) := \frac{1}{t^2} \exp\left(-\frac{x^2}{t} - 3\frac{xy}{t^2} - 3\frac{y^2}{t^3}\right).$$

An easy calculation shows that the function $E$ is $L$-harmonic on the set $\mathbb{R}^3_0 := \mathbb{R}^3 \setminus \{(x,y,t) \in \mathbb{R}^3;\ t = 0\}$. Let $u$ be $L$-harmonic on an open set $U \subset \mathbb{R}^3$. Then, obviously, for $a$, $b$, $r \in \mathbb{R}$, $r > 0$, the function

$$(Su)(x,y,t) := u(rx + a, r^3 y - r^2 at, r^2 t + b), \tag{1.1}$$

is also $L$-harmonic on the open set $V := \{(x,y,t) \in \mathbb{R}^3; x = \frac{1}{r}(x' - a),\ y = \frac{1}{r^3}(y' + at' - ab),\ t = \frac{1}{r^2}(t' - b),\ (x', y', t') \in U\}$. Furthermore, we consider the transformation

$$(Au)(x,y,t) := E(x,y,t)u(\frac{x}{t} + 3\frac{y}{t^2}, \frac{y}{t^3}, -\frac{1}{t}) \tag{1.2}$$

on $\mathbb{R}^3_0$. Put $x' = \frac{x}{t} + \frac{3y}{t^2}$, $y' = \frac{y}{t^3}$, $t' = -\frac{1}{t}$. A straightforward calculation shows that

$$L(Au)(x,y,t) = u(x',y',t')L(E)(x,y,t) + \frac{E(x,y,t)}{t^2} L(u)(x',y',t')$$

on $\mathbb{R}^3_0$. Since the function $E$ is $L$-harmonic on $\mathbb{R}^3_0$, we conclude that $Au$ is $L$-harmonic on $\mathbb{R}^3_0$ provided that the function $u$ is $L$-harmonic on this set. We remark that the transformation in (1.2) plays the same rôle for the Kolmogorov operator as the Appell transformation for the heat equation. It is clear that the composition of both transformations $S$ and $A$ is a transformation which maps $L$-harmonic functions into $L$-harmonic functions. All these transformations are of the form

$$(Tu)(x,y,t) := \varphi(x,y,t)u(\Psi(x,y,t)), \tag{1.3}$$

where $\Psi$ maps some open set $U \subset \mathbb{R}^3$ bijectively onto some open set $V \subset \mathbb{R}^3$, $\varphi$ is a strictly positive function on $U$, and $u$ is a $L$-harmonic function on $V$.
It is a natural question whether all transformations of type (1.3) can be obtained as a composition of transformations of type (1.1) and (1.2). The answer is given in the next section.

## 2. The main result.

Following H.Leutwiler, cf. [8], p. 217, we give the following

**Definition.** Let $U, V \subset \mathbb{R}^3$ be open sets, $\Psi : U \to V$ a bijection from $U$ onto $V$, and let $\varphi : U \to \mathbb{R}$ be a strictly positive function. The transformation $T$, defined by (1.3), is said to be *an $L$-harmonic morphism on $U$*, provided $Tu$ is $L$-harmonic on $U$ whenever $u$ is $L$-harmonic on $V$.

The transformations $A$ and $S$ and their composition are examples for $L$-harmonic morphisms. As shown in the following theorem and its corollary, these describe the whole group of $L$-harmonic morphisms on some open set.

**Theorem.** *Let $\alpha, \beta, \gamma, \delta, \tilde{\gamma}, \tilde{\delta}, \kappa, \nu, \omega$ be real numbers, $\alpha\delta - \beta\gamma = 1$, $\nu > 0$. Put $\Gamma := \gamma\tilde{\delta} - \tilde{\gamma}\delta$. Let*

$$\chi(x,y,t) = \frac{1}{\delta}x - \frac{\tilde{\gamma}}{\delta^3}t^2 - \frac{2\tilde{\delta}}{\delta^3}t - \kappa, \tag{2.1}$$

$$\xi(y,t) = \frac{1}{\delta^3}y + \frac{\tilde{\gamma}}{3\delta^5}t^3 + \frac{\tilde{\delta}}{\delta^5}t^2 + \frac{\kappa}{\delta^2}t + \omega, \tag{2.2}$$

$$\tau(t) = \frac{1}{\delta}\left(\frac{1}{\delta}t + \beta\right), \tag{2.3}$$

$$\varphi(x,y,t) = \nu \exp\left(-\frac{1}{\delta^2}(\tilde{\gamma}t + \tilde{\delta})x - \frac{\tilde{\gamma}}{\delta^2}y + \frac{\tilde{\gamma}^2}{3\delta^4}t^3 + \frac{\tilde{\gamma}\tilde{\delta}}{\delta^4}t^2 + \frac{\tilde{\delta}^2}{\delta^4}t\right), \tag{2.4}$$

*if $\gamma = 0$, and*

$$\chi(x,y,t) = \frac{1}{\gamma t + \delta}x + \frac{3\gamma}{(\gamma t + \delta)^2}y + \frac{\Gamma}{\gamma^2}\frac{1}{(\gamma t + \delta)^2} + \frac{2\tilde{\gamma}}{\gamma^2}\frac{1}{\gamma t + \delta} + \kappa, \tag{2.5}$$

$$\xi(y,t) = \frac{1}{(\gamma t + \delta)^3}y + \frac{\Gamma}{3\gamma^3}\frac{1}{(\gamma t + \delta)^3} + \frac{\tilde{\gamma}}{\gamma^3}\frac{1}{(\gamma t + \delta)^2} + \frac{\kappa}{\gamma}\frac{1}{\gamma t + \delta} + \omega, \tag{2.6}$$

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{2.7}$$

$$\begin{aligned}\varphi(x,y,t) = \frac{\nu}{(\gamma t + \delta)^2}\exp\Bigg(&-\frac{\gamma}{(\gamma t + \delta)}x^2 - \frac{\tilde{\gamma}t + \tilde{\delta}}{(\gamma t + \delta)^2}x - \frac{3\gamma^2}{(\gamma t + \delta)^2}xy \\ &- \frac{3\gamma^3}{(\gamma t + \delta)^3}y^2 - \frac{3\tilde{\gamma}(\gamma t + \delta) + 2\Gamma}{(\gamma t + \delta)^3}y \\ &- \frac{\Gamma^2}{3\gamma^3}\frac{1}{(\gamma t + \delta)^3} - \frac{\tilde{\gamma}\Gamma}{\gamma^3}\frac{1}{(\gamma t + \delta)^2} - \frac{\tilde{\gamma}^2}{\gamma^3}\frac{1}{\gamma t + \delta}\Bigg),\end{aligned} \tag{2.8}$$

*if $\gamma \neq 0$. Then*

$$(Tu)(x,y,t) := \varphi(x,y,t)u(\chi(x,y,t), \xi(y,t), \tau(t)) \tag{2.9}$$

*defines an $L$-harmonic morphism on $\mathbb{R}^3$ if $\gamma = 0$, and an $L$-harmonic morphism on $\mathbb{R}^3$ outside the plane $\gamma t + \delta = 0$, if $\gamma \neq 0$.*
*Conversely, let $U \subset \mathbb{R}^3$ be an open set and let $M$ be an $L$-harmonic morphism on $U$. Then there is a transformation $T$ of form (2.9) such that $M = T|_U$.*

Rewriting (2.9) we get that every transformation of form (2.9) can be obtained as a composition of transformations of type (1.1) and (1.2).

**Corollary.** *Every $L$-harmonic morphism $T$ is a composition of morphisms of type $A$ and $S$.*

## 3. Proof of the theorem.

The proof of the first part of the theorem follows by a straightforward calculation. We omit this.
The proof of the second part is based on the fact that the following polynomials are $L$-harmonic on $\mathbb{R}^3$:

$$1,\ x,\ x^2 + 2t,\ x^3 + 6xt,\ x^4 + 12x^2t + 12t^2, \tag{3.1}$$

$$y + xt,\ xy + tx^2 + t^2,\ 3y^2 + 6txy + 3t^2x^2 + 2t^3. \tag{3.2}$$

Let $u : V \to \mathbb{R}$ be an $L$-harmonic function on an open set $V \subset \mathbb{R}^3$ and let $\Psi : U \to V$ map the open set $U \subset \mathbb{R}^3$ bijectively onto $V$. Further, let

$$(Mu)(x, y, t) := \varphi(x, y, t)u(\Psi(x, y, t)), \quad (x, y, t) \in U, \tag{3.3}$$

be an $L$-harmonic morphism on $U$ and let $\Psi = (\chi, \xi, \tau)$ where the mappings $\chi$, $\xi$, $\tau$ are defined on $U$.
For arbitrary functions $u$, $v \in \mathcal{C}^{2,1,1}(U)$ we have

$$L(uv) = uL(v) + vL(u) + 2u_x v_x. \tag{3.4}$$

From (3.1) and (3.3) it follows that the functions $\varphi$ and $\varphi\chi$ are $L$-harmonic on $U$. By (3.4) we get that

$$\varphi L(\chi) + 2\chi_x\varphi_x = 0. \tag{3.5}$$

We deduce from (3.1) and (3.3) that $L(\varphi(\chi^2 + 2\tau)) = 0$. From (3.4) and (3.5) we have

$$\varphi(\chi_x^2 + L(\tau)) + 2\varphi_x\tau_x = 0. \tag{3.6}$$

By (3.1) and (3.3), $L(\varphi(\chi^3 + 6\chi\tau)) = 0$. Using (3.4), (3.5) and (3.6) we obtain

$$\varphi\chi_x\tau_x = 0. \tag{3.7}$$

4

Similarly, we deduce from (3.1) that $\varphi\tau_x^2 = 0$. Since $\varphi > 0$, we arrive that

$$\tau_x = 0 \quad \text{and} \quad \chi_x^2 + L(\tau) = 0. \tag{3.8}$$

By (3.2) and (3.3), $L(\varphi(\xi + \chi\tau)) = 0$. It follows from (3.4), (3.5) and (3.8) that

$$\varphi L(\xi) - \varphi\chi\chi_x^2 + 2\varphi_x\xi_x = 0. \tag{3.9}$$

Since the polynomial $xy + tx^2 + t^2$ is $L$-harmonic, we have by (3.4)

$$\varphi L(\chi\xi) + \varphi L(\chi^2\tau) + \varphi L(\tau^2) + 2\varphi_x(\chi\xi)_x + 2\varphi_x(\chi^2\tau)_x + 2\varphi_x(\tau^2)_x = 0.$$

Using again (3.4), (3.5), (3.8) and (3.9), we conclude that

$$\varphi\chi_x\xi_x = 0. \tag{3.10}$$

From (3.2) and (3.4) it follows that

$$3\varphi L(\xi^2) + 6\varphi L(\chi\xi\tau) + 3\varphi L(\chi^2\tau^2) + 2\varphi L(\tau^3) + 6\varphi_x(\xi^2)_x + 12\varphi_x(\chi\xi\tau)_x + \varphi_x(\chi^2\tau^2)_x + 4\varphi_x(\tau^3)_x = 0.$$

Further, by (3.4), (3.5) and (3.8) - (3.10), we have $\varphi\xi_x^2 = 0$. Consequently,

$$\xi_x = 0 \quad \text{and} \quad L(\xi) - \chi\chi_x^2 = 0. \tag{3.11}$$

According to (3.8) $\tau = \tau(y,t)$ and further

$$\chi_x^2(x,y,t) + x\tau_y(y,t) = \tau_t(y,t), \qquad (x,y,t) \in U. \tag{3.12}$$

We claim that $\tau_y(y,t) = 0$ for every $(x,y,t) \in U$. Fix $(x,y,t) \in U$. If $\chi_x(x,y,t) = 0$ then differentiating in (3.12) with respect to $x$, we get $\tau_y(y,t) = 0$. If $\chi_x(x,y,t) \neq 0$ then there exists $\varepsilon > 0$ such that $I := \{(x+s,y,t) \in \mathbb{R}^3;\ s \in ]-\varepsilon,\varepsilon[\} \subset U$ and $\chi_x^2(\cdot,y,t) > 0$ on $I$. Consequently, on $I$ we have $L(\tau) \neq 0$. From (3.8) and (3.11) it follows that

$$\chi(\bar{x},y,t) = -\frac{\bar{x}\xi_y(y,t) - \xi_t(y,t)}{\bar{x}\tau_y(y,t) - \tau_t(y,t)}, \quad (\bar{x},y,t) \in I.$$

Differentiating this equality with respect to $x$, we obtain

$$\chi_x(\bar{x},y,t) = \frac{\xi_t(y,t)\tau_y(y,t) - \tau_t(y,t)\xi_y(y,t)}{(\bar{x}\tau_y(y,t) - \tau_t(y,t))^2}, \quad (\bar{x},y,t) \in I.$$

Using (3.8) we derive that

$$-\left(\bar{x}\tau_y(y,t) - \tau_t(y,t)\right)^5 = \left(\xi_y(y,t)\tau_t(y,t) - \tau_y(y,t)\xi_t(y,t)\right)^2, \quad (\bar{x},y,t) \in I. \tag{3.13}$$

Since the left handside in (3.13) depends on $\bar{x}$ but the right handside is independent of $\bar{x}$, it follows that $\tau_y(y,t) = 0$, i.e. $\tau = \tau(t)$. Now, by (3.12), we have

$$\chi_x^2 = \dot{\tau} \tag{3.14}$$

where the dot denotes the derivative with respect to $t$. From (3.14) it follows that $\chi_x$ does not depended on $x$ and $y$. We put

$$\vartheta(t) := \chi_x(x,y,t), \quad \text{i.e.} \quad \vartheta^2 = \dot{\tau}. \tag{3.15}$$

Inserting this in (3.11) and differentiating with respect to $x$, we obtain $\xi_y = \vartheta^3$. Consequently,

$$\xi(y,t) = y\vartheta^3(t) + d(t) \tag{3.16}$$

with some unknown function $d = d(t)$. Inserting $\xi$ in (3.11), we get

$$\vartheta^3(t)x - (\vartheta^3(t))^{\dot{}})y - \dot{d}(t) = \chi(x,y,t)\vartheta^2(t). \tag{3.17}$$

From (3.15) it follows that

$$\chi(x,y,t) = \vartheta(t)x + c(y,t)$$

with some other unknown function $c = c(y,t)$. Since $\Psi = (\chi, \xi, \tau)$ maps $U$ bijectively onto $V$, we find that $\vartheta \neq 0$ on $U$. Consequently, by (3.17), we have

$$\chi(x,y,t) = \vartheta(t)x - 3\dot{\vartheta}(t)y - \frac{\dot{d}(t)}{\vartheta^2(t)}. \tag{3.18}$$

Put

$$\Theta := \frac{1}{2\vartheta}\left(\frac{\dot{d}}{\vartheta^2}\right)^{\dot{}} \quad \text{and} \quad A = A(x,y,t) := \frac{3}{2}\frac{\ddot{\vartheta}(t)}{\vartheta(t)}y - 2\frac{\dot{\vartheta}(t)}{\vartheta(t)}x + \Theta(t). \tag{3.19}$$

Inserting (3.18) in (3.5) we get

$$\varphi_x = -\varphi A. \tag{3.20}$$

Differentiating in (3.20) with respect to $x$, we obtain

$$\varphi_{xx} = -\varphi_x A - \varphi A_x = \varphi(A^2 - A_x). \tag{3.21}$$

From (3.20) it follows that $(\log\varphi)_x = -A$. Consequently,

$$\log\varphi(x,y,t) = -\frac{3}{2}\frac{\ddot{\vartheta}(t)}{\vartheta(t)}xy + \frac{\dot{\vartheta}(t)}{\vartheta(t)}x^2 - \Theta(t)x + g(y,t) \tag{3.22}$$

with some unknown function $g = g(y,t)$. Differentiating in (3.22) with respect to $y$ and $t$, we get

$$\begin{aligned} \varphi_y(x,y,t) &= \varphi(x,y,t)\left(-\frac{3}{2}\frac{\ddot{\vartheta}(t)}{\vartheta(t)}x + g_y(y,t)\right), &(3.23)\\ \varphi_t(x,y,t) &= \varphi(x,y,t)\left(\left(\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)^{\cdot} x^2 - \frac{3}{2}\left(\frac{\ddot{\vartheta}(t)}{\vartheta(t)}\right)^{\cdot} xy - \dot{\Theta}(t)x + g_t(y,t)\right). &(3.24)\end{aligned}$$

According to (3.1) and (3.3), $L(\varphi) = 0$. Further, by (3.21), (3.23) and (3.24) we have

$$\begin{aligned} &\left(4\left(\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)^2 - \frac{3}{2}\frac{\ddot{\vartheta}(t)}{\vartheta(t)} - \left(\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)^{\cdot}\right)x^2 + \frac{9}{4}\left(\frac{\ddot{\vartheta}(t)}{\vartheta(t)}\right)^2 y^2\\ &+3\frac{\ddot{\vartheta}(t)}{\vartheta(t)}\Theta(t)y + \Theta^2(t) + 2\frac{\dot{\vartheta}(t)}{\vartheta(t)} - g_t(y,t)\\ &+\left(\left(\frac{3}{2}\left(\frac{\ddot{\vartheta}(t)}{\vartheta(t)}\right)^{\cdot} - 6\frac{\ddot{\vartheta}(t)}{\vartheta(t)}\cdot\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)y + \dot{\Theta}(t) - 4\frac{\dot{\vartheta}(t)}{\vartheta(t)}\Theta(t) + g_y(y,t)\right)x = 0. \qquad (3.25)\end{aligned}$$

Differentiating twice in (3.25) with respect to $x$ we get

$$4\left(\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)^2 - \frac{3}{2}\frac{\ddot{\vartheta}(t)}{\vartheta(t)} - \left(\frac{\dot{\vartheta}(t)}{\vartheta(t)}\right)^{\cdot} = 0, \quad \text{i.e.} \quad 2(\dot{\vartheta}(t))^2 = \ddot{\vartheta}(t)\vartheta(t).$$

Integration of this equation yields

$$\vartheta(t) = \frac{1}{\gamma t + \delta}$$

with some constants $\gamma$, $\delta \in \mathbb{R}$, $\gamma^2 + \delta^2 > 0$. In order to determine $\tau$, we observe that on account of (3.15) $\vartheta^2 = \dot{\tau}$, therefore

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta} \qquad (3.26)$$

with arbitrary $\alpha$, $\beta \in \mathbb{R}$ satisfying the condition $\alpha\delta - \beta\gamma = 1$. Inserting $\tau$ in (3.25) we obtain

$$\begin{aligned} &\left(\frac{6\gamma^3}{(\gamma t+\delta)^3}y + \dot{\Theta}(t) + \frac{4\gamma}{\gamma t + \delta}\Theta(t) + g_y(y,t)\right)x + \frac{9\gamma^4}{(\gamma t+\delta)^4}y^2\\ &+\frac{6\gamma^2}{(\gamma t+\delta)^2}\Theta(t)y + \Theta^2(t) - \frac{2\gamma}{\gamma t+\delta} - g_t(y,t) = 0. \qquad (3.27)\end{aligned}$$

Differentiating in (3.27) with respect to $x$ we have

$$g_y(y,t) = -\frac{6\gamma^3}{(\gamma t+\delta)^3}y - \dot{\Theta}(t) - \frac{4\gamma}{\gamma t+\delta}\Theta(t),$$

and consequently

$$g(y,t) = -\frac{3\gamma^3}{(\gamma t+\delta)^3}y^2 - \left(\dot{\Theta}(t) + \frac{4\gamma}{\gamma t+\delta}\Theta(t)\right)y + \tilde{g}(t)$$

with some unknown function $\tilde{g} = \tilde{g}(t)$. Inserting $g$ in (3.27), we obtain

$$\left(\ddot{\Theta}(t) + \left(\frac{4\gamma}{\gamma t+\delta}\Theta(t)\right)^{\cdot} + \frac{6\gamma^2}{(\gamma t+\delta)^2}\Theta(t)\right)y + \Theta^2(t) - \frac{2\gamma}{\gamma t+\delta} - \dot{\tilde{g}}(t) = 0. \qquad (3.28)$$

Differentiating in (3.28) with respect to $y$ we get

$$\left((\gamma t+\delta)^2\Theta(t)\right)^{\cdot\cdot} = 0.$$

Integration of this equation yields

$$\Theta(t) = \frac{\tilde{\gamma}t+\tilde{\delta}}{(\gamma t+\delta)^2}$$

with a real number $\tilde{\gamma}$, $\tilde{\delta} \in \mathbb{R}$. Inserting this in (3.19) we get

$$\left((\gamma t+\delta)^2\dot{d}(t)\right)^{\cdot} = 2\frac{\tilde{\gamma}t+\tilde{\delta}}{(\gamma t+\delta)^3}.$$

Let us consider the two cases $\gamma = 0$ and $\gamma \neq 0$ separately. If $\gamma = 0$, then $\delta \neq 0$ and

$$d(t) = \frac{1}{3}\frac{\tilde{\gamma}}{\delta^5}t^3 + \frac{\tilde{\delta}}{\delta^5}t^2 + \frac{\kappa}{\delta^2}t + \omega,$$

with arbitrary constants $\kappa$, $\omega \in \mathbb{R}$. Further, from (3.28) it follows that

$$\dot{\tilde{g}}(t) = \frac{1}{\delta^4}(\tilde{\gamma}t+\tilde{\delta})^2.$$

Consequently,

$$\tilde{g}(t) = \frac{1}{3}\frac{\tilde{\gamma}^2}{\delta^4}t^3 + \frac{\tilde{\gamma}\tilde{\delta}}{\delta^4}t^2 + \frac{\tilde{\delta}^2}{\delta^4}t + \tilde{\nu}$$

with some constant $\tilde{\nu} \in \mathbb{R}$. Finally, from (3.16), (3.18), (3.22) and (3.26) we obtain (2.1) – (2.4) with $\nu := \exp \tilde{\nu}$.
We consider now the case $\gamma \neq 0$. We get

$$d(t) = \frac{\Gamma}{3\gamma^3} \frac{1}{(\gamma t + \delta)^3} + \frac{\tilde{\gamma}}{\gamma^3} \frac{1}{(\gamma t + \delta)^2} + \frac{\kappa}{\gamma} \frac{1}{\gamma t + \delta} + \omega$$

with $\Gamma := \gamma\tilde{\delta} - \tilde{\gamma}\delta$ and some constants $\omega, \kappa \in \mathbb{R}$. Furthermore, it follows from (3.28) that

$$\dot{\tilde{g}}(t) = \left( \frac{\tilde{\gamma}t + \tilde{\delta}}{(\gamma t + \delta)^2} \right)^2 - \frac{2\gamma}{\gamma t + \delta}.$$

Consequently,

$$\tilde{g}(t) = -\frac{\Gamma^2}{3\gamma^3} \frac{1}{(\gamma t + \delta)^3} - \frac{\tilde{\gamma}\Gamma}{\gamma^3} \frac{1}{(\gamma t + \delta)^2} - \frac{\tilde{\gamma}^2}{\gamma^3} \frac{1}{\gamma t + \delta} + \log \frac{1}{(\gamma t + \delta)^2} + \tilde{\nu},$$

with a constant $\tilde{\nu} \in \mathbb{R}$. Finally, from (3.16), (3.18), (3.22) and (3.26) we obtain (2.5) – (2.8) with $\nu := \exp \tilde{\nu}$. This finishes the proof.

# Bibliography


[1] P.Appell, *Sur l'équation $\partial^2 z/\partial x^2 - \partial z/\partial y = 0$ et la théorie du chaleur*, J. Math. Pures. Appl (4)8 (1892), 187–216.

[2] H.Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Mathematics **22**, Springer–Verlag, Berlin, 1966.

[3] N.Garofalo, E.Lanconelli, *Level sets of the fundamental solution and the Harnack inequality for degenerate equations of Kolmogorov type*, Trans. Amer. Math.Soc. **321** (1990), 775–792.

[4] L.L.Helms, *Introduction to potential theory*, Wiley Interscience, New York, 1969.

[5] W.Hoh, N.Jacob, *On the potential theory of the Kolmogorov equation*, Math.Nachr. **154** (1991), 51–66.

[6] O.D.Kellogg, *Foundations of potential theory*, Springer-Verlag, Berlin, 1929 (re-issued 1967).

[7] L.P.Kuptsov, *Mean value theorem and a maximum principle for Kolmogorov's equation*, Math.Notes **15** (1974), 280–286.

[8] H.Leutwiler, *On the Appell transformation*, In: Potential Theory, ed. J.Král et al., Plenum Press, New York, 1987, 215–222.

[9] V.Scornazzani, *Sul problema di Dirichlet per l'operatore di Kolmogorov*, Bollettino U.M.I., Anal.Funz.Appl. **18** (1981), 43–62.



Department of Mathematics, Technical University of Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic

Presently:

Mathematisches Institut, University of Erlangen–Nürnberg, Bismarckstrasse 1a, 91054 Erlangen, Germany

# Essential base, Choquet's boundary and Wiener's test in potential theory

Miroslav Brzezina

**Abstract.** Wiener's type test for Choquet's boundary and essential base is presented in the case when a suitable Wiener's test for regularity is known.

**Keywords:** Wiener's test, essential base, Choquet's boundary, capacity

**Mathematical subject classification:** 31D05, 35J25, 35K20

## Introduction.

Let $(X, \mathcal{W})$ be a balayage space (for definitions, see [1]). At investigating the boundary behaviour of the Perron–Wiener–Brelot solution of the generalized Dirichlet problem, the notion of regular and Choquet's boundary points plays an important role. Every Choquet's point is regular, but the converse needn't be true in general. As shown in [7], these notions are different in the case of the harmonic space associated with the heat equation.

In this note we present Wiener's type test for Choquet's boundary and essential base, if a suitable Wiener's test for regularity is known. The main results are contained in Theorem 1.2 and in Theorem 2.2.

All our considerations will be done in a balayage space $(X, \mathcal{W})$. We shall adopt notations from [1].

## 1. Essential base.

**Definition 1.1.** Let $\mathcal{K}$ denote the set of all compact subsets of $X$. A set function $\gamma : \mathcal{K} \to [0, \infty]$ is said to be *a Choquet capacity on* $X$ if it satisfies the following conditions:

(i) $\gamma(K) \leq \gamma(L)$, whenever $K$, $L \in \mathcal{K}$ , $K \subset L$;

(ii) $\gamma(K \cap L) + \gamma(K \cup L) \leq \gamma(K) + \gamma(L)$, whenever $K$, $L \in \mathcal{K}$ ;

(iii) if $(K_n)_{n=1}^{\infty}$ is a decreasing sequence of compact sets with the intersection $K$, i.e. $K_n \downarrow K$, then $\lim_{n\to\infty} \gamma(K_n) = \gamma(K)$.

For an arbitrary set $E$, we define *the inner capacity* $\gamma_*$ and *the outer capacity* $\gamma^*$ by

$$\begin{aligned}\gamma_*(E) &= \sup\{\gamma(K); K \subset E, K \text{ compact}\},\\ \gamma^*(E) &= \inf\{\gamma_*(U); E \subset U, U \text{ open}\},\end{aligned}$$

respectively. We say that $E \subset X$ is *capacitable* if $\gamma_*(E) = \gamma^*(E)$. For details see e.g. [6], [2], [1]. In what follows, $\gamma$ denotes a Choquet capacity.

**Lemma 1.1.** *Every Borel subset of $X$ is capacitable.*

PROOF. See [2], p. 72. □

**Remark 1.1.** It follows from Definition 1.1 that $\gamma(K) = \gamma_*(K)$, whenever $K$ is a compact subset of $X$. We can extend the set function $\gamma$ which is defined for compact sets only to the capacitable sets $E \subset X$ by defining $\gamma(E) = \gamma_*(E)$. In particular, we write $\gamma(E)$ instead of $\gamma_*(E)$ and $\gamma^*(E)$, whenever $E$ is capacitable.

**Definition 1.2.** Let $E$ be an arbitrary subset of $X$ and $z \in X$. Then $E$ is said to be *semipolar at $z$* if there exists a fine neighborhood $V$ of $z$ such that the set $E \cap V$ is semipolar. Let $\beta(E)$ be the set of all points $z \in X$ such that $E$ is not semipolar at $z$. The set $\beta(E)$ is called *the essential base of $E$*. For details and fundamental properties of $\beta(E)$ see e.g. [1]; we note only that $\beta(E)$ is a $G_\delta$-set.

**Definition 1.3.** Let $K \subset X$ be a compact set. The *$\alpha$-capacity of $K$* is defined as

$$\alpha(K) := \gamma(\beta(K)).$$

Let $E \subset X$ be an arbitrary set. Then

$$\alpha_*(E) = \sup\{\alpha(K); K \subset E, K \text{ compact}\}$$

is called *the inner $\alpha$-capacity of $E$*. (For more details see [5].)

**Remark 1.2.** It follows from Definition 1.3 that $\alpha_*(K) = \alpha(K)$, whenever $K \subset X$ is a compact set. Further, for an arbitrary subset $A$ of $X$, the inequality $\alpha_*(A) \leq \gamma_*(A)$ holds. (Indeed, if $K \subset A$ is a compact set, then it follows from [5], p. 2, that $\beta(K) \subset K$. The rest follows easily from the definition of $\alpha_*$ and a monotonicity of $\gamma_*$.) The set function $\alpha_*$ is clearly increasing.

From now, we assume that $\gamma$ is a Choquet capacity on $X$ satisfying the following condition:

(R) if $A$ is a relatively compact Borel subset of $X$ then $\gamma(A) = \gamma(\overline{A}^f)$.

**Proposition 1.1.** *Let $B$ be a Borel subset of $X$ and $S \subset X$ be semipolar. Then*

$$\alpha_*(B) = \alpha_*(B \setminus S).$$

PROOF. See [5], p. 5. □

**Proposition 1.2.** *Let $B$ be a Borel subset of $X$. Then there exists a Borel semipolar set $S$ such that*

$$\alpha_*(B) = \gamma(B \setminus S).$$

PROOF. See [5], p. 6.

**Notation.** For $z \in X$, $r \in ]0,1]$, let $B^r(z)$ denote a compact set in $X$ such that:

(i) $B^r(z) \subset B^s(z)$ for $r < s$;

(ii) $\bigcap_{0<r\leq 1} B^r(z) = \{z\}$.

For $\lambda \in ]0,1[$ and $k \in \mathbb{N}$ write $B_k(z,\lambda)$ instead of $B^{\lambda^k}(z)$.

**Theorem 1.1.** *Let $z \in X$, $E$ be a subset of $X$ and let $\gamma$ be a Choquet capacity on $X$. Suppose that $\{z\}$ is thin at $z$ and that the following condition (P) holds:*

> *There exists a sequence of positive numbers $(c_k(z))_{k=1}^{\infty}$ such that the following statements are equivalent, whenever $F \subset X$ is compact:*
>
> *$F$ is thin at $z$;*
> $\sum_{k=1}^{\infty} c_k(z)\gamma(F \cap B_k(z,\lambda)) < \infty$.

*Then $E$ is thin at $z$ if and only if the series*

$$\sum_{k=1}^{\infty} c_k(z)\gamma^*(E \cap B_k(z,\lambda)) \tag{1}$$

*is convergent.*

PROOF. We omit the proof of Theorem 1.1, since it differs from that of [4], p. 229, only in minor details. □

**Theorem 1.2.** *Let $z \in X$, let $B$ be a Borel subset of $X$ and let $\gamma$ be a Choquet capacity on $X$. Suppose that $\{z\}$ is thin at $z$ and that the condition (P) from Theorem 1.1 holds. Then $B$ is semipolar at $z$ if and only if the series*

$$\sum_{k=1}^{\infty} c_k(z)\alpha_*(B \cap B_k(z,\lambda)) \tag{2}$$

*is convergent.*

PROOF. Assume that the series in (2) is convergent. According to Proposition 1.2, there exists, for every $k \in \mathbb{N}$, a Borel semipolar set $S_k$ such that

$$\gamma((B \cap B_k(z,\lambda)) \setminus S_k) = \alpha_*(B \cap B_k(z,\lambda)). \tag{3}$$

Clearly, the set $S = \{z\} \cup \bigcup_{k=1}^{\infty} S_k$ is semipolar. For $k \in \mathbb{N}$ we have $(B \setminus S) \cap B_k(z,\lambda) \subset (B \cap B_k(z,\lambda)) \setminus S_k$. Thus (3) and the monotonicity of $\gamma$ yield

$$\gamma((B \setminus S) \cap B_k(z,\lambda)) \leq \alpha_*(B \cap B_k(z,\lambda)).$$

Since the series in (2) is convergent we obtain that the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma((B \setminus S) \cap B_k(z,\lambda))$$

is convergent. According to Theorem 1.1, the set $B \setminus S$ is thin at $z$, hence $V = \complement(B \setminus S)$ is a fine neighborhood of the point $z$ by [8], p. 335. The set $V \cap B$, being a subset of $S$, is semipolar. Consequently, $B$ is semipolar at $z$ by Definition 1.2.

Let $B$ be a Borel set semipolar at $z$. Then there exists a fine neighborhood $V$ of the point $z$ such that the set $V \cap B$ is semipolar. Since $z$ has a fundamental system of fine neighborhoods which are compact in the initial topology of $X$, see [1], p. 56, we can assume that $V$ is compact. Since $B \cap B_k(z,\lambda)$ and $(B \setminus V) \cap B_k(z,\lambda)$ differ for a semipolar set, we have by Proposition 1.1

$$\alpha_*(B \cap B_k(z,\lambda)) = \alpha_*((B \setminus V) \cap B_k(z,\lambda)). \tag{4}$$

As $V$ is a fine neighborhood of the point $z$, the series

$$\sum_{k=1}^{\infty} c_k(z)\gamma(\complement V \cap B_k(z,\lambda)) \tag{5}$$

is convergent by Theorem 1.1 and by [8], p. 335. Using the equality (4), Remark 1.2, the inclusion $(B \setminus V) \cap B_k(z,\lambda) \subset \complement V \cap B_k(z,\lambda)$ and the monotonicity of $\gamma$, we obtain

$$\alpha_*(B \cap B_k(z,\lambda)) \leq \gamma(\complement V \cap B_k(z,\lambda)).$$

Convergence of the series in (5) now implies that the series (2) is convergent. □

**Remark 1.3.** If the assumption (P) holds and if $B$ is an arbitrary set semipolar at $z$, then the series in (2) is convergent. Indeed, from [1], p. 285, it follows that there exists a Borel set $B'$ such that $B \subset B'$ and $B'$ is semipolar at $z$. Applying the assertion proved above on the Borel set $B'$ and using the monotonicity of $\alpha_*$ we obtain the convergence of series in (2).

**Remark 1.4.** An example from [3], p. 99, shows that Theorem 1.2 fails if $B$ is not supposed to be a Borel set.

## 2. Choquet's boundary.

Let $\mathcal{C}(X)$ denote the set of all continuous functions on $X$ and let $\mathcal{P}$ denote the set of all continuous potentials on $X$.
For an open set $U \subset X$, let $\mathcal{S}(U)$ denote the set of all superharmonic functions on $U$ (for the definition, see [1], p. 342) and let

$$\mathcal{C}_{\mathcal{P}}(U) := \{f \in \mathcal{C}(X);\ \text{there exists } p \in \mathcal{P} \text{ with } |f| \leq p\}.$$

Further, put

$$S(U) := \mathcal{C}_{\mathcal{P}}(U) \cap \mathcal{S}(U).$$

**Definition 2.1.** Let $U$ be an open subset of $X$. For $z \in X$, let $\mathcal{M}_z$ denote the set of all positive Radon measures $\mu$ on $X$ such that $-\infty < \mu(s) \leq s(z)$, whenever $s \in S(U)$. (Obviously, the Dirac measure $\varepsilon_z$ concentrated at $z$ belongs to $\mathcal{M}_z$.) The set

$$Ch_{S(U)}X := \{z \in X;\ \mathcal{M}_z = \{\varepsilon_z\}\}$$

is called *Choquet's boundary* of $X$ (with respect to $S(U)$).

**Proposition 2.1.** *Let $U$ be an open subset of $X$. Then*

$$Ch_{S(U)}X = \beta(\complement U).$$

PROOF. See [1], p. 364. □

**Remark 2.1.** From Proposition 2.1 and [5], p. 2, it follows that

$$\complement\overline{U} \subset Ch_{S(U)}X \subset \complement U.$$

Also, only for points from $\partial U$ it is not obviously determined whether it belongs to $Ch_{S(U)}X$.

**Theorem 2.2.** *Let the assumptions of Theorem 1.2 be satisfied. Further, let $U$ be an open subset of $X$ and $z \in \partial U$ such that the set $\{z\}$ is thin at $z$. Then $z \in Ch_{S(U)}X$ if and only if the series*

$$\sum_{k=1}^{\infty} c_k(z)\alpha(\complement U \cap B_k(z,\lambda))$$

*is divergent.*

PROOF. The assertion follows from Theorem 1.2 and Lemma 2.1. □

## References


[1] J.Bliedtner, W.Hansen, *Potential theory. An Analytic and Probabilistic Approach to Balayage*, Springer-Verlag, Berlin, 1986.

[2] M.Brelot, *Éléments de la théorie classique du potentiel*, $2^e$ ed., Centre de Documentation Universitaire, Paris, 1961.

[3] M.Brzezina, *On the base and the essential base in parabolic potential theory*, Czechoslovak Math. J. **40** (1990), 87–103.

[4] M.Brzezina, *Wiener's test of thinness in potential theory*, Comm. Math. Univ. Carolinae **31** (1990), 227–232.

[5] M.Brzezina, *On continuous capacities*, (preprint).

[6] L.L.Helms, *Introduction to potential theory*, Wiley Interscience, New York, 1969.

[7] J.Köhn, M.Sieveking, *Reguläre und extremale Randpunkte in der Potentialtheorie*, Rev. Roumaine Math. Pures Appl. **12** (1967), 1489–1502.

[8] J.Lukeš, J.Malý, L.Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Mathematics 1189, Springer-Verlag, 1986.



Department of Mathematics, Technical University of Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic

# On continuous capacities

Miroslav Brzezina

**Abstract.** Let $(X, \mathcal{W})$ be a balayage space and $\gamma$ be a Choquet capacity on $X$. Let further $\beta(E)$ denote the essential base of $E \subset X$. For a compact set $K \subset X$ put $\alpha(K) := \gamma(\beta(K))$. Some properties of the set function $\alpha$ are investigated. In particular, it is shown when $\alpha$ is the Choquet capacity. Further, the relation $\alpha$ to so called continuous capacity deduced from a kernel on $X$ are given. Some open problem from the book by G. Anger from 1967 are solved.

**Keywords:** capacities, continuous capacities, semipolar sets, essential base

**Mathematical subject classification:** 31B15, 31C15

## Introduction

The negligible (or small) sets play an important role in the potential theory. In this note, we will investigate semipolar sets and some set functions describing them.
All our consideration will be done in a balayage space $(X, \mathcal{W})$. (For its definition, basic properties and notion used bellow we recommend the monograph by J. Bliedtner and W. Hansen [4].)
We introduce the notion of $\alpha$-capacity and we study some its properties. Especially, we give the relation to Borel semipolar sets (see Definition 2.1 and Corollary 2.3). In Theorem 2.1, we solve the problem when an $\alpha$-capacity is a capacity in Choquet's sense.
In the last section, we deal with a continuous capacity introduced by G.Anger in [1]. We give this notion to the connection to an $\alpha$-capacity and we solve some open problems from the book by G. Anger [1].

## 1. Essential base

In this part we recall the notion of essential base and we give some basic properties needed in following.

**Definition 1.1.** Let $E \subset X$ and $z \in X$. Then $E$ is said to be *semipolar at* $z$ if there exists a fine neighborhood $V$ of $z$ such that the set $E \cap V$ is semipolar. Let $\beta(E)$ denote the set of all points $z \in X$ such that $E$ is not semipolar at $z$. The set $\beta(E)$ is called the *essential base* of $E$.

The following lemma is an easy consequence of Definition 1.1.

**Lemma 1.1.** *Let* $A$, $B \subset X$. *Then*

*(i) if* $A \subset B$ *then* $\beta(A) \subset \beta(B)$;

*(ii)* $\beta(A \cup B) = \beta(A) \cup \beta(B)$;

*(iii)* $\beta(A \cap B) \subset \beta(A) \cap \beta(B)$.

**Remark 1.1.** The notion of essential base was introduced into potential theory by J. Bliedtner and W. Hansen in [3].

**Proposition 1.1.** *Let* $E \subset X$. *The essential base* $\beta(E)$ *is the smallest finely closed set* $F \subset X$ *such that* $E \setminus F$ *is semipolar.* □

Proof. See [4], p. 296.

**Lemma 1.2.** *Let* $E$ *be an arbitrary subset of* $X$. *Then*

(i) *if $E$ is fine closed then $\beta(E) \subset E$;*

(ii) *if $E$ is fine open then $E \subset \beta(E)$;*

(iii) *if $E$ is fine open and if $A \subset X$ then $\beta(A) \cap E \subset \beta(A \cap E)$.*

*In particular, for a compact set $L \subset X$, int $L \subset \beta(L) \subset L$.*

PROOF. Easy consequence of Proposition 1.1. □

**Proposition 1.2.** *Let $B$ be a Borel subset of $X$. Then there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of $B$ such that*

$$\beta(B) = \beta(\bigcup_{n=1}^{\infty} K_n) = \overline{\bigcup_{n=1}^{\infty} \beta(K_n)}^{f}.$$

*(Here, for a set $E \subset X$, the symbol $\overline{E}^f$ denotes the fine closure of $E$.)*

PROOF. From [4], p. 301, it follows the existence of a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of $B$ such that

$$\beta(\bigcup_{n=1}^{\infty} K_n) = \beta(B).$$

The second equality of the assertion follows from [4], p. 297. □

**Proposition 1.3.** *Let $(X, \mathcal{W})$ be a balayage space, let $1 \in \mathcal{W}$ and $B$ be a Borel subset of $X$. Then*

$$R_1^{\beta(B)} = \sup\{p \in \mathcal{P};\, p \leq 1 \text{ on } X \text{ and } C(p) \subset B\}, \tag{1}$$

*where set of functions on the right side is increasingly filtered. In particular,*

$$\hat{R}_1^{\beta(B)} = R_1^{\beta(B)} \quad \text{and} \quad C(\hat{R}_1^{\beta(B)}) \subset \overline{\beta(B)}.$$

*(Here, for $u \in \mathcal{W}$, the $C(u)$ denotes the superharmonic carrier of $u$.)*

PROOF. The first part of the assertion follows from [8], p. 502. Since the function on the right side of (1) is lower semicontinuous it follows that $\hat{R}_1^{\beta(B)} = R_1^{\beta(B)}$. By [4], p. 252, $C(R_1^{\beta(B)}) \subset \overline{\beta(B)}$, i.e., $C(\hat{R}_1^{\beta(B)}) \subset \overline{\beta(B)}$. □

## 2. $\alpha$-capacity

**Definition 2.1.** Let $\mathcal{K}$ denote the set of all compact subsets of $X$. A set function $\gamma : \mathcal{K} \to [0, \infty]$ is said to be a *Choquet capacity* on $X$ if it satisfies the following conditions:

(i) $\gamma(K) \leq \gamma(L)$, whenever $K, L \in \mathcal{K}$, $K \subset L$,
*(monotonicity of $\gamma$)*;

(ii) $\gamma(K \cap L) + \gamma(K \cup L) \leq \gamma(K) + \gamma(L)$, whenever $K, L \in \mathcal{K}$,
*(strong subadditivity of $\gamma$)*;

(iii) if $(K_n)_{n=1}^{\infty}$ is decreasing sequence of compact sets with the intersection $K$, i.e. $K_n \downarrow K$, then $\lim_{n\to\infty} \gamma(K_n) = \gamma(K)$,
*(right continuity of $\gamma$)*.

For an arbitrary set $E$, we define the *inner capacity* $\gamma_*$ and the *outer capacity* $\gamma^*$ by

$$\gamma_*(E) = \sup\{\gamma(K); K \subset E, K \text{ compact}\}, \quad \gamma^*(E) = \inf\{\gamma_*(U); E \subset U, U \text{ open}\}.$$

We say that $E \subset X$ is *capacitable* if $\gamma_*(E) = \gamma^*(E)$. For details see e.g. [10], [5], [4]. In what follows, $\gamma$ denotes a Choquet capacity.

**Lemma 2.1.** *Every Borel subset of $X$ is capacitable.*

PROOF. See [5], p. 72. □

**Lemma 2.2.** *Let $(M_n)_{n=1}^{\infty}$ be an increasing sequence of subsets of $X$ with union $M$. Then*

$$\lim_{n\to\infty} \gamma^*(M_n) = \gamma^*(M).$$

PROOF. See [5], p. 70. □

**Remark 2.1.** It follows from Definition 2.1 that $\gamma(K) = \gamma_*(K)$, whenever $K$ is a compact subset of $X$. We can extend the set function $\gamma$ which is defined for compact sets only to the capacitable sets $E \subset X$ by defining $\gamma(E) = \gamma_*(E)$. In particular, we write $\gamma(E)$ instead of $\gamma_*(E)$ and $\gamma^*(E)$, whenever $E$ is capacitable.

**Definition 2.2.** Let $K \subset X$ be a compact set. The *$\alpha$-capacity of* $K$ is defined as $\alpha(K) = \gamma(\beta(K))$. Let $E \subset X$ be an arbitrary set. Then $\alpha_*(E) = \sup\{\alpha(K); K \subset E, K \text{ compact}\}$ is called *the inner $\alpha$-capacity of* $E$.

**Remark 2.2.** It follows from [4], pp. 272, 297, that the set $\beta(K)$ is a Borel set. Consequently, the set function $\alpha$ is well defined.

**Remark 2.3.** It follows from Definition 2.2 that $\alpha_*(K) = \alpha(K)$, whenever $K \subset X$ is a compact set. Further, for an arbitrary subset $A$ of $X$, the inequality $\alpha_*(A) \leq \gamma_*(A)$ holds. (Indeed, if $K \subset A$ is a compact set, then it follows from Lemma 1.2 that $\beta(K) \subset K$. The rest follows easily from the definition of $\alpha_*$ and monotonicity of $\gamma_*$.) The set function $\alpha_*$ is clearly increasing.

**Lemma 2.3.** *Let $K$, $K_1$, $K_2$ be compact subsets of $X$. Then*

*(i)* $0 \leq \alpha(K) \leq \gamma(K)$;

*(ii) if* $K_1 \subset K_2$ *then* $\alpha(K_1) \leq \alpha(K_2)$,
*(monotonicity of $\alpha$-capacity)*;

*(iii)* $\alpha(K_1 \cup K_2) + \alpha(K_1 \cap K_2) \leq \alpha(K_1) + \alpha(K_2)$,
*(strong subadditivity of $\alpha$-capacity)*.

PROOF. The assertion (i) follows from Lemma 1.2 and from the monotonicity of $\gamma$. The monotonicity of the set operator $\beta$ and the monotonicity of $\gamma$ gives (ii). From Lemma 1.2 (ii) and (iii) and from the strong subadditivity of $\gamma$ the assertion (iii) follows. □

**Remark 2.4.** In Lemma 2.3, we did not prove the right continuity of $\alpha$-capacity on compact sets. The following example shows that this is not true in general.
Consider the potential theory for the heat operator in $\mathbb{R} \times \mathbb{R}$. Let $K = [0,1] \times \{0\}$ and $K_j \subset \mathbb{R} \times \mathbb{R}$, $j \in \mathbb{N}$, be compact sets such that $K_{j+1} \subset \operatorname{int} K_j$, $j \in \mathbb{N}$, and $K = \bigcap_{j=1}^{\infty} K_j$. Let further $\alpha$-${}^h\text{cap}$ denote the $\alpha$-capacity deduced from the heat capacity ${}^h\text{cap}$. Then obviously $\alpha$-${}^h\text{cap}(K) = 0$, since

the set $K$ is semipolar. By Lemma 1.2, $K \subset \operatorname{int} K_j \subset \beta(K_j)$, $j \in \mathbb{N}$. Consequently, ${}^h\mathrm{cap}(K) \leq {}^h\mathrm{cap}(\beta(K_j)) := \alpha\text{-}{}^h\mathrm{cap}(K_j)$, $j \in \mathbb{N}$. But the heat capacity of $K$ is equal to the Lebesgue measure of $K$, i.e. ${}^h\mathrm{cap}(K) = \lambda^1(K) = 1$ (see e.g. [12]), and hence the $\alpha$-capacity $\alpha\text{-}{}^h\mathrm{cap}$ is not right continuous on compact sets.

It is natural to ask when the $\alpha$-capacity is a Choquet capacity and when both notions are identical, i.e., when the equation $\gamma = \alpha$ holds. Theorem 2.1 give us the answer to this question.

**Theorem 2.1.** *Let $(X, \mathcal{W})$ be a balayage space and $\gamma$ be a Choquet capacity on $X$. Assume that the following condition holds:*

*(P) compact set $K \subset X$ is polar if and only if $\gamma(K) = 0$.*

*Then the following conditions are equivalent:*

*(i) $\alpha$ is a Choquet capacity on $X$;*

*(ii) $\alpha = \gamma$;*

*(iii) the balayage space $(X, \mathcal{W})$ satisfies the axiom of polarity, i.e. the semipolar sets in $X$ are polar.*

Proof. Let the condition (iii) be satisfied and let $K$ be a compact subset of $X$. By Proposition 1.1, the set $K \setminus \beta(K)$ is semipolar. Using the condition (P) we obtain

$$\alpha(K) \leq \gamma(K) \leq \gamma(K \setminus \beta(K)) + \gamma(\beta(K)) = \alpha(K),$$

i.e. $\alpha = \gamma$. (From the validity of the condition (P) for compact sets the validity of (P) for Borel subsets of $X$ follows.)
The implication $(ii) \Rightarrow (i)$ is obvious.
Assume that $\alpha$ is a Choquet capacity on $X$ and that in $(X, \mathcal{W})$ the axiom of polarity does not hold, i.e. there exists nonpolar semipolar set $S \subset X$. According to [4], p. 285, there exists a Borel semipolar set such that $S' \supset S$. By [4], p. 284 there exists a nonpolar compact set $K \subset S'$. Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets in $X$ such that

$$K_{n+1} \subset \operatorname{int} K_n,\ n \in \mathbb{N}, \text{ and } \bigcap_{n=1}^{\infty} K_n = K.$$

According to Lemma 1.2, $K \subset \operatorname{int} K_n \subset \beta(K_n)$ for all $n \in \mathbb{N}$. Consequently, $\alpha(K_n) \geq \gamma(K) > 0$, $n \in \mathbb{N}$, since the set $K$ is nonpolar. From the assumption that $\alpha$ is a Choquet capacity on $X$ it follows that $\alpha(K) > 0$. This is a contradiction since the set $K$ is semipolar. □

**Remark 2.5.** Let $(X, \mathcal{W})$ be a balayage space and $\gamma$ be a Choquet capacity on $X$ satisfying the condition (P) from Theorem 2.1. For a compact set $K \subset X$, let $K \setminus \beta(K)$ be polar. Then the $\alpha$-capacity $\alpha$ is right continuous on $K$, i.e. $\lim_{n\to\infty} \alpha(K_n) = \alpha(K)$ whenever $(K_n)_{n=1}^{\infty}$ is a sequence of compact subsets of $X$ such that $K_n \downarrow K$.
(Indeed, from the assumption (P) of Theorem 2.1 it follows that $\gamma(K) = \gamma(\beta(K))$. According to Lemma 1.2, $\beta(K_n) \subset K_n$, $n \in \mathbb{N}$. Consequently,

$$\alpha(K) \leq \alpha(K_n) = \gamma(\beta(K_n)) \leq \gamma(K_n),\quad n \in \mathbb{N}.$$

Since $\gamma$ is a Choquet capacity, the following relations hold:

$$\alpha(K) \leq \lim_{n\to\infty} \alpha(K_n) \leq \gamma(K) = \gamma(\beta(K)) = \alpha(K).)$$

For a nonpolar semipolar set $K$, the $\alpha$-capacity $\alpha$ is not right continuous on $K$.

**Theorem 2.2.** *Let $\gamma$ be a Choquet capacity on $X$ satisfying the following condition:*

*(R) if $A$ is a relatively compact Borel subset of $X$ then $\gamma(A) = \gamma(\overline{A}^f)$.*

*Further, let $B$ be a Borel subset of $X$. Then $\alpha_*(B) = \gamma(\beta(B))$.*

PROOF. First let $\alpha_*(B) = \infty$. Choose $s \in \mathbb{R}$, $s > 0$, arbitrary. Then there exists a compact set $K \subset B$ such that $s < \alpha(K)$. Consequently, $s < \gamma(\beta(B))$ and $\alpha_*(B) = \gamma(\beta(B))$ since $s$ is arbitrary.
Let now $\alpha_*(B) < \infty$ and $B$ be a relatively compact set. Further, let $(K_n)_{n=1}^{\infty}$, $K_n \subset B$, $n \in \mathbb{N}$, be as in Proposition 1.2. By Definition 2.2, there exist compact sets $L_n \subset B$ such that $\alpha_*(B) \leq \alpha(L_n) + \frac{1}{n}$ for every $n \in \mathbb{N}$. It follows from Proposition 1.2 and from monotonicity of the operator $\beta$ that

$$\overline{(\bigcup_{n=1}^{\infty} \beta(L_n \cup K_n))}^f = \beta(B). \tag{2}$$

We can assume that $K_n \subset K_{n+1}$ and $L_n \subset L_{n+1}$ for every $n \in \mathbb{N}$. Further,

$$\alpha_*(B) \leq \alpha(L_n \cup K_n) + \frac{1}{n}, \qquad \alpha(L_n \cup K_n) \leq \alpha_*(B).$$

This together with Definition 2.2 yields

$$\alpha_*(B) \leq \gamma(\beta(K_n \cup L_n)) + \frac{1}{n} \leq \alpha_*(B) + \frac{1}{n}$$

for every $n \in \mathbb{N}$. By Lemma 2.2, by relation (2) and by the assumption (R), we obtain

$$\alpha_*(B) = \gamma(\bigcup_{n=1}^{\infty} \beta(K_n \cup L_n)) = \gamma(\overline{\bigcup_{n=1}^{\infty} \beta(K_n \cup L_n)}^f) = \gamma(\beta(B)),$$

i.e. the desired equality.
Let $\alpha_*(B) < \infty$ and $B$ be an arbitrary Borel set. Further, let $(U_n)_{n=1}^{\infty}$ be a sequence of open relatively compact subsets of $X$ such that $U_n \uparrow X$. As proved above,

$$\gamma(\beta(B \cap U_n)) = \alpha_*(B \cap U_n) \leq \alpha_*(B),$$

for all $n \in \mathbb{N}$. By monotonicity of $\gamma$ and by Lemma 1.2 (iii) it follows that $\gamma(\beta(B) \cap U_n) \leq \alpha_*(B)$ for all $n \in \mathbb{N}$. Since $\beta(B) \cap U_n \uparrow \beta(B)$, we get according to Lemma 2.2 $\gamma(\beta(B)) \leq \alpha_*(B)$. The converse inequality follows easily from Definition 2.2. □

**Corollary 2.1.** *Let $\gamma$ be a Choquet capacity on $X$ satisfying the condition (R) from Theorem 2.2. Further, let $B$ be a Borel subset of $X$ and let $S \subset X$ be semipolar. Then $\alpha_*(B) = \alpha_*(B \setminus S)$.*

PROOF. First let $S$ be a Borel semipolar set. Obviously, $\beta(B) = \beta(B \setminus S)$. Consequently

$$\alpha_*(B) = \gamma(\beta(B)) = \gamma(\beta(B \setminus S)) = \alpha_*(B \setminus S).$$

If $S$ is an arbitrary semipolar subset of $X$ then there exists a Borel semipolar set $S'$ such that $S \subset S'$; see [4], p. 285. It follows from the monotonicity of $\alpha_*$ that

$$\alpha_*(B \setminus S') \leq \alpha_*(B \setminus S) \leq \alpha_*(B). \tag{3}$$

As proved above, $\alpha_*(B) = \alpha_*(B \setminus S')$. This together with (3) yields the desired equality. □

**Corollary 2.2.** *Let $\gamma$ be a Choquet capacity on $X$ satisfying the condition (R) from Theorem 2.2. Let $B$ be a Borel subset of $X$. Then there exists a Borel semipolar set $S$ such that $\alpha_*(B) = \gamma(B \setminus S)$.*

PROOF. Let $S = B \setminus \beta(B)$. It follows from [4], pp. 297, 272 and 271, that $S$ is a Borel semipolar set. Further, $B \setminus S \subset \beta(B)$. From the monotonicity of $\gamma$ and from Theorem 2.2 we obtain $\gamma(B \setminus S) \leq \alpha_*(B)$. According to Corollary 2.1 and Remark 2.3 $\alpha_*(B) \leq \gamma(B \setminus S)$. □

**Corollary 2.3.** *Let $\gamma$ be a Choquet capacity on $X$ satisfying the following condition:*

*if, for a compact set $K \subset X$, $\gamma(K) = 0$ then $K$ is polar.*

*Further, let $B$ be a Borel subset of $X$. Then $B$ is semipolar if and only if $\alpha_*(B) = 0$.*

PROOF. Let $B$ be a Borel set and $\alpha_*(B) = 0$. For a compact set $K \subset B$, $\alpha_*(K) = \gamma(\beta(K)) = 0$. Since $\beta(K)$ is a Borel set, it follows from the assumption and from [4], p. 248, that the set $\beta(K)$ is polar. But $K = (K \setminus \beta(K)) \cup \beta(K)$. According to Proposition 1.1 the set $K \setminus \beta(K)$ is semipolar. Consequently, every compact set $K \subset B$ is semipolar. By [4], p. 301, it follows that $B$ is semipolar. The rest of assertion is an easy consequence of the definition of $\alpha_*$. □

**Corollary 2.4.** *Let $\gamma$ be a Choquet capacity on $X$ satisfying the condition (R) from Theorem 2.2. Let $B_1$ and $B_2$ be Borel subsets of $X$. Then*

$$\alpha_*(B_1 \cup B_2) + \alpha_*(B_1 \cap B_2) \leq \alpha_*(B_1) + \alpha_*(B_1).$$

PROOF. The assertion follows from Lemma 2.3 and Theorem 2.2. □

**Remark 2.6.** As the following example shows, the assumption in Corollary 2.3 that $B$ is a Borel set can not be omitted.

Consider the potential theory for the heat operator in $\mathbb{R} \times \mathbb{R}$. Let $T$ be a set of Berstein's type (see e.g. [11] for the existence) and $S_1 = \mathbb{R} \times T$, $S_2 = \mathbb{R} \times \complement T$. Let $K$ be an arbitrary compact subset of $S_1$. Then $K \subset \mathbb{R} \times L$ for a suitable countable set $L \subset T$. Consequently, $\mathbb{R} \times L$ is semipolar. Let $\alpha\text{-}^h\text{cap}_*$ denote the inner $\alpha$-capacity deduced from the heat capacity $^h\text{cap}$. (The condition from Corollary 2.3 is of course fulfilled.) From the monotonicity and from the definition of $\alpha\text{-}^h\text{cap}_*$ it follows that $\alpha\text{-}^h\text{cap}(K) = 0$. Consequently, $\alpha\text{-}^h\text{cap}_*(S_1) = 0$. Similarly, $\alpha\text{-}^h\text{cap}_*(S_2) = 0$. Since $S_1 \cup S_2 = \mathbb{R} \times \mathbb{R}$, at least one of the sets $S_i$, i=1, 2, is not semipolar. Consequently, there exists a nonsemipolar set $A \subset \mathbb{R} \times \mathbb{R}$ such that $\alpha\text{-}^h\text{cap}_*(A) = 0$. By Corollary 2.3, this set cannot be a Borel set.

The sets $S_1$ and $S_2$ are an example of sets which does not hold the assertion of Corollary 2.4.

**Remark 2.7.** Let $\gamma$ be a Choquet capacity on $X$ satisfying the condition (R) of Theorem 2.2. For a compact set $K \subset X$, put

$$\tilde{\alpha}(K) := \inf\{\gamma_*(K \setminus S); S \subset X, S \text{ semipolar }\}.$$

Then $\alpha(K) = \tilde{\alpha}(K)$. (Indeed, since $S_K = K \setminus \beta(K)$ is semipolar (see Proposition 1.1), we have $\tilde{\alpha}(K) \leq \gamma_*(K \setminus S_K) \leq \gamma(\beta(K)) = \alpha(K)$. Let $S \subset X$ be an arbitrary semipolar set. According to Corollary 2.1 and Remark 2.3 $\alpha(K) = \alpha_*(K) = \alpha_*(K \setminus S) \leq \gamma_*(K \setminus S)$. Taking infimum with respect to $S \subset X$, $S$ semipolar, we get $\alpha(K) \leq \tilde{\alpha}(K)$.) The proof above shows that the infimum in the definition of $\tilde{\alpha}$ is actually attained.

## 3. Continuous capacities

In [6], we have investigated the so called $\mathbf{K}$-capacity. We recall the basic definitions (cf. [6]). In the following let $X$ be a locally compact Haussdorff space with a countable base and let $\mathcal{M}^+$ stand for the set of all nonnegative Radon measures on $X$. For a set $E \subset X$, let us denote by $\mathcal{M}^+(E)$ the collection of all nonnegative Radon measures on $X$ with *compact* support in $E$. The support of a measure is denoted by supp. A lower semicontinuous function $\mathbf{K} : X \times X \to [0, \infty]$ is called a *kernel on* $X$. The $\mathbf{K}$-*potential* of a measure $\mu \in \mathcal{M}^+$ is defined as

$$\mathbf{K}_\mu(x) := \int_X \mathbf{K}(x, y)\,\mu(dy), \quad x \in X.$$

For a compact set $L \subset X$, the $\mathbf{K}$-*capacity* (corresponding to the kernel $\mathbf{K}$) is defined by

$$\operatorname{cap}(L) := \sup\{\mu(X);\ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \le 1 \text{ on } X\}.$$

The adjoint kernel $\tilde{\mathbf{K}}$ of a kernel $\mathbf{K}$ is defined by $\tilde{\mathbf{K}}(x, y) := \mathbf{K}(y, x)$, $x, y \in X$. Corresponding notions will be noted by a tilde.

By an easy modification of this definition we obtain the notion of a continuous capacity.

**Definition 3.1.** Let $\mathbf{K}$ be a kernel on $X$. A set function $\sigma : \mathcal{K} \to [0, \infty]$ defined by $(L \in \mathcal{K})$

$$\sigma(L) := \sup\{\mu(X);\ \mu \in \mathcal{M}^+(L),\ \mathbf{K}_\mu \le 1 \text{ and continuous } \mathbf{K}\text{-potential on } X\}$$

is called *continuous* $\mathbf{K}$-*capacity* (corresponding to the kernel $\mathbf{K}$) on $X$.
For $E \subset X$, we define *inner continuous* $\mathbf{K}$-*capacity* by

$$\sigma_*(E) := \sup\{\sigma(K);\ K \subset E,\ K \text{ compact }\}.$$

**Remark 3.1.** The continuous capacity was first introduced into potential theory by G. Anger, see [1]. As we will see in Remark 3.3, the continuous capacity is not a Choquet capacity in general. The notation 'continuous capacity' is deduced from the requirement of continuity of $\mathbf{K}$-potentials in Definition 3.1.

The following Lemma is an easy consequence of Definition 3.1.

**Lemma 3.1.** *Let $c$ and $\sigma$ denote the $\mathbf{K}$-capacity and the continuous $\mathbf{K}$-capacity on $X$, respectively. Let $K, L \in \mathcal{K}$. Then*

*(i)* $0 \le \sigma(L) \le c(L) \le \infty$;

*(ii) if* $K \subset L$ *then* $\sigma(K) \le \sigma(L)$;

*(iii)* $\sigma(K \cup L) \le \sigma(K) + \sigma(L)$.

The question of the relation between a continuous $\mathbf{K}$-capacity and an $\alpha$-capacity deduced from a $\mathbf{K}$-capacity is natural. Because $\alpha$-capacity is defined by using a structure of a balayage space and a continuous capacity does not, the kernel $\mathbf{K}$ on $X$ must be in a relation with a balayage space.
From now, we will consider the balayage space $(X, \mathcal{W})$ and the kernel $\mathbf{K}$ on $X$ having the following property:
there exists a balayage space $(X, \widetilde{\mathcal{W}})$ such that:

- $1 \in \mathcal{W} \cap \widetilde{\mathcal{W}}$;
- for every $p \in \mathcal{P}(X)$ there exists exactly one measure $\mu \in \mathcal{M}^+$ such that $\mathbf{K}_\mu = p$ and $\operatorname{supp} \mu = C(p)$;
- if $\mu \in \mathcal{M}^+$ and $\overline{\{\mathbf{K}_\mu < \infty\}} = X$, then $\mathbf{K}_\mu \in \mathcal{P}(X)$;
- for every $\widetilde{p} \in \widetilde{\mathcal{P}}(X)$ there exists exactly one measure $\mu \in \mathcal{M}^+$ such that $\widetilde{\mathbf{K}}_\mu = \widetilde{p}$ and $\operatorname{supp} \mu = C(\widetilde{p})$, ($\widetilde{\mathbf{K}}$ is the adjoint kernel of the kernel $\mathbf{K}$);
- if $\mu \in \mathcal{M}^+$ and $\overline{\{\widetilde{\mathbf{K}}_\mu < \infty\}} = X$, then $\widetilde{\mathbf{K}}_\mu \in \widetilde{\mathcal{P}}(X)$.

(Here and in the following $\mathcal{P}(X)$ and $\widetilde{\mathcal{P}}(X)$ stand for the set of all potentials (with respect to $\mathcal{W}$ and $\widetilde{\mathcal{W}}$, respectively) on $X$; $C(u)$ and $C(\widetilde{u})$ denote the carrier of $u \in \mathcal{W}$ and $\widetilde{u} \in \widetilde{\mathcal{W}}$, respectively.)

The following theorem deal with the question formulated above.

**Theorem 3.1.** *Let $\gamma$ be a $\mathbf{K}$-capacity on $X$, $\sigma$ be a continuous $\mathbf{K}$-capacity on $X$ (both corresponding to the kernel $\mathbf{K}$) and let $\alpha$ denote the $\alpha$-capacity deduced from the capacity $\gamma$. Then, for every compact set $L \subset X$,*

$$\alpha(L) = \sigma(L).$$

Proof. Let $L \in \mathcal{K}$. According to Proposition 1.3, $\hat{R}_1^{\beta(L)} = R_1^{\beta(L)}$ and $C(\hat{R}_1^{\beta(L)}) \subset \overline{\beta(L)}$. By Lemma 1.2, $\beta(L) \subset L$. Consequently, $\overline{\beta(L)} \subset L$. From here and from Remark 2.2 it follows that $\beta(L)$ is a relatively compact Borel set. Let $\mu \in \mathcal{M}^+(\overline{\beta(L)})$ be a measure, which existence follows from [6], p. 97, with the properties

$$\hat{R}_1^{\beta(L)} = \mathbf{K}_\mu \quad \text{and} \quad \gamma(\beta(L)) = \mu(X) = \alpha(L).$$

Further, let $U \subset X$ be an open, relatively compact set, $L \subset U$. The existence of a measure $\nu \in \mathcal{M}^+(U)$ such that

$$\tilde{\mathbf{K}}_\nu = 1 \text{ on the neighborhood of } L \tag{4}$$

follows from the [6], p. 91. Using the Fubini's theorem, we get

$$\alpha(L) = \int_X \tilde{\mathbf{K}}_\nu \, d\mu = \int_X \mathbf{K}_\mu \, d\nu = \int_X \hat{R}_1^{\beta(L)} \, d\nu.$$

By Proposition 1.3 and [5], p. 7, we obtain

$$\begin{aligned}\alpha(L) &= \int_X \sup\{\mathbf{K}_{\mu'};\ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\}\, d\nu \\ &= \sup\{\int_X \mathbf{K}_{\mu'}\, d\nu;\ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\}.\end{aligned}$$

Using the Fubini's theorem and the equality (4), we get

$$\alpha(L) = \sup\{\mu'(L);\ \mu' \in \mathcal{M}^+(L),\ \mathbf{K}_{\mu'} \le 1 \text{ and continuous on } X\},$$

and hence $\alpha(L) = \sigma(L)$, what we wanted to prove. □

**Remark 3.2.** Let the assumption of Theorem 3.1 be fulfilled. Then the condition (R) of Theorem 2.2 is satisfied. (Indeed, let $B$ be a Borel relatively compact subset of $X$ then, by [4], p. 273, $\hat{R}_1^B = \hat{R}_1^{\overline{B}^f}$. It follows from this and [6], p. 97, that $\gamma(B) = \gamma(\overline{B}^f)$.)

**Remark 3.3.** From Theorem 3.1 and from Theorem 2.1 it follows that a continuous capacity is a Choquet capacity if and only if the corresponding balayage space satisfies the axiom of polarity. It is known that the balayage space generated by the heat operator does not satisfy this axiom. Consequently, the continuous heat capacity is not a Choquet capacity.

**Corollary 3.1.** *Let the assumptions of Theorem 3.1 be fulfilled. If $K$, $L \in \mathcal{K}$ then*

$$\sigma(K \cup L) + \sigma(K \cap L) \leq \sigma(K) + \sigma(L).$$

PROOF. The assertion of the corollary follows from Theorem 3.1 and from Lemma 2.3(iii). □

**Corollary 3.2.** *For all $L \in \mathcal{K}$, let $C(\hat{R}_1^L) \subset L$, $C(\hat{\tilde{R}}_1^L) \subset L$ and for all $x \in X$, let the set $\{x\}$ be $\mathcal{W}$- and $\tilde{\mathcal{W}}$-totally thin. Let $\sigma$ and $\tilde{\sigma}$, respectively, denote the continuous $\mathbf{K}$-capacity and the continuous $\tilde{\mathbf{K}}$-capacity on $X$, respectively. Then, for all $L \in \mathcal{K}$, the following equality holds*

$$\sigma(L) = \tilde{\sigma}(L).$$

PROOF. Let $\mathcal{P}$, $\tilde{\mathcal{P}}$, $\mathcal{S}$ and $\tilde{\mathcal{S}}$ denote the system of all $\mathcal{W}$-polar, $\tilde{\mathcal{W}}$-polar, $\mathcal{W}$-semipolar and $\tilde{\mathcal{W}}$-semipolar subsets in $X$, respectively. Further, denote $c$ and $\tilde{c}$ the $\mathbf{K}$-capacity and the $\tilde{\mathbf{K}}$-capacity on $X$, respectively. By [6], p. 97, $\mathcal{P} = \tilde{\mathcal{P}}$. According to [9], p. 510, it follows that $\mathcal{S} = \tilde{\mathcal{S}}$. Now, by [6], p. 92, and by Remark 2.7 it follows that for $L \in \mathcal{K}$ the following equalities hold

$$\sigma(L) = \inf\{c_*(L \setminus S);\ S \in \mathcal{S}\} \quad \text{and} \quad \tilde{\sigma}(L) = \inf\{\tilde{c}_*(L \setminus S);\ S \in \tilde{\mathcal{S}}\}.$$

By [6], p. 92, $c = \tilde{c}$. From this and above the desired equality follows. □

**Corollary 3.3.** *Let the assumption of Corollary 3.2 be fulfilled. Further let $S$ be a Borel subset of $X$. Then the following conditions are equivalent:*

*(i)* $S$ *is* $\mathcal{W}$*-semipolar;* *(iii)* $S$ *is* $\tilde{\mathcal{W}}$*-semipolar;*
*(ii)* $\sigma_*(S) = 0$; *(iv)* $\tilde{\sigma}_*(S) = 0$.

PROOF. The equivalence (ii)⇔(iv) follows from Corollary 3.2, the equivalence (i)⇔(iii) from [9], p. 510, and the equivalence (i)⇔(ii) from Corollary 2.3. □

**Remark 3.4.** In this remark we will give the partial answer to some unsolved problems from the book [1], pp. 94, 95.
Let $c$ and $\sigma$ denote the $\mathbf{K}$-capacity and the continuous $\mathbf{K}$-capacity (corresponding the kernel $\mathbf{K}$ on $X$), respectively, and let the assumption of Theorem 3.1 be fulfilled.
**Problem 19.** Which kernels on $X$ satisfy C-maximum principle? (The kernel $\mathbf{K}$ on $X$ satisfies *C-maximum principle*, if the following holds:

if $\mu \in \mathcal{M}^+(X)$ and if $\mathbf{K}_\mu$ is a continuous $\mathbf{K}$-potential on $X$, $\mathbf{K}_\mu \leq M$ on supp $\mu$ ($M \in \mathbb{R}$) then $\mathbf{K}_\mu \leq M$ on $X$.)

In the considered situation all kernels satisfy C-maximum principle (see [4], p. 116).
**Problem 21.** For which compact sets $L \subset X$ the continuous $\mathbf{K}$-capacity $\sigma$ is right continuous on $L$?
Let $\mathcal{K}_1$ be a system of all compact subsets of $X$ such that:

if $L \in \mathcal{K}_1$, $L_n \in \mathcal{K}$, $n \in \mathbb{N}$, and $L_n \downarrow L$, then $\sigma(L_n) \to \sigma(L)$ for $n \to \infty$.

From Remark 2.5 we get:

$$\{L \in \mathcal{K};\, L \setminus \beta(L) \text{ polar }\} \subset \mathcal{K}_1 \subset \complement\{L \in \mathcal{K};\, L \text{ nonpolar semipolar }\}.$$

The inclusions given above solve partially our problem.

**Problem 22.** For which kernel $\mathbf{K}$ the continuous $\mathbf{K}$-capacity $\sigma$ is right continuous?
Full answer to this question gives Theorem 2.1. They are only this kernels on $X$ for which the corresponding balayage space satisfies the axiom of polarity.

**Problem 23.** Does there exist a kernel $\mathbf{K}$ on $\mathbb{R}$ such that the continuous $\mathbf{K}$-capacity $\sigma$ is right continuous?
The Riesz kernels $N_\alpha$, $\alpha \in ]0, 1[$, on $\mathbb{R}$ have desired property (by Theorem 2.1).

**Problem 29.** Which relation between a continuous $\mathbf{K}$-capacity and a continuous $\tilde{\mathbf{K}}$-capacity holds?
The answer to this problem gives Corollary 3.2.

**Problem 30.** Which sets $B \subset X$ are $\sigma$-capacitable? (The set $B \subset X$ is *$\sigma$-capacitable* if

$$\sigma_*(B) = \inf\{\sigma_*(U);\, U \supset B,\, U \text{ open }\} =: \sigma^*(B).)$$

If $P \subset X$ is polar, then by [6], p. 97, $c^*(P) = 0$. Further, $0 \le \sigma_*(P) \le \sigma^*(P) \le c^*(P) \le 0$. Hence, the polar sets are $\sigma$-capacitable.
Let

$$\Sigma := \{L \cup P;\, L \in \mathcal{K},\, P \subset X,\, P \text{ and } L \setminus \beta(L) \text{ polar }\}.$$

Let $A \in \Sigma$, $A = L \cup P$, $L \in \mathcal{K}$, $P$ and $L \setminus \beta(L)$ be polar. Further, let $L_n \in \mathcal{K}$, $n \in \mathbb{N}$, $L_n \downarrow L$, $L \subset \text{int } L_n$, $n \in \mathbb{N}$. Then:

$$\sigma^*(L \cup P) \le \sigma^*(L) + \sigma^*(P) \le \sigma_*(\text{int } L_n) \le \sigma_*(L_n) = c(\beta(L_n)) \le c(L_n).$$

Since $c$ is a Choquet capacity, $\sigma^*(L \cup P) \le c(L)$. Further,

$$\sigma_*(L \cup P) \le \sigma^*(L \cup P) \le c(L) = c(\beta(L)) = \sigma(L) = \sigma_*(L) \le \sigma_*(L \cup P).$$

The sets $A \in \Sigma$ are hence $\sigma$-capacitable. Further, nonpolar semipolar sets are not $\sigma$-capacitable (see Remark 2.5).

# References


[1] G.Anger, *Funktionalanalytische Betrachtungen bei Differentialgleichungen unter Verwendung von Methoden der Potentialtheorie I*, Akademie-Verlag, Berlin, 1967.

[2] H.Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Mathematics **22**, Springer–Verlag, Berlin, 1966.

[3] J.Bliedtner, W.Hansen, *Simplicial cones in potential theory*, Inventiones Math. **29** (1975), 83–110.

[4] J.Bliedtner, W.Hansen, *Potential theory. An Analytic and Probabilistic Approach to Balayage*, Springer-Verlag, Berlin, 1986.

[5] M.Brelot, *Éléments de la théorie classique du potentiel*, 2$^e$ ed., Centre de Documentation Universitaire, Paris, 1961.

[6] M.Brzezina, *Kernels and Choquet capacities*, Aequationes Math. **45** (1993), 89–99.

[7] C.Constantinescu, A.Cornea, *Potential theory on harmonic spaces*, Springer-Verlag, Berlin, 1972.

[8] W.Hansen, *Semi-polar sets are almost negligible*, J. reine angew. Math. **314** (1980), 217–220.

[9] W.Hansen, *Semi-polar sets and quasi-balayage*, Math. Ann. **257** (1981), 495-517.

[10] L.L.Helms, *Introduction to potential theory*, Wiley Interscience, New York, 1969.

[11] C.J.Oxtoby, *Measure and category*, Springer Verlag, Berlin, 1971.

[12] N.A.Watson, *Thermal capacity*, Proc. London Math. Soc. **(3)37** (1978), 342–362.

Department of Mathematics, Technical University of Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic