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ROBUST CONTROL METHODS

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metody robustního řízení

Contents

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Robust control - Robustní řízení

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The discrete time state model - Stavový model diskrétního systému

Controllability and observability - Řiditelnost a Pozorovatelnost

Internal stability - Vnitřní stabilita

The singular value decomposition(SVD) - Rozklad matice podél singulárních čísel

Cost function - Kriteriální funkce

Quadratic Cost Functions - Kvadratická kriteriální funkce

The system 2-norm cost function - Kritériální funkce 2-norm systému

The system ∞-norm cost function - Kritériální funkce ∞-norm systému

Nominal stability - Nominální stabilita

Nominal performance - Nominální chování

Robust stability - Robustní stabilita

Robust performance - Robustní chování

Uncertainty - Neurčitost

Unstructured uncertainty - Nestrukturovaná neurčitost

Structured uncertainty - Strukturovaná neurčitost

The structured singular value - Strukturované singulární číslo

performance robustness analysis using the SSV - Analýza robustního chování pomocí SSV

The Hamiltonian equations - Hamiltonovy rovnice

The Riccati equation - Riccatiho rovnice

Performance bound - mez chování

Steady state full information control - Ustálený stav s úplnou řídící informací

Saddle point - Sedlový bod

Full information control - Řízení s plnou informací

H_∞ suboptimal controller - H_∞ suboptimální regulátor

H_∞ estimation - H_∞ estimace

Adjoint system - Adjungovaný systém

Finite time estimation - Estimace v konečném čase

H_∞ optimal estimator - H_∞ optimálni estimator

Annotation

Robustness of control systems to disturbances and uncertainties has always been the central issue in feedback control. Feedback would not be needed for most control systems if there were no disturbances and uncertainties.

A constant gain feedback controller may be designed to cope with parameter changes provided that such changes are within certain bounds. This motivated many researchers to study and accomplish several kinds of control system, requiring mathematical theory.

H_∞ optimal control theory is one of the off-line modern feedback control method that make possible to design a constant gain feedback controller which is able to stabilize the real plant in each of its modes.

I must emphasize that H_∞ Control is an extended topics which involves the study of various fields of mathematics, and control. In this thesis I have tried to present a brief discussion of H_∞ control theory and its applications.

List of Symbols

Symbol	Meaning
$1(t)$	Unit step function
α	Real part of pole, coefficient
$\alpha(s)$	Characteristic equation
$\mathbf{A}(t)$, $\mathbf{A}$	State matrix
$\mathbf{A}_c$	Controller state matrix
$\mathbf{A}_{cl}$	Closed-loop system state matrix
adj(•)	Adjugate of a matrix
$\mathbf{A}_f$	State matrix for shaping filter
$\mathcal{B}$	Linear space on which a norm is defined (Banach space)
$\mathbf{B}(t)$, $\mathbf{B}$	Input matrix
$\mathbf{B}_{cl}$	Closed-loop system input matrix
$\mathbf{B}_{cr}$	Controller input matrix for the input r
$\mathbf{B}_f$	Input matrix for shaping filter
$\mathbf{B}_w$	Input matrix for the input w
$\mathbf{B}_{wo}$	Input matrix associated with a constant input
$C(t)$	Correction term in unbiased estimator
$\mathbf{C}(t)$, $\mathbf{C}$	Output matrix
$\mathbf{C}_c$	Controller output matrix
$\mathbf{C}_{cl}$	Closed-loop system output matrix
$\mathbf{C}_e$	Output matrix associated with the output error
$\mathbf{C}_f$	Output matrix for shaping filter
$\mathbf{C}_y$	Output matrix for the output y
$\mathcal{C}^{m\times n}$	Set of complex m x n matrices
$\mathcal{C}^n$	Set of complex n-dimensional vectors
$d(t)$	Output disturbance
$\mathbf{D}(t)$, $\mathbf{D}$	Input-to-output coupling matrix
$\mathbf{D}_{cl}$	Closed-loop system input -to-output coupling matrix
$\mathbf{D}_{cr}$	Controller input-to output coupling matrix from r to u
den(s)	Transfer function denominator
det(•)	Determinant of a matrix
$\mathbf{D}_{yw}$	Input-to-output coupling matrix from w to y

$\mathbf{D}_L(s)$	Diagonal scaling matrix (left D-scaling matrix)
$\mathbf{D}_R(s)$	Diagonal scaling matrix (right *D-scaling matrix)*
$\mathbf{\Delta}(s)$	Normalized general perturbation
$\mathbf{\Delta}'(s)$	Unnormalized general perturbation
$\mathbf{\Delta}_a(s)$	Additive perturbation
$\mathbf{\Delta}_{fi}(s)$	Input freeback perturbation
$\mathbf{\Delta}_{fo}(s)$	Output freedback perturbation
$\mathbf{\Delta}_{G_{yr}}$	Change in the closed-loop transfer function due to a perturbation
$\mathbf{\Delta}_i(s)$	Input-multiplicative perturbation
$\mathbf{\Delta}_{max}(j\omega)$	Frequency-dependent perturbation bound
$\mathbf{\Delta}_o(s)$	Output multiplicative perturbation
$\mathbf{\Delta}_p$	Normalized perturbation augmented with performance block
$\widetilde{\mathbf{\Delta}}$	Specific perturbation
$\overline{\mathbf{\Delta}}$	Set of perturbations with a given block diagonal structure
$\overline{\mathbf{\Delta}}_s$	Set of block diagonal perturbations (square blocks)
$\Delta \mathrm{J}(\bullet, \delta\bullet)$	Increment of J
$\mathrm{d_i}$	D-scale
$\delta \mathrm{J}(\bullet, \delta\bullet)$	Variation of J
$\delta \mathrm{x}$	Variation of x, differential of x
δ_{ij}	Kronecker delta function
$\delta(t)$	Dirac delta (impulse) function
ε	Small, positive constant
$\mathrm{e}(t)$	Tracking error, estimation error
e_I	Integral of the tracking error
$\mathrm{E}[\bullet]$	Expectation operator
ϕ	Phase perturbation
$\mathbf{\Phi}(t,t_0)$, $\mathbf{\Phi}(t)$	State-transition matrix of a time-varying, time-invariant system
$\mathbf{\Phi}$, $\mathbf{\Phi}(T)$	Discrete-time state matrix
$\mathbf{F}$, $\mathbf{F}(t, \tau)$	Linear estimator weights
ϕ_{max}	Maximum-phase perturbation
ϕ_{min}	Minimum-phase perturbation
g	Uncertain gain
γ	∞-norm performance bound
$\mathbf{G}(s)$	Laplace transfer function of a generic system or a plant
$\mathbf{G}(t)$	Kalman gain

$\mathbf{g}(t)$	Impulse response matrix (generic system)
$\widetilde{\mathbf{G}}_c(s)$	Reduced-order controller transfer function
$\zeta(\bullet)$	Linear system (possibly time-varying)
$\Gamma, \Gamma(T)$	Discrete-time input matrix
$\mathbf{G}_0(s)$	Nominal plant transfer function
$\mathbf{G}_1(t)$	State feedback gain for normalized measurement
$\mathbf{G}_{ab}(s)$	Transfer function from input b to output a
$\mathbf{G}_{cl}(s)$	Closed-loop system transfer function
$\mathbf{g}_{cl}(s)$	Closed-loop system impulse response
G_Δ	System with input w - $w - \gamma^{-2}\mathbf{B}_w^T\mathbf{P}\times$ and output $u + \mathbf{B}_w^T\mathbf{P}\times$
$\mathbf{G}_l(s)$	Loop transfer function
$\mathbf{G}_s(s)$	Stable part of a transfer function
$\mathbf{G}_u(s)$	Unstable part of a transfer function
H	Hardy space
$\mathbf{H}(s)$	Perturbed closed-loop system
$\mathcal{H}$	Hamiltonian matrix (LQR)
$\mathcal{H}_\infty$	Hamiltonian matrix (full information control)
$\mathbf{I}$	Identity matrix
$\inf(\bullet)$	Infimum
j	Square root of - 1, discrete index
$J(\bullet)$	Cost function, objective function
J_2	2-norm cost function
$J_a(\bullet, p)$	Augmented cost function augmented objective function
J_γ	Objective function for suboptimal control
$J_{\gamma a}$	Augmented objective function for suboptimal control
J_{LQR}	LQR cost function
J_{ss}	Cost for suboptimal steady-state control
J_{TV}	Cost for time-varying optimal control
k	Discrete time index
$\mathbf{K}(s)$, $\mathbf{K}$	Controller transfer function, controller gain matrix
$\mathbf{K}_r$	Feedforward control gain (tracking input)
$\mathbf{K}_w$	Feedforward control gain (disturbance feedforward)
$\mathbf{A}$	Diagonal matrix containing eigenvalues
$\mathcal{L}$	Laplace transform

ℓ_2	Space of signals with finite 2-norms
λ	Eingenvalue
$\mathbf{L}$	Controllability matrix
$L_l \times n_i$	Matrix demensions
$\mathbf{L}_c$	Controllability grammian
$\lambda_i(\bullet)$	ith eigenvalue of a matrix
lim	Limit
$\mathbf{L}_o$	Observability grammian
$m(t)$	Measured output
$\max\{\bullet\}$	Maximum operator
$\mathbf{M}_{ij}$	Element of the matrix **M** (row i, column j)
$\min\{\bullet\}$	Minimum operator
$m_x(t)$	Mean of the random process x(t)
$m_{x/y}$	Conditional mean
$\widetilde{m}(t)$	Kalman innovations process
$\mu_{\bar{\Delta}}(\bullet)$	Structured singular value
$\mathbf{N}$	Observability matrix
n	Discrete-time index, matrix dimension
$\mathbf{N}_s$	Square version of a matrix
$\mathbf{N}(s)$	Nominal closed-loop transfer function (standard form)
$\mathrm{num}(s)$	Transfer function numerator
n_v	Dimension of the vector **v**
N_z	Number of right falf-plane zeros of $\{1 + G(s)K(s)\}$
$\mathbf{P}(s)$	Plant transfer function in standard form
$\mathbf{P}(t)$, $\mathbf{P}$	Riccati solution (LQR, H_∞ full information control)
ρ, $\rho(t)$	Lagrange multiplier
$\rho_l(t)$	Pulse function
p_i	Poles of a system
$\mathbf{Q}(t)$	State weighting function
$\mathbf{Q}_R$	Combined state and control weighting matrix
$\mathbf{R}$	Factor in Cholsky decomposition of controllability grammian
R	Rank
ρ	Measurement weighting coefficient (output LTR)
$\mathbf{R}(t)$	Control weighting function
$r(t)$	Reference input
$p(\bullet)$	Spectral radius
$\mathrm{rank}(\bullet)$	Rank of a matrix
$\mathrm{Ric}(\bullet)$	Ricci operator

$\Re^{m \times n}$	Set of real m $\times$ n matrices
$\Re^n$	Set of real n-dimensional vectors
$\mathbf{S}$	Matrix of singular values
s	Laplace variable
σ_i	Singular value
$\overline{\sigma}$	Maximum singular value
$\underline{\sigma}$	Minimum singular value
$\sup(\bullet)$	Supremum
$\mathbf{S}_w(\omega)$	Spectral density of the random process w(t)
T	Sampling time(for discrete-time systems)
t	Time
τ	Time variable; time difference in correlation function
$\mathbf{T}, \mathbf{T}(t)$	Transformation matrix for change of basis
t_0	Initial time
T_1, T_2	Components of matrix fraction decomposition
T_f	Final time
$tr(\bullet)$	Trace operator
T_s	Settling time
$\mathbf{U}$	Matrix of left singular vectors
$u(t)$	Control input; generic system imput
U_i	Normalized control input
$\overline{u}_i$	Upper bound on the *i*th element of the control
$\mathbf{V}$	Matrix of right singular vectors
v	Generic vector, measurement error, measurement noise
$v(t)$	Measurement noise
v_1	White shaping filter input, normalized measurement input
v_2	Noise on constructed measurement
v_k	*k*th element of the vector **v**
v^T	Transpose of the vector v
$v^{\dagger}$	conjugate transpose of v
Ω	$(s\mathbf{I} - \mathbf{A}_{11})^{-1}$
$\mathbf{W}$	Weighted matrix
ω	Frequency
$\mathbf{W}(t)$	Weighted function
$w(t)$	Disturbance input

w_0	Constant disturbance input
$\mathbf{w}_0(t)$	Output weighted function
w_1	While shaping filter input, normalized plant input
w_f	Fictitious noise for LTR
$\mathbf{W}_i(t)$	Input-weighted function
$\hat{x}$	Estimate of x
$\hat{x}_\varepsilon$	Estimate of x given data through x - ε
$X(s)$	Laplace transform of x(t)
$X(t)$	State
x^*	Extremal of x
x_0	Initial state
x_d	Desired state
x_d	Desired state (tracking systems)
x_f	Final state
$\tilde{x}(t)$	State after change of basis or coordinate translation, adjoint state
$\mathbf{\Psi}$	Eigenvector matrix
$\mathbf{Y}(t)$	Reference output weighting function
$y(t)$	Reference output; generic system output
y_1	Normalized reference output
y_w	Reference output due to w
$\mathbf{y}$	Hamiltonian matrix (Kalman filter)
$\mathbf{y}_\infty$	Hamiltonian matrix (H_∞ estimation)
$\mathbf{Z}(t)$	Output error weighting function
$z(t)$	Constructed measurement (nonwhite measurement noise)
$z(t)$	Coordinate translation of the state
$z(t)$	Transformed state
z_i	Zeros of a system
$[a,b)$	Real interval, closed at a and open at b
∞	Infinity
$\lVert\bullet\rVert_2$	2-norm(vector, signal, or system)
$\lVert\bullet\rVert$	Euclidean vector norm, generic norm
$\lVert\bullet\rVert_{2,[t0,\,tf]}$	Finite-time signal 2-norm
$\lVert\bullet\rVert_{\infty,[t0,\,tf]}$	Finite-time signal ∞-norm
$\lVert\bullet\rVert_E$	Vector Euclidian norm
$\lVert\bullet\rVert_{W(t)}$	Weighted signal 2-norm
$\lVert\bullet\rVert_W$	Weighted vector 2-norm

$\|\|\bullet\|\|_\infty$	∞-norm(vector, signal, or system)
$\angle\{\bullet\}$	Argument of a complex number
$\otimes$	Convolution operator
$\in$	Element of a set
$\|\bullet\|$	Magnitude of a complex number

1 Introduction

The purpose of thesis is to present modern feedback control method based on H_∞ optimal control theory in a concise way, requiring only a moderate mathematical background. We consider a control system with possibly multiple sources of uncertainties, noises, and disturbances as show in Figure 1.1.

Many industrial systems may work in several different modes of operation. Each of the modes is naturally described by a different transfer function. In such a case, it is highly desirable to find a single fixed and off-line designed controller that is able to stabilize the real plant in each of its modes. If it exists, such a controller is definitely robust.

This thesis is divided into several chapters, which each chapter includes following subjects:

Chapter 2 provides a summary of the continuous-time and discrete-time linear state models along with their general solutions in a MIMO using state space models. The transfer function and and general models of feedback control system are discussed. Internal stability of linear feedback systems is examined in depth. Controllability and observability are defined, and tests for these properties are given.

An overview of control system performance analysis is given in Chapter 3. This chapter begins by defining norms for signals and systems. A more detailed presentation is given of the singular values and principal gains. A number of cost functions that are useful in control system design are also presented. This chapter concludes with a discussion of methods for computing the various performance criteria and costs.

Control system robustness analysis is presented in Chapter 4. Two general types of perturbations are presented: transfer functions (unstructured uncertainty) and multiple transfer functions and/or parameters (structured uncertainty). The stability and performance robustness of general systems subject to these perturbations is then addressed.

H_∞ full information control is developed along with H_∞ estimation in Chapter 5. The chapter begins with a brief summary of the mathematics of differential games. These results are then used to develop suboptimal solutions to the H_∞ full information control problem. The H_∞ full information control problem is defined: Find a feedback controller that yields a given bound on the closed-loop system ∞-norm from the disturbance input to the reference output. The measurements available to the controller are assumed to be the entire plant state and the disturbance input (hence the term *full information*).

The suboptimal H_∞ estimation problem is defined: Find an estimator that yields a given bound on the gain from the plant disturbance inputs (including measurement error) to the estimation error. The H_∞ estimation results are developed using the full information results and duality.

The H_∞ output feedback controller is developed in Chapter 6. The application of H_∞ control is also discussed in detail. Tracking system design, integral control, and designing for robustness are all addressed. Performance limitations are also presented, since these limitations frequently affect the specification of H_∞ control problems. A very general procedure for the development of robust controllers, known as μ -synthesis, is also presented in Chapter 6. This chapter concludes with a design case study of μ-synthesis methodologies.

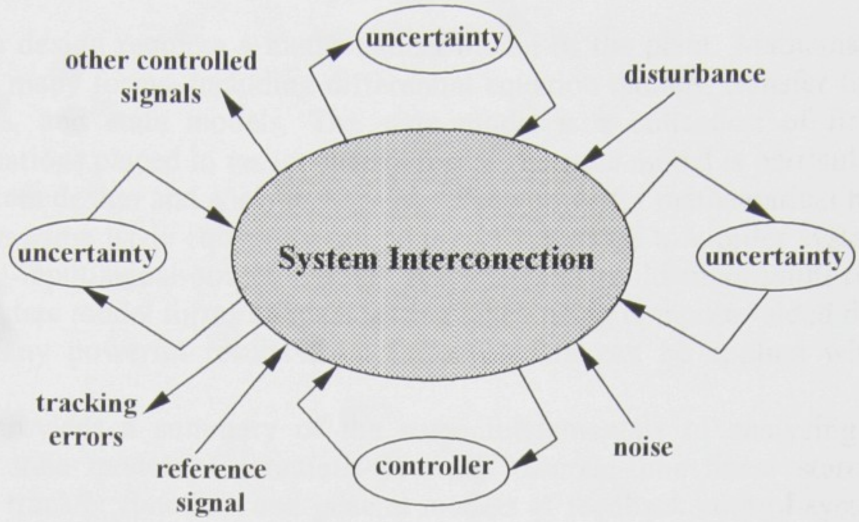


FIGURE. 1.1 General system interconnection

2 Fundamental Issues of Control and Important Application

Control system design requires a mathematical model of the plant. Mathematical models of plants come in many forms, including differential equation models, transfer function models, block diagrams, and state models. The state model is a collection of first-order, linear differential equations placed in vector-matrix form. The state model is particularly convenient for control system design and analysis because of the powerful mathematical results available and because the same basic equations can be used to describe low-order systems, high-order systems, signal-input/signal-output (SISO) systems, and multi-input/multi-output (MIMO) systems. The state model forms an ideal setting when using computer-aided design software. In addition, many powerful results from linear algebra can be applied when using state models.

This chapter provides a summary of the some fundamentals of analysing MIMO linear systems using state models. Continuous-time and discrete-time linear state equations are presented. The transfer function and general models of feedback control system are defined and discussed. The general models of feedback control system includes the desired outputs and disturbance inputs. Internal stability of linear feedback systems is examined. The chapter concludes with a definitions of controllability and observability along with a practical means of testing a state model for controllability and observability.

2.1 The Continuous-Time State Model

A continuous-time linear system is a transformation between the input (and the initial conditions) and the output, both being vector functions of time in general. A mathematical model of a continuous-time linear system is typically in the form of one or more ordinary differential equations involving the input, the output, and possibly some additional variables that are intermediary between the input and output. The mathematical model can be put into a standard form known as the stale model:

$$\dot{x}(t) = \mathbf{A}(t)\,x(t) + \mathbf{B}(t)\,u(t) \tag{2.1a}$$

$$y(t) = \mathbf{C}(t)\,x(t) + \mathbf{D}(t)\,u(t) \tag{2.1b}$$

The variable $x(t) \in \Re^{n_x}$ is called the stale of the system, $x(t) \in \Re^{n_u}$ is the input to the system, and $y(t) \in \Re^{n_y}$ is the output of the system. The matrix $\mathbf{A}(t) \in \Re^{n_x * n_x}$ is the stale matrix, $\mathbf{B}(t) \in \Re^{n_x * n_u}$ is the input matrix, $\mathbf{C}(t) \in \Re^{n_y * n_x}$ is the output matrix, and $\mathbf{D}(t) \in \Re^{n_y * n_u}$ is the input-to-output coupling matrix. Equation (2.1a) is called the stale equation, and (2.1b) is the measurement or output equation.

The state model and its solution can be simplified when the system is time-invariant (i.e., the system matrixes in the state model do not depend on time):

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{2.2a}$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \tag{2.2b}$$

[Kuo-1991]

2.2 The Discrete-Time State Model and Simulation

Computer simulation is frequently used to evaluate control system performance. Computer simulation can be used to solve for outputs when the system is of high order or is subject to complicated inputs that are not easily amenable to analytic solutions. In addition, the effects of time-variations, delays, and nonlinearities can be easily evaluated using simulation.

Computer simulation is also typically used to verify analytic results prior to hardware implementation.
Computer simulation requires that the continuous-time plant and controller models be approximated by discrete-time systems, That is, by difference equations. Digital controllers also utilize difference equation models when operating. One method of generating a digital controller is to design a continuous-time controller and then approximate this controller in discrete time. The generation of discrete-time approximations of continuous-time systems is, therefore, of fundamental importance in simulation and control system design.
A number of methods ran be used to form a discrete-time approximation of a continuous-time system: Euler's method, the zero-order hold approximation, the bilinear transformation, the impulse invariant approximation, and so on. Two of these approximation methods are considered here: Euler's method and the zero-order Hold approximation. Euler's method provides a means of generating discrete-time state equations for a wide range of applications, including systems that are time-varying and nonlinear. The zero-order hold approximation is also presented because this method tends to provide better results for linear, time-invariant systems.
A discrete-time state equation can be obtained from the stale equation (2.2a) using Euler's method of approximating the derivative:

$$\frac{x(kT+T)-x(kT)}{T} \approx \dot{x}(kT) = \mathbf{A}x(kT) + \mathbf{B}u(kT) \tag{2.3}$$

A difference equation for the stale can then be obtained:

$$x(kT+T) = x(kT) + T\{\mathbf{A}x(kT) + \mathbf{B}u(kT)\}. \tag{2.4}$$

A discrete-time state equation can also be obtained from (2.2a) using the zero-order hold approximation. In this cane, the stale difference equation is obtained by solving the state equation (2.2a) while assuming the input is constant over the sampling time:

$$u(t) = u(0) \text{ for all } t \in [0, T).$$

The stale at the sampling time T is then

$$x(T) = e^{\mathbf{A}T}x(0) + \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}u(0)d\tau \tag{2.5}$$

The input can be taken out of the integral:

$$x(T) = [e^{\mathbf{A}T}]x(0) + [\int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}d\tau]u(0) = [e^{\mathbf{A}T}]x(0) + [\int_0^T e^{\mathbf{A}\tau}\mathbf{B}d\tau]u(0), \tag{2.6}$$

where the second integral in this equation is generated from the first by performing a change of variables. This equation can be written as follows:

$$x(T) = \mathbf{\Phi}(T)x(0) + \mathbf{\Gamma}(T)u(0), \tag{2.7}$$

where

$$\mathbf{\Phi}(T) = e^{\mathbf{A}T}, \quad \mathbf{\Gamma}(T) = \int_0^T e^{\mathbf{A}\tau}\mathbf{B}d\tau$$

Note that $\mathbf{\Phi}(T)$ is the state-transition matrix evaluated at one sampling time. The matrix $\mathbf{\Gamma}(T)$ is the discrete-time input matrix, which is often referred to as simply the input matrix when the system can be implicitly assumed to be discrete-time. The state at any time $(k+1)T$ can be found given the initial conditions on the state at time kT because the system (2.2) is time-invariant:

$$x(kT+T) = \mathbf{\Phi}x(kT) + \mathbf{\Gamma}u(kT) \tag{2.8a}$$

The output equation in the continuous-time state model is an algebraic equation. A discrete-time version of this equation is obtained by simply evaluating (2.2b) at the sample times:

$$y(kT) = \mathbf{C}x(kT) + \mathbf{D}u(kT) \tag{2.8b}$$

The response of a continuous-time system can be approximated by implementing the discrete-time system on a digital computer, a process known as simulation. The system is simulated by recursively implementing either (2.4) or (2.8a) to generate a time history, or trajectory, of the stale. The output trajectory is then generated from the state trajectory by applying (2.8b) [Franklin , Powell, Workman-1998].

2.3 Transfer Functions

The Laplace transform is a useful tool for the analysis and solution of time-invariant differential equations. Taking The Laplace transform of the stale and output equations (2.2) yields

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{2.9}$$

$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s) \tag{2.10}$$

where $Y(s)$, $U(s)$, and $Y(s)$ are the Laplace transforms of $x(t)$, $u(t)$, and $y(t)$, respectively. Note that $x(0)$ is the initial condition on the state. Solving for $Y(s)$ as a function of $U(s)$ and $x(0)$ yields

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) + \left\{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right\}U(s) \tag{2.11}$$

The Laplace transfer function (or simply the transfer function) of the linear system is the matrix of gains between the Laplace transform of the input vector and the Laplace transform of the output vector:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\,. \tag{2.12}$$

Assuming the initial conditions are zero

$$Y(s) = \mathbf{G}(s)U(s)\,. \tag{2.13}$$

Note that this equation is often used to define the transfer function.

The transfer function can be written as a matrix of ratios of polynomials in s:

$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1n_u}(s) \\ \vdots & \ddots & \vdots \\ G_{n_y1}(s) & \cdots & G_{n_yn_u}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\mathrm{num}_{11}(s)}{\mathrm{den}_{11}(s)} & \cdots & \dfrac{\mathrm{num}_{1n_u}(s)}{\mathrm{den}_{1n_u}(s)} \\ \vdots & \ddots & \vdots \\ \dfrac{\mathrm{num}_{n_y1}(s)}{\mathrm{den}_{n_y1}(s)} & \cdots & \dfrac{\mathrm{num}_{n_yn_u}(s)}{\mathrm{den}_{n_yn_u}(s)} \end{bmatrix}. \tag{2.14}$$

Note that each term in $\mathbf{G}(s)$ is a proper ratio of polynomials; that is, the order of the numerator is less than or equal to the order of the denominator. When there is no input-to-output coupling ($\mathbf{D} = \mathbf{0}$), each term in $\mathbf{G}(s)$ is a strictly proper ratio of polynomials; that is, the order of the numerator is less than the order of the denominator. [Kuo-1991]

2.4 General Models of Feedback Control Systems

A block diagram of a general feedback control system is shown in Figure 2.1. The disturbance input $w_o(t)$ is the vector of inputs to the plant that are not generated by the control system. Measurement noise is included as a disturbance input since the measurement process is included as part of the plant in this model. The reference input $r(t)$ is an input to the control system that specifies the desired behavior of some or all of the plant outputs, and is equal to the desired value of these outputs. Note that reference inputs are only included for those outputs that have nonzero desired values. The reference input and the disturbance input are both external inputs to the plant and can be combined into a single input:

$$w(t) = \begin{bmatrix} r(t) \\ \cdots\cdots \\ w_0(t) \end{bmatrix}, \tag{2.15}$$

where $w(t)$ is called a generalized disturbance input [or simply a disturbance input when this does not lead to confusion with $w_0(t)$]. The control input $u(t)$ is the vector of inputs to the plant that are generated by the control system. The reference output $y(t)$ is the vector of plant outputs that are of interest. These outputs include the errors between plant states and the desired values of these states (or linear combinations of the plant states), which are termed the output, errors. Additionally, the control input can be incorporated into the reference output when desired. The measured output $m(t)$ is the vector of plant outputs that are directly measured and, therefore, available for feedback to the controller. In general, the measured output is distinct from the reference output, although they may be identical in some applications.

The fundamental objective of the control system is to keep the reference output close to the desired value, that is, the reference input. Tracking performance, or steady state performance, is used to describe the control system's ability to meet this objective during normal operation when the reference input is slowly varying. Abrupt changes in the reference input generate transients in the system, since physical systems cannot react instantaneously. The controller should force these transients to decay out in a reasonable period of time. *Transient performance* is the term used to describe the decay of transients in the control system. Note that the feedback system is required to be stable in order to guarantee this decay. The disturbance inputs tend to force the plant to exhibit undesirable behavior. The ability of the control system to mitigate the effects of disturbance inputs is called disturbance rejection.

A well-designed control system should keep the output errors small in the presence of both changing reference inputs and disturbance inputs. Intuitively, disturbances are inputs that tend to generate undesirable behavior in the plant. The reference input, on the other band, is an input that tends to generate desirable behavior in the plant. These

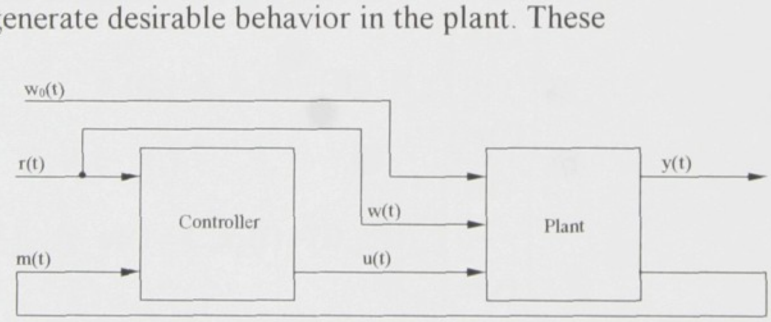


FIGURE 2.1 A general closed-loop control system

inputs are lumped together as generalized disturbances since they both tend to generate nonzero output errors: the disturbances by changing the plant state, and the reference input by changing the desired behavior of the plant state. Further, a well-designed control system should use a reasonable amount of control, that is, maintain the control inputs at sufficiently small levels That the actuators are not saturated and do not utilize excessive amounts of energy, fuel, and so on. Therefore, it may be desirable to include the control input as part of the reference output in some applications. The reference output can then be used exclusively to evaluate control system performance.

The state model and the transfer function model for the closed-loop system in Figure 2.1 are developed below. The state model of the plant is

$$\dot{x}(t) = \mathbf{A}x(t) + \left[\mathbf{B}_u \;\;\vdots\;\; \mathbf{B}_w\right]\begin{bmatrix} u(t) \\ \cdots\cdots \\ w(t) \end{bmatrix}; \tag{2.16}$$

$$\begin{bmatrix} m(t) \\ \cdots\cdots \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_m \\ \cdots \\ \mathbf{C}_m \end{bmatrix} x(t) + \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{D}_{mw} \\ \cdots & & \cdots\cdots \\ \mathbf{D}_{yu} & \vdots & \mathbf{D}_{yw} \end{bmatrix}\begin{bmatrix} u(t) \\ \cdots\cdots \\ w(t) \end{bmatrix}. \tag{2.17}$$

In general, $\boldsymbol{w}(t) = [r(t)\;\; w_0(t)]^T$ includes disturbance inputs to the plant that enter through $\mathbf{B}_w$, errors in the measurement that enter through $\mathbf{D}_{mw}$, and reference inputs that enter through $\mathbf{D}_{yw}$. Note that any number of these disturbances may be absent. The term $\mathbf{D}_{yu}$ results in input-to-output coupling between the control input and the reference output. This coupling is typically used to incorporate the control input into the reference output. It is assumed that, there is no input-to-output coupling between the control input and the measured output. This assumption is valid in almost and applications and simplifies the algebra in deriving future results. The transfer function model of the plant is

$$\begin{bmatrix} M(s) \\ \cdots\cdots \\ Y(s) \end{bmatrix} = \mathbf{G}(s)\begin{bmatrix} U(s) \\ \cdots\cdots \\ W(s) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{mu}(s) & \vdots & \mathbf{G}_{mw}(s) \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{G}_{yu}(s) & \vdots & \mathbf{G}_{yw}(s) \end{bmatrix}\begin{bmatrix} U(s) \\ \cdots\cdots \\ W(s) \end{bmatrix} \tag{2.18}$$

Where

$$\mathbf{G}_{mu}(s) = \mathbf{C}_m(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_u; \tag{2.19a}$$
$$\mathbf{G}_{mw}(s) = \mathbf{C}_m(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_w + \mathbf{D}_{mw}; \tag{2.19b}$$
$$\mathbf{G}_{yu}(s) = \mathbf{C}_v(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_u + \mathbf{D}_{yu}; \tag{2.19c}$$
$$\mathbf{G}_{yw}(s) = \mathbf{C}_y(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_w + \mathbf{D}_{yw}; \tag{2.19d}$$

The stale model of the controller in Figure 2.1 is

$$\dot{x}_c(t) = \mathbf{A}_c x_c(t) + [\mathbf{B}_{cr} \vdots \mathbf{B}_{cm}]\begin{bmatrix} r(t) \\ \cdots\cdots \\ m(t) \end{bmatrix}; \tag{2.20a}$$

$$u(t) = \mathbf{C}_c x_c(t) + [\mathbf{D}_{cr} \vdots \mathbf{D}_{cm}]\begin{bmatrix} r(t) \\ \cdots\cdots \\ m(t) \end{bmatrix}. \tag{2.20b}$$

The transfer function model of the controller is

$$U(s) = \mathbf{G}_c(s)\begin{bmatrix} R(s) \\ \cdots\cdots \\ M(s) \end{bmatrix} = [\mathbf{G}_{cr}(s) \vdots \mathbf{G}_{cm}(s)]\begin{bmatrix} R(s) \\ \cdots\cdots \\ M(s) \end{bmatrix} \tag{2.21}$$

Where

$$\mathbf{G}_{cr}(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_{cr} + \mathbf{D}_{cr}; \tag{2.22a}$$
$$\mathbf{G}_{cm}(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_{cm} + \mathbf{D}_{cm}; \tag{2.22b}$$

The difference, $[r(t) - m(t)]$, is the only input to the controller in many applications. This is an error feedback system and is a special case of the controller described above with

$$\mathbf{B}_{cm} = -\mathbf{B}_{cr}\;;\;\; \mathbf{D}_{cm} = -\mathbf{D}_{cr}\;;\;\; \mathbf{G}_{cm}(s) = -\mathbf{G}_{cr}(s).$$

A state model of the closed-loop system is obtained by combining the model of the plant with the model of the controller. Appending (2.20a) to (2.16) yields

$$\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_u \\ \cdots \\ \mathbf{0} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{B}_w \\ \cdots \\ \mathbf{0} \end{bmatrix} w(t) + \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{B}_{cr} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{B}_{cm} \end{bmatrix} m(t), \tag{2.23}$$

where all of the inputs are displayed as individual terms. The control input is the output of the controller (2.20b):

$$\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_u \\ \cdots \\ \mathbf{0} \end{bmatrix} \{\mathbf{C}_c x_c(t) + \mathbf{D}_{cr} r(t) + \mathbf{D}_{cm} m(t)\} + \begin{bmatrix} \mathbf{B}_w \\ \cdots \\ \mathbf{0} \end{bmatrix} w(t) + \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{B}_{cr} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{B}_{cm} \end{bmatrix} m(t). \tag{2.24}$$

Combining like terms yields

$$\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \vdots & \mathbf{B}_u\mathbf{C}_c \\ \cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_w \\ \cdots \\ \mathbf{0} \end{bmatrix} w(t) + \begin{bmatrix} \mathbf{B}_u\mathbf{D}_{cr} \\ \cdots\cdots \\ \mathbf{B}_{cr} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{B}_u\mathbf{D}_{cm} \\ \cdots\cdots \\ \mathbf{B}_{cm} \end{bmatrix} m(t). \tag{2.25}$$

The measured output is given by (2.17):

$$\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \vdots & \mathbf{B}_u\mathbf{C}_c \\ \cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_w \\ \cdots \\ \mathbf{0} \end{bmatrix} w(t) + \begin{bmatrix} \mathbf{B}_u\mathbf{D}_{cr} \\ \cdots\cdots\cdots \\ \mathbf{B}_{cr} \end{bmatrix} r(t) + \begin{bmatrix} \mathbf{B}_u\mathbf{D}_{cm} \\ \cdots\cdots\cdots \\ \mathbf{B}_{cm} \end{bmatrix} [\mathbf{C}_m x(t) + \mathbf{D}_{mw} w(t)]. \tag{2.26}$$

Again, combining like terms,

$$\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_u\mathbf{D}_{cm}\mathbf{C}_m & \vdots & \mathbf{B}_u\mathbf{C}_c \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots \\ \mathbf{B}_{cm}\mathbf{C}_m & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_w + \mathbf{B}_u\mathbf{D}_{cm}\mathbf{D}_{mw} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{B}_{cm}\mathbf{D}_{mw} \end{bmatrix} w(t) + \begin{bmatrix} \mathbf{B}_u\mathbf{D}_{cr} \\ \cdots\cdots \\ \mathbf{B}_{cr} \end{bmatrix} r(t). \tag{2.27}$$

Noting that the reference input forms the first part of the generalized disturbance input, the state equation of the closed-loop system is

$$
\begin{bmatrix} \dot{x}(t) \\ \cdots\cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}+\mathbf{B}_u\mathbf{D}_{cm}\mathbf{C}_m & \vdots & \mathbf{B}_u\mathbf{C}_c \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots \\ \mathbf{B}_{cm}\mathbf{C}_m & \vdots & \mathbf{A}_c \end{bmatrix} \begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix}
$$

$$
+\begin{bmatrix} \mathbf{B}_w+\mathbf{B}_u\mathbf{D}_{cm}\mathbf{D}_{mw}+[\mathbf{B}_u\mathbf{D}_{cr}\,\vdots\,\mathbf{0}] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{B}_{cm}\mathbf{D}_{mw}+[\mathbf{B}_{cr}\,\vdots\,\mathbf{0}] \end{bmatrix} w(t) = \mathbf{A}_{cl}\begin{bmatrix} x(t) \\ \cdots\cdots \\ x_c(t) \end{bmatrix} + \mathbf{B}_{cl}w(t) \tag{2.28}
$$

where $\mathbf{A}_{cl}$ and $\mathbf{B}_{cl}$ are defined above, and the partitioning of the matrices within $\mathbf{B}_{cl}$, is defined by the partitioning of the generalized disturbance input (2.15).

The output of the closed-loop system is the reference output as given in (2.17):

$$y(t) = \mathbf{C}_y x(t) + \mathbf{D}_{yu}u(t) + \mathbf{D}_{yw}w(t). \tag{2.29}$$

Nothing that the control input is the output of the controller (2.20b):

$$y(t) = \mathbf{C}_y x(t) + \mathbf{D}_{yu}\{\mathbf{C}_c x_c(t) + \mathbf{D}_{cr}r(t) + \mathbf{D}_{cm}m(t)\} + \mathbf{D}_{yw}w(t). \tag{2.30}$$

Rearranging and substituting for $m(t)$ using (2.17) gives:

$$y(t) = \mathbf{C}_y x(t) + \mathbf{D}_{yu}\mathbf{C}_c x_c(t) + \mathbf{D}_{yu}\mathbf{D}_{cr}r(t) + \mathbf{D}_{yu}\mathbf{D}_{cm}\{\mathbf{C}_m x(t) + \mathbf{D}_{mw}w(t)\} + \mathbf{D}_{yw}w(t). \tag{2.31}$$

Combining terms yields the measurement equation for the closed-loop system:

$$
y(t) = [\mathbf{C}_y + \mathbf{D}_{yu}\mathbf{D}_{cm}\mathbf{C}_{cm}\,\vdots\,\mathbf{D}_{yu}\mathbf{C}_c]\begin{bmatrix} x(t) \\ \cdots \\ x_c(t) \end{bmatrix}
$$

$$
+\{\mathbf{D}_{yw}+\mathbf{D}_{yu}\mathbf{D}_{cm}\mathbf{D}_{mw}+[\mathbf{D}_{yu}\mathbf{D}_{cr}\,\vdots\,\mathbf{0}]\}w(t) = \mathbf{C}_{cl}\begin{bmatrix} x(t) \\ \cdots \\ x_c(t) \end{bmatrix} + \mathbf{D}_{cl}w(t) \tag{2.32}
$$

where $\mathbf{C}_{cl}$ and $\mathbf{D}_{cl}$, are defined above, and the partitioning of the matrix within $\mathbf{D}_{cl}$ is defined by the partitioning of the generalized disturbance input (2.15). Equations (2.28) and (2.32) form the state model of the closed-loop system. The input to this system is the generalized disturbance input, which consists of both the disturbance inputs and the reference inputs. The output of this system is the reference output of the plant, which may contain the output errors, the control input, and other linear combinations of the plant states [Burel-1999].

2.5 Stability

An important property of feedback control systems (indeed, one that is absolutely essential) is stability. Stability guarantees that the system output remains finite if the input is finite.

DEFINITION: A system is bounded input/bounded output-stable (or simply stable) if for every bounded input,

$$|u_i(t)| < M_1 \quad \text{for all } t \text{ and all } i,$$

the output is bounded:

$$|y_j(t)| < M_2 \quad \text{for all } t \text{ and all } j,$$

provided that the initial conditions are zero.

The stability of a causal, linear, time-invariant system depends on the system impulse response. The jth element of the output is a function of the input and the impulse response:

$$\left|y_j(t)\right| = \left|\sum_{i=1}^{n_u}\left\{\int_0^t g_{ji}(\tau)u_i(t-\tau)d\tau\right\}\right|. \tag{2.33}$$

Each element of the output is bounded by the expression

$$\left|y_j(t)\right| \le M_1\sum_{i=1}^{n_u}\left\{\int_0^t \left|g_{ji}(\tau)\right|d\tau\right\}, \tag{2.34}$$

if and only if each element of the impulse response matrix is absolutely integrable:

$$\int_0^\infty \left|g_{ji}(\tau)\right|d\tau < \infty . \tag{2.35}$$

This equation provides a test for stability and demonstrates that the impulse response of a stable system most approach zero as time approaches infinity; that is, the output of a stable system returns to zero after being subjected to a temporary disturbance.

The elements of the impulse response matrix can be expanded as a linear combination of the natural modes and the impulse function:

$$g_{ji}(t) = \alpha_1 e^{\mathrm{Re}[p_1]t}e^{j\,\mathrm{Im}[p_1]t} + \ldots + \alpha_{n_x} e^{\mathrm{Re}[p_{n_x}]t}e^{j\,\mathrm{Im}[p_{n_x}]t} + D_{ji}\delta(t) \quad \text{for} \quad t \ge 0 \tag{2.36}$$

The term involving the delta function is absolutely integrable. The α_i terms are constants, and the exponential of an imaginary number has a magnitude of 1 for all time. Therefore, the terms involving each pole are absolutely integrable if and only if the real part of the pole is negative, and

A causal, linear, time-invariant system is stable if and only if all of its poles have negative real parts.

This test for stability is used extensively for linear, time-invariant systems [Burel-1999].

2.5.1 Internal Stability

The definitions of stability are based on the input/output behavior of the system. A system may be input/output stable and still have internal signals that are unbounded. This situation is typically catastrophic to a real-world system, since these unbounded signals cause loss of linearity, damage to the system, or both. Unbounded internal signals in stable, linear, time-invariant systems are the result of internal, unstable pole-zero cancellations.

A linear feedback system is termed internally stable if all internal signals and all possible outputs remain bounded given that all possible inputs are bounded. Interval stability is evaluated by considering all of the possible transfer functions associated with the feedback system. A general feedback system is shown in Figure 2.2. The inputs u_1 and u_2 are applied at each of the two possible locations between blocks. The four possible outputs y_1, y_2, e_1, and e_2 are the output of each block and the output of each summing junction. The inputs can represent input disturbances, output disturbances, reference inputs, and measurement noise. The outputs represent the plant output, the controller output, the plant input, and the controller input, respectively. The eight possible transfer functions are

$$\begin{bmatrix} y_1 \\ \cdots \\ y_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{G} & \vdots & -(\mathbf{I}+\mathbf{GK})^{-1}\mathbf{GK} \\ \cdots\cdots\cdots\cdots & \ddots & \cdots\cdots\cdots\cdots \\ (\mathbf{I}+\mathbf{KG})^{-1}\mathbf{KG} & \vdots & (\mathbf{I}+\mathbf{KG})^{-1}\mathbf{K} \end{bmatrix}\begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{y_1u_1} & \vdots & \mathbf{G}_{y_1u_2} \\ \cdots\cdots & \ddots & \cdots\cdots \\ \mathbf{G}_{y_2u_1} & \vdots & \mathbf{G}_{y_2u_2} \end{bmatrix}\begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix}; \tag{2.37}$$

$$\begin{bmatrix} e_1 \\ \cdots \\ e_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{I}+\mathbf{KG})^{-1} & \vdots & -(\mathbf{I}+\mathbf{KG})^{-1}\mathbf{K} \\ \cdots\cdots\cdots & \ddots & \cdots\cdots\cdots \\ (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{G} & \vdots & (\mathbf{I}+\mathbf{GK})^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{e_1u_1} & \vdots & \mathbf{G}_{e_1u_2} \\ \cdots\cdots & \ddots & \cdots\cdots \\ \mathbf{G}_{e_2u_1} & \vdots & \mathbf{G}_{e_2u_2} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix}. \tag{2.38}$$

The concept of interval stability is formally defined as follows.

DEFINITION: The feedback system consisting of the plant $\mathbf{G}(s)$ and the controller $\mathbf{K}(s)$ (either in the feedback path or the forward path) is internally stable if each of the eight transfer functions in (2.37) and (2.38) are stable.

Internal stability is a stronger condition than stability and will be required when designing feedback systems.

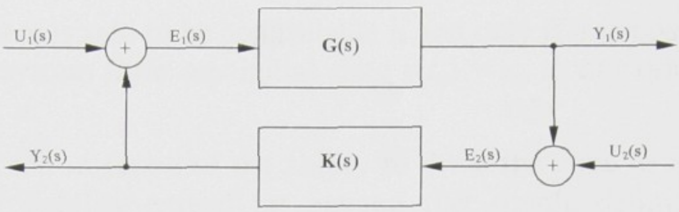


FIGURE 2.2 Block diagram for evaluating internal stability of a feedback system

The determination of interval stability is simplified by considering:

$$\begin{bmatrix} e_1 \\ \cdots \\ e_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} + \begin{bmatrix} -y_2 \\ \cdots \\ y_1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \vdots & -\mathbf{I} \\ \cdots & & \cdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1 \\ \cdots \\ y_2 \end{bmatrix}. \tag{2.39}$$

Solving for the vector y yields:

$$\begin{bmatrix} y_1 \\ \cdots \\ y_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & & \cdots \\ -\mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} e_1 \\ \cdots \\ e_2 \end{bmatrix} - \begin{bmatrix} \mathbf{I} & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} \right). \tag{2.40}$$

The transfer functions from u to y are then related to the transfer functions from u to e:

$$\begin{bmatrix} \mathbf{G}_{y_1u_1} & \vdots & \mathbf{G}_{y_1u_2} \\ \cdots & & \cdots \\ \mathbf{G}_{y_2u_1} & \vdots & \mathbf{G}_{y_2u_2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & & \cdots \\ -\mathbf{I} & \vdots & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{G}_{e_1u_1} & \vdots & \mathbf{G}_{e_1u_2} \\ \cdots & & \cdots \\ \mathbf{G}_{e_2u_1} & \vdots & \mathbf{G}_{e_2u_2} \end{bmatrix} - \begin{bmatrix} \mathbf{I} & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \right). \tag{2.41}$$

Since all of the transformation matrices relating $\mathbf{G}_{e_iu_j}$ to $\mathbf{G}_{y_iu_j}$ are constants and therefore stable, the following result is obtained:

The feedback system consisting of the plant $\mathbf{G}(s)$ and the controller $\mathbf{K}(s)$ (either in the feed-back path or in the forward path) is internally stable if and only if each of the four transfer functions, $\mathbf{G}_{e_1u_1}$, $\mathbf{G}_{e_1u_2}$, $\mathbf{G}_{e_2u_1}$ and $\mathbf{G}_{e_2u_2}$ are stable.

The feedback system is also internally stable if and only if the four transfer functions from the inputs u_1 and u_2 to the outputs y_1 and y_2 are stable. As a matter of convention, however, the transfer functions from the inputs to the outputs e_1 and e_2 are typically used to evaluate interval stability [Burel-1999].

2.6 Controllability and Observability

The present section addresses the following questions: Under what conditions can a system be controlled from the input, and under what conditions can the state of a system be estimated from the knowledge of the input and output? The answers to these questions depend on the system being controlled/estimated and the actuators/sensors that are employed. The theory developed in answering these questions provides a guide to the selection of actuators and sensors, and also proves useful in generating controllers and estimators.

2.6.1 Controllability

The question of controllability arises in a number of applications. Controllability is formally defined as follows:

DEFINITION: A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system from *any* initial state $x(t_o) = x_o$ to *any* other state $x(t_f) = x_f$ in a finite time $0 \leq t_f - t_o < \infty$.

Controllability is defined in terms of the ability to drive the state to a given value, but the implications of controllability extend far beyond this simple definition. The concept of controllability is frequently encountered in linear system theory and controller design. An example of particular importance to controller design is the fact that the closed-loop poles can be placed at any desired location using state feedback if and only if the plant is controllable. Other applications of controllability will be encountered tater in this text.

A simple test for controllability exists when the system is linear and time-invariant. A linear, time-invariant system is controllable if and only if the controllability matrix, which is defined as

$$\mathbf{L} = \left[\mathbf{B} \vdots \mathbf{AB} \vdots \cdots \vdots \mathbf{A}^{(n_x-1)}\mathbf{B} \right] \tag{2.42}$$

has full rank, that is, a rank equal to the system order n_x. The controllability matrix $\mathbf{L} \in \Re^{n_x \times n_x n_u}$ is square when the system has a single input (i.e., $n_u = 1$). In this case, the system is controllable if and only if $\det(\mathbf{L})$ is not equal to zero [Leondes-1996].

2.6.2 Observability

The question of observability arises in a number of applications. Observability is formally defined as follows:

DEFINITION: A system is said to be observable if and only if its state $x(t_o)$, at any time t_o, can be determined from knowledge of the input and output over a finite period of time, that is, $u(t)$ and $y(t)$, where $t_o \leq t \leq t_f$.

Observability is defined in terms of the ability to estimate the state, but the implications of observability extend far beyond this simple definition. The concept of observability is frequently encountered in linear system theory, estimator design, and controller design. An example of particular importance occurs in the design of Luenberger observers. The observer poles, which control the rate of convergence of the estimates, can be placed at any location if and only if the plant is observable. Other applications of observability will be encountered later in this text.

A simple test for observability exists for the case where the system is linear and time-invariant. A linear, time-invariant system is observable if and only if the observability matrix, which is defined as

$$
\mathbf{N} = \begin{bmatrix} \mathbf{C} \\ \cdots \\ \mathbf{CA} \\ \cdots \\ \vdots \\ \cdots \\ \mathbf{CA}^{(n_x-1)} \end{bmatrix} \tag{2.43}
$$

has full rank. The observability matrix $\mathbf{N} \in \Re^{n_x n_y * n_x}$ is square when the system has a single output (i.e., $n_y = 1$). In this case, the system is observable if and only if $\det(\mathbf{N})$ is not equal to zero [Leondes-1996].

2.7 Examples

Example 2.1

The block diagram of a control system for a field-controlled dc motor is given in Figure 2.3. Disturbance inputs are included, and a distinction is made between the actual position of the motor shaft and the measured position of the motor shaft. The plant (the motor) has four inputs: The control input, which is the field voltage $u(t)$; two true disturbance inputs, which are the load torque $d(t)$ and the measurement noise $v(t)$; and the reference input $r(t)$, which specifies the desired motor-shaft angle. The measured output is the measured angular position of the motor shaft. This output is available for feedback to the controller and is denoted $m(t)$. The reference output consists of both the error $e(t)$ between the actual position of the motor shaft and the desired position of the motor shaft, and the control input. A state model of this plant is

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & 0 \\ k_2 & -p_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} k_1 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 0 & k_2 k_3 & 0 \\ 0 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \cdots \\ r(t) \\ d(t) \\ v(t) \end{bmatrix};
$$

$$
\begin{bmatrix} m(t) \\ \cdots \\ e(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 & 0 & 1 \\ \cdots & & \cdots & \cdots & \cdots \\ 0 & \vdots & 1 & 0 & 0 \\ 1 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \cdots \\ r(t) \\ d(t) \\ v(t) \end{bmatrix},
$$

where the matrices are partitioned as in (2.16). The controller is a lead network with the following state model:

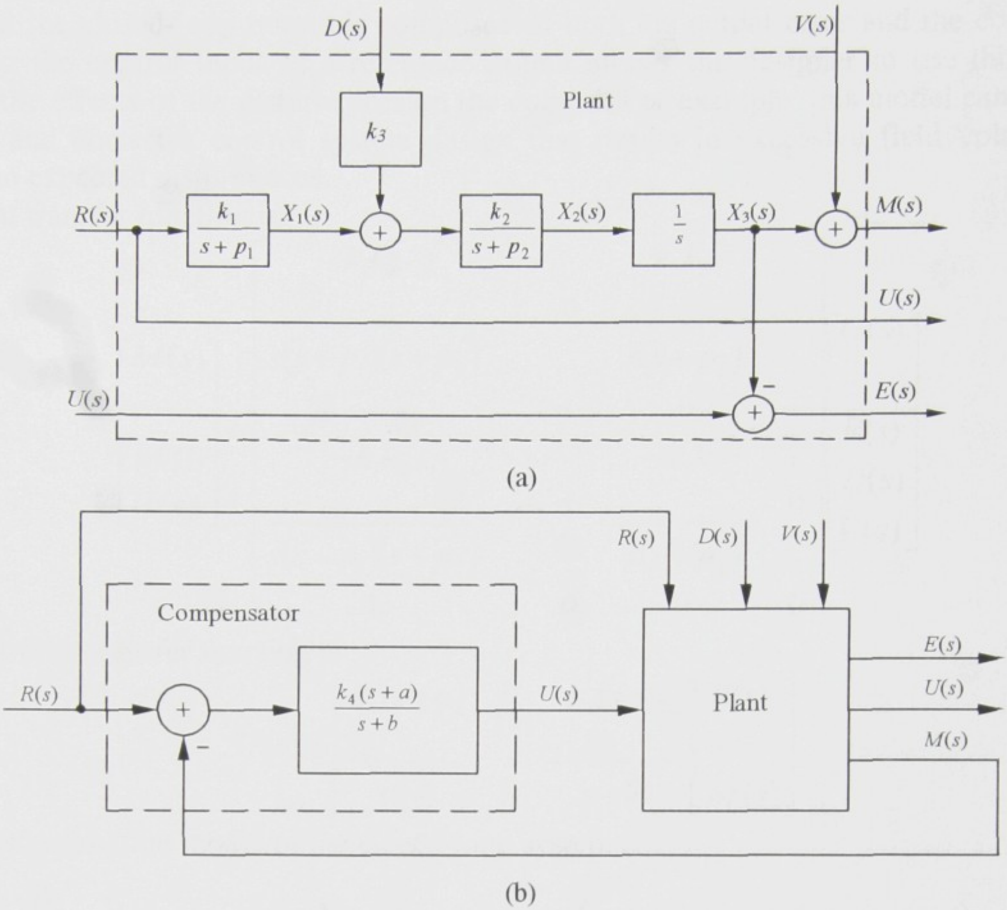


Figure 2.3 Block diagram for a control system for a field-controlled DC motor (a) plant model; (b) control system

$$\dot{x}_c(t) = -bx_c(t) + \begin{bmatrix} k_4(a-b) & \vdots & -k_4(a-b) \end{bmatrix} \begin{bmatrix} r(t) \\ \cdots \\ m(t) \end{bmatrix};$$

$$u(t) = x_c(t) + \begin{bmatrix} k_4 & \vdots & -k_4 \end{bmatrix} \begin{bmatrix} r(t) \\ \cdots \\ m(t) \end{bmatrix}.$$

The state model of the closed-loop system is given by (2.28) and (2.32):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \cdots \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} -p_1 & 0 & -k_1k_4 & k_1 \\ k_2 & -p_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & k_4(b-a) & -b \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} k_1k_4 & 0 & -k_1k_4 \\ 0 & k_2k_3 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ k_4(a-b) & 0 & k_4(b-a) \end{bmatrix} \begin{bmatrix} r(t) \\ d(t) \\ v(t) \end{bmatrix};$$

$$\begin{bmatrix} e(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & \vdots & 0 \\ 0 & 0 & -k_4 & \vdots & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \cdots \\ x_c(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ k_4 & 0 & -k_4 \end{bmatrix} \begin{bmatrix} r(t) \\ d(t) \\ v(t) \end{bmatrix}.$$

Note that the closed-loop output is composed of both the output error and the control input. Including the control input as a reference output allows the designer to use this model to analyse the effects of the disturbances on the control. For example, this model can be used to identify and correct a control system design that results in excessive field voltages when subject to expected disturbances.

The plant transfer function is

$$\begin{bmatrix} M(s) \\ \cdots \\ E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \dfrac{k_1k_2}{s(s+p_1)(s+p_2)} & \vdots & 0 & \dfrac{k_2k_3}{s(s+p_2)} & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ \dfrac{-k_1k_2}{s(s+p_1)(s+p_2)} & \vdots & 1 & \dfrac{-k_2k_3}{s(s+p_2)} & 0 \\ 1 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U(s) \\ \cdots \\ R(s) \\ D(s) \\ V(s) \end{bmatrix}.$$

The controller transfer function is

$$U(s) = \begin{bmatrix} \dfrac{k_4(s+a)}{s+b} & \vdots & \dfrac{-k_4(s+a)}{s+b} \end{bmatrix} \begin{bmatrix} R(s) \\ \cdots \\ M(s) \end{bmatrix}.$$

The transfer function model of the closed-loop system is:

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \dfrac{s(s+p_1)(s+p_2)(s+b)}{\alpha(s)} & \dfrac{-k_2k_3(s+p_1)(s+b)}{\alpha(s)} & \dfrac{k_1k_2k_4(s+a)}{\alpha(s)} \\ \dfrac{k_4s(s+p_1)(s+p_2)(s+a)}{\alpha(s)} & \dfrac{-k_2k_3k_4(s+p_1)(s+a)}{\alpha(s)} & \dfrac{-k_4s(s+p_1)(s+p_2)(s+a)}{\alpha(s)} \end{bmatrix} \begin{bmatrix} R \\ D \\ V \end{bmatrix}$$

Where the denominator is

$$\alpha(s) = s^4 + (p_1 + p_2 + b)s^3 + (p_1p_2 + p_1b + p_2b)s^2 + (p_1p_2b + k_1k_2k_4)s + k_1k_2k_4a$$

The determination of the transfer function matrix is quite tedious in this example. Fortunately, the transfer function matrix can be generated using a computer. Symbolic math software can be used to find the transfer function matrix; alternatively, the state model (2.28) and (2.32) can be generated and used to find this matrix.

Example 2.2

We are given the system described by the following state model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix};$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The controllability text matrix is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & -1 & 8 \\ 0 & 4 & 0 & -12 \end{bmatrix},$$

which has full rank (a rank of 2). The system is therefore controllable. The observabillity test matrix is

$$\mathbf{N} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 5 \end{bmatrix},$$

which has full rank (a rank of 2). The system is therefore observable.

Table 2.1 MATLAB commands to generate closed loop state model and transfer functions for Example 2.1

```
% Define the plant model.
Ap=[-t1 0 0
    k2 -t2 0
    0 1 0];
Bu=[k1 0 0]';
Bw=[0   0  0
   0 k2*k3 0
   0   0  0];
Cm=[0 0 1];
Cy=[0 0 -1
   0 0 0];
Dmw=[0 0 1];
Dyu=[0 1]';
Dyw=[1 0 0
    0 0 0];
% Define the controller model.
Ac=-b;
Bcr=k4*(a-b);
Bcm=-k4*(a-b);
Cc=1;
Dcr=k4;
Dcm=-k4;
% Generate the closed loop state model.
Acl=[Ap+Bu*Dcm*Cm Bu*Cc
     Bcm*Cm     Ac ]
temp1=[Bu*Dcr zeros(3,2)];
temp2=[Bcr zeros(1,2)];
Bcl=[Bw+Bu*Dcm*Dmw+temp1
     Bcm*Dmw+temp2]
Ccl=[Cy+Dyu*Dcm*Cm Dyu*Cc]
Dcl=Dyw+Dyu*Dcm*Dmw+[Dyu*Dcr zeros(2)]
% Generate the closed loop transfer fucntion.
[nu1,de]=ss2tf(Acl,Bcl,Ccl,Dcl,1)
[nu2,de]=ss2tf(Acl,Bcl,Ccl,Dcl,2)
[nu3,de]=ss2tf(Acl,Bcl,Ccl,Dcl,3)
```

3 Introduction to Basic Concepts

This chapter begins by defining a vector norms and matrix norms. Singular value decomposition and principal gain are then discussed. A number of cost functions, which find application in control system design, are also presented and discussed, along with methods for computing the various performance criteria and cost.

3.1 Norms for Signals and Systems

One way to describe the performance of a control system is in terms of the size of certain signals of interest. A quantitative treatment of the performance and robustness of control systems requires the introduction of appropriate signal and system norms, which give measures of the magnitudes of the involved signals and system operators. In this chapter we give a brief description of the most important norms for signals and systems.

3.1.1 Norms for Signals

We consider signals mapping $(-\infty, \infty)$ to $\Re$. They are assumed to be piecewise continuous.

Of course, a signal may be zero for $t<0$ (i.e., it may start at time $t=0$).

The norm, denoted $\|\bullet\|_p$, is a real-valued function of the elements of a linear space $\Re$. A linear space is a set where any linear combination of elements is also an element of the set, and can be composed of vectors, signals, systems, or other possible collections of elements. A norm has the following properties:

$$\|x\|_p \geq 0; \tag{3.1a}$$

$$\|x\|_p = 0 \text{ if and only if } x = 0; \tag{3.1b}$$

$$\|\alpha x\|_p = |\alpha| \|x\|_p; \tag{3.1c}$$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p. \tag{3.1d}$$

where $x, y \in \Re$, and α is a scalar.

3.1.2 Vector Norms

A familiar example of a norm is the Euclidean vector norm, or the vector 2-norm, which appears in elementary geometry and vector analysis. The Euclidean norm is defined as follows:

$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{k=1}^{n_x} x_k^2} \tag{3.2}$$

on the space of real vectors. This norm can be generalized to operate on the space of complex vectors:

$$\|x\|_2 = \sqrt{x^\dagger x} = \sqrt{\sum_{k=1}^{n_x} |x_k|^2}. \tag{3.3}$$

Note that this definition is identical to the previous definition over the space of real vectors.

A modification of the Euclidean vector norm can be obtained by adding a positive definite weighting matrix $\mathbf{W}$:

$$\|x\|_w = \sqrt{x^\dagger \mathbf{W} x}. \tag{3.4}$$

Note that the weighting matrix in this expression must be positive definite to ensure that this function satisfies (3.1a) and (3.1b).
The vector ∞-norm is defined as follows:

$$\|x\|_\infty = \max_i |x_i| . \tag{3.5}$$

Additional vector norms can be defined as required in applications. Note that the above norms all specify the length of a vector (in some sense), and satisfy the properties given in (3.1).

3.1.3 Signal Norms

The Euclidean norm can be generalized to operate on signals:

$$\|x(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} x^\dagger(t)x(t)dt} . \tag{3.6}$$

This norm is referred to as the signal 2-norm. Note that all signals do not have finite 2-norms. When using norms, the signal will be assumed to be an element of the linear space over which the norm is defined (in this case, the space ℓ_2) unless otherwise specified.
Weighted signal 2-norms are defined as follows:

$$\|x(t)\|_{\mathbf{W}(t)} = \sqrt{\int_{-\infty}^{\infty} x^\dagger(t)\mathbf{W}(t)x(t)dt} , \tag{3.7}$$

where $\mathbf{W}(t)$ is positive definite at all times. An additional generalization of the signal 2-norm is obtained by defining this norm over a finite time interval:

$$\|x(t)\|_{2,[t_0,t_f]} = \sqrt{\int_{t_0}^{t_f} x^\dagger(t)x(t)dt} . \tag{3.8}$$

The signal ∞-norm is defined as follows:

$$\|x(t)\|_\infty = \sup_t \max_i |x_i(t)| . \tag{3.9}$$

The supremum is used in this expression since the set of times is infinite. In this case, $x_i(t)$ may asymptotically approach a value but never actually achieve the maximum. The signal ∞-norm can also be defined on a finite time interval:

$$\|x(t)\|_{\infty,[t_0,t_f]} = \max_{t\in[t_0,t_f]} \max_i |x_i(t)| . \tag{3.10}$$

The same notation is used for the vector, signal, and system (after defining this norm) 2-norms (and ∞-norms). The distinction between the given norms is provided by the set over which they operate. This should be apparent when these norms are used.

3.1.4 The Singular Value Decomposition

The singular value decomposition (SVD) is a matrix factorization that has found a number of applications to engineering problems. The SVD of a matrix $\mathbf{M} \in \ell^{n_y * n_u}$ is defined as follows:

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^\dagger = \sum_{i=1}^{p} \sigma_i U_i V_i^\dagger , \tag{3.11}$$

where $\mathbf{U} \in \ell^{n_y * n_u}$ and $\mathbf{V} \in \ell^{n_y * n_u}$ are unitary matrices, p equals the minimum of n_y and n_u, superscript † denotes the conjugate transpose, and U_i and V_i are the ith columns of **U** and **V**, respectively. A unitary matrix **V** is a matrix which has the property

$$\mathbf{V}^\dagger\mathbf{V}=\mathbf{V}\mathbf{V}^\dagger=\mathbf{I}, \tag{3.12a}$$

or equivalently, a matrix whose columns are orthonormal:

$$V_i^\dagger V_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} . \tag{3.12b}$$

The expression δ_{ij} denotes the Kronecker delta function, which is defined by (3.12b). The vectors $\{U_i\}$ and $\{V_i\}$ are called the left and right singular vectors of **M,** respectively. The matrix $\mathbf{S} \in \Re^{n_y * n_u}$ is diagonal:

$$\mathbf{S} = \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \ddots & & \vdots & \mathbf{0} \\ 0 & & \sigma_p & \vdots & \end{bmatrix} or \ \mathbf{S} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_p \end{bmatrix} or \ \mathbf{S} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_p \\ \cdots & \cdots & \cdots \\ & \mathbf{0} & \end{bmatrix} \tag{3.13}$$

when $n_y < n_u$, $n_y = n_u$, an $n_y > n_u$, respectively. The parameters $\{\sigma_i\}$ are called the singular values of **M.** These singular values are ordered (by convention):

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n_u} \geq 0\,.$$

The singular value decomposition provides a detailed picture of how the matrix operates on a vector (termed the input):

$$\mathbf{M}x = \sum_{i=1}^{p} \sigma_i U_i V_i^{\dagger} x\,. \tag{3.14}$$

The term $V_i^{\dagger} x$ gives the length of the input in the direction defined by the given right singular vector. This length is then multiplied by the associated singular value. This product is the length of the output vector in the direction defined by the left singular vector. The matrix-vector product is then the sum of these terms over the various input directions specified by the right singular vectors. Note that the gain for an input in the direction of a right singular vector is given by the associated singular value.

The gains for a complete orthonormal basis for the input are useful in generating the range of matrix gains. The p singular values (given above) provide this information when the number of inputs is less than or equal to the number of outputs. When the number of inputs is greater than the number of outputs, there are input basis vectors that do not appear in the summation in (3.11). In this case, the gain for these additional input basis vectors is zero. To simplify future expressions, additional zero singular values are defined for these input directions. The complete set of singular values (numbering n_u) is then defined as the positive square roots of the diagonal elements of $\mathbf{S}^T\mathbf{S}$, where the superscript T denotes the transpose.

The singular vectors and singular values can be computed by solving the following pair of eigenvalue problems:

$$\mathbf{M}\mathbf{M}^{\dagger} U_i = \sigma_i^2 U_i; \tag{3.15a}$$

$$\mathbf{M}^{\dagger}\mathbf{M} V_i = \sigma_i^2 V_i. \tag{3.15b}$$

The nonzero singular values are the non-negative square roots of these eigenvalues, and they can be found from either (3.15a) or (3.15b). The complete set of singular values, including all zero singular values, consists of the non-negative square roots of the eigenvalues in (3.15b). The eigenvectors in (3.15) are the singular vectors, provided they are normalized as given in (3.12). Note that a sign ambiguity exists for the normalized eigenvectors since multiplication by -1 yields a distinct normalized eigenvector. For the SVD, the sign of the product $U_i V_i^{\dagger}$ is constrained by (3.11) for singular vectors associated with nonzero singular values. Therefore, the sign of one of these vectors can be chosen arbitrarily, while the sign of the other is specified by the following constraint equation:

$$\mathbf{M}\mathrm{V}_i = \sigma_i U_i. \tag{3.16}$$

This constraint equation car also be used to simplify the computation of the SVD. For example, the right singular vectors can be computed using (3.15b), and then the left singular vectors (associated with nonzero singular values) can be found:

$$U_i = \frac{1}{\sigma_i}\mathbf{M}V_i\,. \tag{3.17}$$

There is no sign constraint on singular vectors associated with zero singular values, and any of the associated eigenvectors can be used. Other numerically reliable and computationally efficient algorithms exist for finding the SVD, but a discussion of these algorithms is beyond the scope of this text.

The singular value decomposition has the property that the matrix gain for all inputs is less than the largest singular value (denoted $\bar{\sigma}$):

$$\frac{\|\mathbf{M}x\|}{\|x\|} \le \sigma_1 = \bar{\sigma}\,. \tag{3.18}$$

The gain bound (3.18) can be intuitively understood by noting that σ_1 is the largest gain over the orthogonal set of input directions defined by the right singular vectors. As such, the maximum gain of the matrix is achieved when the input x is proportional to V_1. The direct calculation of matrix gain with the input (αV_1) yields

$$\frac{\|\mathbf{M}(\alpha V_1)\|}{\|\alpha V_1\|} = \frac{\left\|\sum_{i=1}^{p}\sigma_i U_i V_i^{\dagger}(\alpha V_1)\right\|}{\|\alpha V_1\|} = \frac{\|\alpha\sigma_1 U_1\|}{\|\alpha V_1\|} = \frac{|\alpha|\sigma_1}{|\alpha|} = \sigma_1, \tag{3.19}$$

where the orthonormality of the singular vectors (3.12b) is used in generating this result.

The minimum matrix gain can also be determined from the singular value decomposition:

$$\frac{\|\mathbf{M}x\|}{\|x\|} \ge \sigma_{n_u} = \underline{\sigma} = \begin{cases}\sigma_p \text{ if } n_y \ge n_u \\ 0 \text{ if } n_y < n_u\end{cases}. \tag{3.20}$$

The minimum gain equals zero whenever the rank of **M** is less than the number of inputs. A gain of zero indicates that there is an input that yields a zero output; that is, the null space of **M** is not empty. The dimension of this null space is greater than or equal to $(n_u - n_y)$, by the fundamental theorem of linear algebra. Therefore, the minimum gain is always zero whenever $n_y < n_u$. The minimum matrix gain is achieved when the input x is proportional to V_{nu}. Further, the output for this input is proportional to U_{nu} whenever $n_y \ge n_u$, or the output is zero (as discussed previously) when $n_y < n_u$.

An additional property of the singular value decomposition will prove useful in deriving subsequent results. The maximum singular value of a product of matrices is bounded by the maximum singular values of the individual matrices:

$$\bar{\sigma}(\mathbf{MA}) \le \bar{\sigma}(\mathbf{M})\bar{\sigma}(\mathbf{A})\,. \tag{3.21}$$

This fact is derived by noting that the maximum gain of **(MA)** occurs when the input vector is the first right singular vector of **(MA),** which is the singular vector associated with the largest singular value:

$$\bar{\sigma}(\mathbf{MA}) = \frac{\|\mathbf{MA}V_1\|}{\|V_1\|} = \frac{\|\mathbf{M}(\mathbf{A}V_1)\|}{\|\mathbf{A}V_1\|}\frac{\|\mathbf{A}V_1\|}{\|V_1\|} \le \bar{\sigma}(\mathbf{M})\bar{\sigma}(\mathbf{A})\,. \tag{3.22}$$

Intuitively, this inequality results from the fact that the vector V_1 maximizes the gain of the product (**MA**) but may not maximize the gain of the individual matrices **A** and **M** [Watkins-1991].

3.1.5 The Principal Gains

The steady-state output resulting from a pure tone input $u(t) = u_o e^{j\omega t}$ is given in terms of the transfer function matrix:

$$y(t)=\mathbf{G}(j\omega)u_0 e^{j\omega t}. \tag{3.23}$$

The matrix-vector product $\mathbf{G}(j\omega)u_o$ defines the output amplitude and, therefore, the system gain at a particular frequency and input direction. For a given frequency, the range of gains over input direction can be found from the singular value decomposition of the transfer function matrix.

The SVD of the transfer function matrix is

$$\mathbf{G}(j\omega) = \sum_{i=1}^{p} \sigma_i(\omega) U_i(\omega) V_i^{\dagger}(\omega), \tag{3.24}$$

where p equals the minimum of n_y and n_u. The frequency-dependent singular values of the transfer function matrix are called the principal gains of the system. A plot of the principal gains (which are continuous functions of frequency) provides frequency-dependent information on the maximum and minimum system gain. The principal gains also provide an indication of the likelihood of observing a particular gain for a system with many inputs and many outputs. When many of the principal gains are clustered together, this indicates that many inputs yield a similar gain and that this gain is more likely to occur.

The right singular vectors of the transfer function matrix define which inputs yield the maximum and minimum gains. The left singular vectors of the transfer function matrix define what outputs result when the maximum and minimum gains are achieved [Burel-1999].

3.2 Cost Functions

The performance of a control system can be quantified in many applications by a cost function. A cost function is, in general, a real-valued, non-negative function of the system, or of the time histories of the states, reference output, and control input, subject to a given set of initial conditions and inputs. The cost can be used to evaluate the performance of a system, where superior performance is indicated by a smaller cost. The cost can also be used to compare the performance of multiple controller designs; that is, the decision on which of several alternative designs is superior can be made by comparing their respective costs. The controller that minimizes the cost, over all possible designs or a set of possible candidate designs, is know as an optimal controller. The selection of a cost function for particular application is a useful art in control system design. The cost functions given are all based on mathematical objects called norms.

3.2.1 Quadratic Cost Functions

The goals of the control system are to drive the output errors to zero, and to do this while using a reasonable amount of control. These goals are typically at odds. The tighter the control of the output errors, the more control is required. The more reasonable the control used (i.e., the less control used), the larger the output errors. A typical control design represents a compromise between keeping the output errors small and keeping the controls small. The cost function should, therefore, include a measure of both the size of the output errors and the size of the control. One such cost function is

$$J = \int_0^{t_f} y^T(t)\mathbf{Y}(t)y(t)dt = \|y(t)\|^2_{\mathbf{Y}(t)}, \tag{3.25}$$

where the reference output is assumed to include both the output errors and the control inputs.

The cost function (3.25) is called quadratic since it is a quadratic function of the reference output. The weighting function $\mathbf{Y}(t)$ is a positive definite matrix, selected to quantify the relative importance of the various output errors and control inputs. This weighting function is also time-dependent which allows this relative importance to change with time. The parameter t_f is the final time, which can be infinity if the control system is intended to operate indefinitely. The norm in (3.25) operates on the set of real signals, and is a finite-time, weighted signal 2-norm.

The cost function (3.25) can be expanded to yield a more detailed description of the cost:

$$J = \int_0^{t_f} \left[e^T(t) \vdots u^T(t) \right] \begin{bmatrix} \mathbf{Z}(t) & \vdots & \mathbf{0} \\ \cdots & & \cdots \\ \mathbf{0} & \vdots & \mathbf{R}(t) \end{bmatrix} \begin{bmatrix} e(t) \\ \cdots \\ u(t) \end{bmatrix} dt = \int_0^{t_f} e^T(t)\mathbf{Z}(t)e(t) + u^T(t)\mathbf{R}(t)u(t)dt, \qquad (3.26)$$

where the weighting functions $\mathbf{Z}(t)$ and $\mathbf{R}(t)$ are positive definite matrix functions of time. Note that this form of the cost function is not as general as the one given in (3.25), but is sufficiently general to be useful in an abundance of applications. The output error can be written

$$e(t) = r(t) - \mathbf{C}_e x(t), \qquad (3.27)$$

where $\mathbf{C}_e$ is the portion of $\mathbf{C}_y$ that generates the output error, and the parts of the $\mathbf{D}$ matrices that contribute to the error are assumed to be zero. The cost (3.26) can then be written in terms of the system state:

$$J = \int_0^{t_f} \{r(t) - \mathbf{C}_e x(t)\}^T \mathbf{Z}(t)\{r(t) - \mathbf{C}_e x(t)\} + u^T(t)\mathbf{R}(t)u(t)dt. \qquad (3.28)$$

This version of the quadratic cost function can be readily simplified in a number of special cases.

The state can often be defined in such a manner that good control is synonymous with linear combinations of the states being close to zero. For example, the state of a servomotor may include the angular displacement of the shaft from the desired position. The goal of the control system is then to drive this element of the state to zero while using a small amount of control. A control system designed to drive the state, or linear combinations of the state, to zero is termed a *regulator*. The quadratic cost function for a regulator is

$$J = \int_0^{t_f} x^T(t)\mathbf{Q}(t)x(t) + u^T(t)\mathbf{R}(t)u(t)dt, \qquad (3.29)$$

where $\mathbf{Q}(t)$ is a positive semidefinite matrix,

$$\mathbf{Q}(t) = \mathbf{C}_e^T \mathbf{Z}(t)\mathbf{C}_e, \qquad (3.30)$$

selected to weight the appropriate states. The weighting on the control $\mathbf{R}(t)$ is used to impose a penalty on the use of excessive amounts of control. Note that in many applications, the cost function is generated by directly selecting $\mathbf{Q}$ as opposed to both defining $\mathbf{C}_e$ and defining the weighting matrix on the output $\mathbf{Z}(t)$.

The weighting matrices in (3.28) or (3.29) are constant in many applications. Constant weighting matrices are used whenever output errors (and nonzero control inputs) at any point in time are equally undesirable. For example, an autopilot for an airplane is required to maintain a heading during the entire flight, and the suppression of heading errors is equally important at all times. Constant weighting matrices are also typically used when the control system is designed to operate indefinitely or for extended periods of time.

The output error may be important only at the final time in some applications. For example, a missile autopilot is designed to position the missile as close as possible to a desired target at impact (the final time). The cost function (3.28) then simplifies to

$$J = \{ r(t_f) - \mathbf{C}_e x(t_f)\}^T \mathbf{V}\{ r(t_f) - \mathbf{C}_e x(t_f)\} + \int_0^{t_f} u^T(t)\mathbf{R}(t)\, u(t)dt. \qquad (3.31)$$

Note that excessive control is undesirable at all times, so the term involving the control input is still integrated over all time.

The cost functions given above all represent compromises between the output and the control input. Performance can also be evaluated using only one of these vectors while the other is subject to a constraint. For example, a satellite may be required to change from one orbit to another orbit while using the minimum amount of fuel. The performance of a candidate system can be evaluated with the following cost function:

$$J = \int_0^{t_f} u^T(t)\mathbf{R}u(t)dt \tag{3.32}$$

as long as this system yields the desired final state:

$$x(t_f) = x_d. \tag{3.33}$$

As a second example, consider the missile that is designed to hit a target. A reasonable cost function is the square of the miss distance (the nearest approach of the missile to the target):

$$J = \left[x(t_f) - x_d\right]^T \left[x(t_f) - x_d\right], \tag{3.34}$$

subject to the constraints that each of the control inputs are bounded:

$$|u_i(t)| \leq \overline{u}_i. \tag{3.35}$$

These constraints on the control inputs quantify the limitations of the control actuators.

The quadratic cost is dependent on the reference input applied, the disturbance input applied, the initial conditions, the final conditions, and/or constraints on the state and control. Collectively these inputs, conditions, and constraints are known as the test conditions. The performance of a system can be quantified by the cost when the system is subject to worst-case test conditions or nominal test conditions. *Nominal* refers to test conditions that are representative of normal operating conditions, and *worst-case* is self explanatory. The test conditions may be simplified to yield information on the effects of one initial condition or the effects of one disturbance input. Performance is also often evaluated for simple test conditions, which include step inputs, impulse inputs, initial conditions, and so on. Quadratic cost functions can be used in a wide range of engineering design applications due to the flexibility provided by the specification of weighting matrices and test conditions.

3.2.2 The System 2-Norm Cost Function

The computation of the cost functions presented above requires that the excitation (initial conditions, final conditions, reference inputs, and disturbance inputs) be known. In many applications, neither the inputs, the initial conditions, not the final conditions are known *a priori*. An alternative type of cost function focuses on the gain between the inputs and the outputs, as defined by the Fourier transfer function of the closed-loop system.

The system 2-norm is proportional to the root mean square gain of the system, and can be thought of as an average gain for the system. This average is performed both over all the elements of the matrix transfer function and over all frequencies:

$$\|\mathbf{G}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} tr\left\{\mathbf{G}^\dagger(j\omega)\mathbf{G}(j\omega)\right\}d\omega}. \tag{3.36}$$

Note that the tr$\{\mathbf{G}^\dagger(j\omega)\mathbf{G}(j\omega)\}$ is the sum of the magnitudes squared of all of the elements of $\mathbf{G}(j\omega)$. The system 2-norm is actually only proportional to the true average since it is not normalized to either the number of elements in the transfer function matrix or to the frequency range. The system 2-norm can also be written in terms of the impulse response matrix by using Parseval's theorem:

$$\|\mathbf{G}\|_2 = \sqrt{\int_0^{\infty} tr\left\{\mathbf{g}^\dagger(t)\mathbf{g}(t)\right\}dt}. \tag{3.37}$$

The indefinite integral in (3.37) only exists when the system is stable. In fact, the whole concept of gain is not very useful for an unstable system whose transient persists indefinitely. Therefore, the system 2-norm is only defined for stable systems.

Computation of the system 2-Norm

The 2-norm of generic stable system is a function of the impulse response:

$$g(t) = \left\{ \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t) \right\} 1(t). \tag{3.38}$$

Substituting his expression into(3.37) yields

$$\begin{aligned}\|\mathbf{G}\|_2 &= \sqrt{\int_0^\infty tr\left\{ [\mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)]^T [\mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\delta(t)] \right\} dt} = \\ &\sqrt{\int_0^\infty tr\left\{ \mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{D}\delta(t) + \mathbf{D}^T\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\delta(t) + \mathbf{D}^T\mathbf{D}\delta^2(t) \right\} dt}.\end{aligned} \tag{3.39}$$

This expression includes an integral of the square of the Dirac delta function. This integral is infinite, and the system 2-norm is therefore not defined, expect when the input-to-output coupling matrix is zero.

For the close-loop system of (2.28) and (2.32), the input-to-output coupling matrix is

$$\mathbf{D}_{cl} = \mathbf{D}_{yw} + \mathbf{D}_{yu}\mathbf{D}_{cm}\mathbf{D}_{mw} + \left[\mathbf{D}_{yu}\mathbf{D}_{cr} \quad \vdots \quad \mathbf{0} \right]. \tag{3.40}$$

The term $\mathbf{D}_{yw}$ in this matrix is typically zero when the reference input is zero and can be deleted from the general control system model. Note that the reference input is part of the generalized disturbance $w(t)$. The second term in this input-to-output coupling matrix is zero when at least one of the matrices $\mathbf{D}_{yu}$, $\mathbf{D}_{cm}$ or $\mathbf{D}_{mw}$ are zero; that is when the reference output does not contain a control term, the controller is strictly proper, or there is no measurement noise. The final term in the input-to-output coupling matrix is zero when either $\mathbf{D}_{yu}$ or $\mathbf{D}_{cr}$ are zero, that is, when either the reference output does not contain a control term, there is no reference input, or the controller is strictly proper. Therefore, the close-loop system 2-norm is defined when this system is stable, there is no reference input, and either the reference output dose not contain a control term, the controller is strictly proper, or there is no measurement noise.

The 2-norm of a generic system reduces to

$$\|\mathbf{G}\|_2 = \sqrt{\int_0^\infty tr\left\{ \mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t}\mathbf{B} \right\} dt} = \sqrt{\int_0^\infty tr\left\{ \mathbf{C}e^{\mathbf{A}^T t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{C}^T \right\} dt} \tag{3.41}$$

When the input-to-output coupling matrix equals zero. Interchanging the order of the trace and the integration operators yields

$$\|\mathbf{G}\|_2 = \sqrt{tr\left\{ \mathbf{B}^T \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t}dt\ \mathbf{B} \right\}} = \sqrt{tr\left\{ \mathbf{C}\int_0^\infty e^{\mathbf{A}^T t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt\ \mathbf{C}^T \right\}} \tag{3.42}$$

Defining the observability grammian,

$$\mathbf{L}_0 = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}^T t}dt \tag{3.43}$$

and the controllability grammian

$$\mathbf{L}_c = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}dt \tag{3.44}$$

The 2-norm can be computed using either expression:

$$\|\mathbf{G}\|_2 = \sqrt{tr\{\ \mathbf{B}^T\mathbf{L}_0\mathbf{B}\}} = \sqrt{tr\{\ \mathbf{C}\mathbf{L}_c\mathbf{C}^T\ \}} \tag{3.45}$$

The observability and controllability grammian can be found by solving the following Lyapunov equations:

$$\mathbf{A}^T\mathbf{L}_0 + \mathbf{L}_0\mathbf{A} = -\mathbf{C}^T\mathbf{C};$$
$$\mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T. \tag{3.46}$$

3.2.3 The System ∞-Norm Cost Function

The maximum gain of a generic system over all frequencies is given by the system x-norm:

$$\|\mathbf{G}\|_\infty = \sup_\omega \bar{\sigma}[\mathbf{G}(j\omega)] \tag{3.47}$$

The fact that this cost function is a norm, that is, possesses the properties given in (3.1), can be easily verified. This cost function is particularly applicable to the design of systems where the performance is specified by bounds on the output error and the control, and reasonable bounds can be generated for sinusoidal disturbance inputs. This is a very intuitive way of specifying the cost, since most specifications for control systems take the form of bounds on the errors and controls. The ∞-norm also finds application in robustness analysis, as will be seen in the next chapter.

The ∞-norm of a system provides a bound on the maximum system gain, where the gain is defined in terms of the signal 2-norm:

$$\|\mathbf{g}(t)\otimes w(t)\|_2 \le \|\mathbf{G}\|_\infty \|w(t)\|_2 \tag{3.48}$$

where $\mathbf{g}(t)\otimes w(t)$ is the input convolved with the impulse response matrix, which yields the time-domain system output. This result is surprising at first glance, since the gain used in the definition of the ∞-norm is based on the transfer function gain. This transfer function gain, in turn, is based on the gain of the system with a sinusoidal input, but the signal 2-norm (over an infinite time interval) of a sinusoidal input does not exist. The bound (3.48) can be intuitively justified by noting that the gain of the system with a sinusoidal input is defined by the signal 2-norm gain when the norm is evaluated over a finite time period. The bound then makes sense for truncated sinusoids, which can be used to construct more complex functions with finite signal 2-norms via the Fourier series.

The bound (3.48) is a tight bound, that is, the equality is nearly achieved for some input signal. In fact, the ∞-norm equals

$$\|\mathbf{G}\|_\infty = \sup_{w\neq 0} \frac{\|\mathbf{g}(t)\otimes w(t)\|_2}{\|w(t)\|_2} \tag{3.49}$$

Equation (3.49) is often used as a definition of the system ∞-norm. This definition can be generalized to finite time intervals:

$$\|\mathbf{G}\|_{\infty,[t_0,t_f]} = \sup_{w\neq 0} \frac{\|\mathbf{g}(t)\otimes w(t)\|_{2,[t_0,t_f]}}{\|w(t)\|_{2,[t_0,t_f]}} \tag{3.50}$$

This car be ∞-norm is interpreted as the maximum system gain over the given time interval.

The ∞-norm (both finite time and infinite time) of a series combination of subsystems is bounded:

$$\|\mathbf{G}_1\mathbf{G}_2\|_\infty \leq \|\mathbf{G}_1\|_\infty \|\mathbf{G}_2\|_\infty \tag{3.51}$$

This result is reasonable since the maximum overall gain of a series combination of subsystems cannot exceed the product of the maximum gains of each subsystem. More formally, (3.54) can be demonstrated by noting that for any nonzero input,

$$\|\mathbf{g}_1(t) \otimes \mathbf{g}_2(t) \otimes w(t)\|_2 \leq \|\mathbf{G}_1\|_\infty \|\mathbf{g}_2(t) \otimes w(t)\|_2 \leq \|\mathbf{G}_1\|_\infty \|\mathbf{G}_2\|_\infty \|w(t)\|_2 \tag{3.52}$$

Dividing by the 2-norm of the input and taking the supremum of the result yields (3.51):

$$\|\mathbf{G}_1\mathbf{G}_2\|_\infty = \sup_{w \neq 0} \frac{\|\mathbf{g}_1(t) \otimes \mathbf{g}_2(t) \otimes w(t)\|_2}{\|w(t)\|_2} \leq \|\mathbf{G}_1\|_\infty \|\mathbf{G}_2\|_\infty \tag{3.53}$$

This property, known as the submultiplicative property, will prove very useful in subsequent chapters.

Computation of the System ∞ - Norm

The ∞ - Norm of a system described by a state model is

$$\sup_\omega \bar{\sigma}[\mathbf{G}(j\omega)] = \sup_\omega \bar{\sigma}\left[\mathbf{C}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]. \tag{3.54}$$

The ∞ - Norm can be computed by iterating over frequency to find the maximum [Burel-1999].

3.3 Examples

Example 3.1

A field-controlled dc motor is being used to position an antenna that is required to track a satellite in low earth orbit. A lead compensator being used to control is motor. Models of this motor and controller are given in Example 2.1. The closed-loop state model of this combination also given in Example 2.1 is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \cdots\\ \dot{x}_c(t)\end{bmatrix}=\begin{bmatrix}-p_1 & 0 & -k_1k_4 & \vdots & k_1\\ k_2 & -p_2 & 0 & \vdots & 0\\ 0 & 1 & 0 & \vdots & 0\\ \cdots & \cdots & \cdots & & \cdots\\ 0 & 0 & k_4(b-a) & \vdots & -b\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ \cdots\\ x_c(t)\end{bmatrix}+\begin{bmatrix}k_1k_4 & 0 & -k_1k_4\\ 0 & k_2k_3 & 0\\ 0 & 0 & 0\\ \cdots & \cdots & \cdots\\ k_4(a-b) & 0 & k_4(b-a)\end{bmatrix}\begin{bmatrix}r(t)\\ d(t)\\ v(t)\end{bmatrix}$$

$$\begin{bmatrix}e(t)\\ u(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & -1 & \vdots & 1\\ 0 & 0 & -k_4 & \vdots & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ \cdots\\ x_c(t)\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ k_4 & 0 & -k_4\end{bmatrix}\begin{bmatrix}r(t)\\ d(t)\\ v(t)\end{bmatrix}$$

Where $r(t)$ is the reference input, $d(t)$ is the disturbance torque, $v(t)$ is the measurement noise, $e(t)$ is the difference between the motor shaft angle and the reference input, and $u(t)$ is the field voltage, which is the control input. Further, the parameters are defined as follows:

$$p_1 = 4; p_2 = 0.1; k_1 = 10; k_2 = 0.01; k_3 = 10; k_4 = 150; a = 0.4; b = 4$$

It is desirable to keep the output error less than 1 degree, while using an input of less than 15 volts. A reasonable cost function is then

$$J = \frac{1}{300}\int_0^{150} e^2(t) + \frac{1}{15^2}u^2(t)dt.$$

Note that a factor of 1/150 is used to normalize the cost with respect to the integration time, a factor of ½ is used to normalize the cost with respect to the number of terms within the integral, and a factor of $1/15^2$ is used to normalize the desired size of the control to 1 (which equals the desired size of the error). The test conditions consist of a nominal path for the satellite:

$$r(t) = \tan^{-1}\left(\frac{h}{562.5 - vt}\right),$$

where h = 320 km and v = 7.5km/s are the height of the satellite and the velocity of the satellite, respectively. The initial conditions of the closed-loop plant are

$$x(0) = \begin{bmatrix}0 & 0 & 25 & 0\end{bmatrix}^T$$

The initial elevation angle of the antenna is 25 degrees, which is selected by estimating the initial satellite elevation angle. The true initial elevation angle of the satellite (the angle when the satellite is first detected) is 30 degrees. This yields a 5 degree error in the initial angle estimate. The disturbance torque is generated by gravity, and is due to imperfect balancing of the antenna. This torque depends on the elevation angle, but can be approximated by the following time-varying function:

$$d(t) = \cos\left[\tan^{-1}\left(\frac{h}{562.5 - vt}\right)\right]$$

The measurement noise is small enough that it can be ignored, that is, set equal to zero. These test conditions represent normal operating conditions. The resulting error trajectory and control trajectory are plotted in Figure 3.1. The cost is

$$J = 1.74,$$

which is greater than 1, indicating that the specifications are not achieved with this controller. Looking at the plots, the control input is seen to exceed the specifications during the initial transient, but meets that specification after this transient decays to zero. Care must be used in interpreting the meaning of the cost when the test conditions include both initial conditions and inputs.

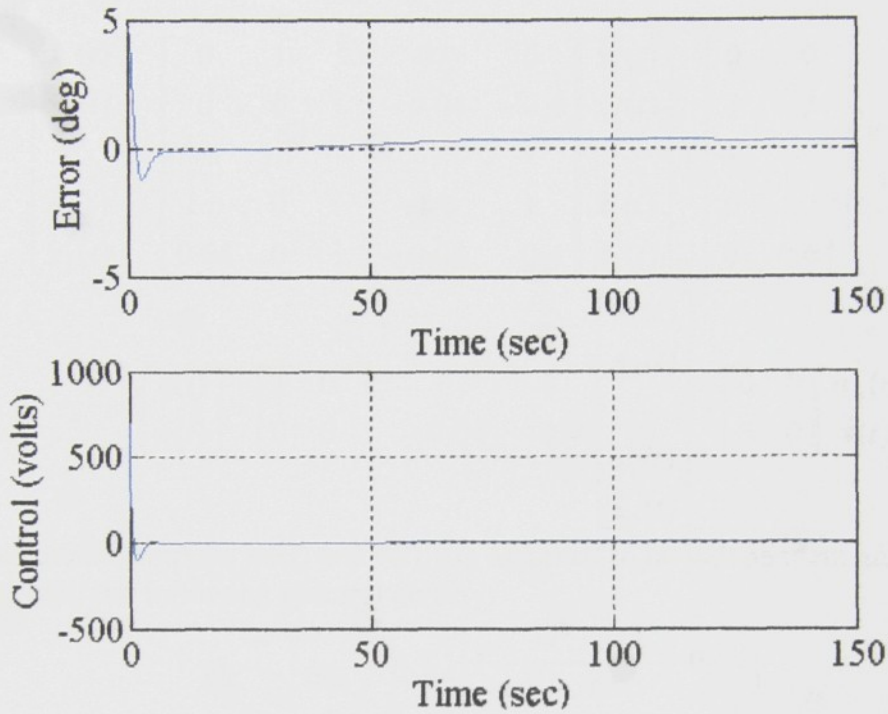


FIGURE 3.1 The reference output for example 2.2

Example 3.2
An autopilot is used to control the altitude of a helicopter while hovering. The altitude dynamics of the helicopter are

$$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 & 0 \\ 1 & \vdots & 1 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \cdots \\ w_0(t) \\ v(t) \end{bmatrix};$$

$$\begin{bmatrix} m(t) \\ \cdots \\ e(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cdots & \cdots \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & 0 \\ 1 & \vdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \cdots \\ w_0(t) \\ v(t) \end{bmatrix},$$

where $e(t)$ is the altitude error (the zero on the coordinate system is set at the desired altitude, so the reference input is zero) and $u(t)$ is the vertical acceleration, which is proportional to the throttle setting. The disturbance $w_0(t)$ is a vertical acceleration caused by wind gusts and $v(t)$ is the measurement noise. Note that the delay due to engine spin-up is ignored in this example. An observer feedback controller for this plant is described by the following state model:

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1.6 & 1 \\ -0.68 & -0.4 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1.6 \\ 0.64 \end{bmatrix} m(t);$$

$$u(t) = [-0.04 \quad -0.4] \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix},$$

where $\hat{x}(t)$ is an estimate of $x(t)$. The close–loop system is then described by the following state model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdots \\ \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & -0.04 & -0.4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1.6 & 0 & \vdots & -1.6 & 1 \\ 0.64 & 0 & \vdots & -0.68 & -0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \cdots & \cdots \\ 0 & 1.6 \\ 0 & 0.64 \end{bmatrix} \begin{bmatrix} w_0(t) \\ v(t) \end{bmatrix};$$

$$\begin{bmatrix} e(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & -0.04 & -0.4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_0(t) \\ v(t) \end{bmatrix}$$

The two disturbance inputs, $w_0(t)$ and $v(t)$, are assumed to be independent and are modelled as white noise with the following spectral density:

$$S_w = \begin{bmatrix} S_{w0} & 0 \\ 0 & S_v \end{bmatrix} = \begin{bmatrix} 1\dfrac{m^2}{\sec^2 - Hz} & 0 \\ 0 & 400\dfrac{m^2}{Hz} \end{bmatrix}.$$

The control input should remain ± 10 m/sec^2, and desired altitude should be maintained to within ± 20 m on average during steady-state operation. The 2-norm of the closed-loop system in this example is

$$\|\mathbf{G}_{cl}\|_2 = 11.2.$$

This number provides an indication of the average system gain. Note that this average gain is dominated by the largest gain, that is, the gain from the disturbance input to the control input. Weighting matrices and weighting functions will be subsequently incorporated within the framework of the system 2-norm to allow greater control over the definition of this average.

Example 3.3

The ∞-norm of the closed-loop system in Example 3.1 is computed by plotting the principal gains of the system, as shown in Figure 3.2. The maximum in Figure 3.2 is

$$\|\mathbf{G}_{cl}\|_\infty = 212$$

This number provides an indication of the maximum system gain. Note that this maximum gain is dominated by the gain from the disturbance input to the control input. Weighting matrices and weighting functions are subsequently incorporated within the plant to allow control over which inputs and outputs are most significant.

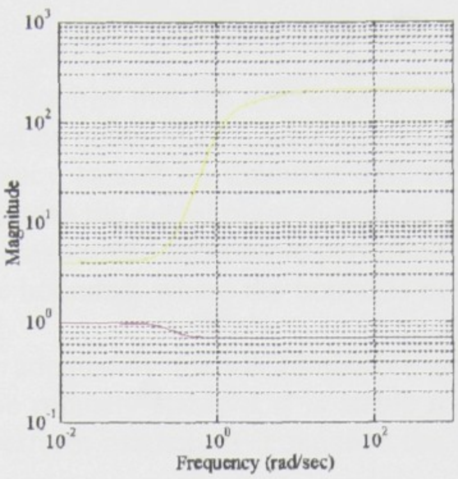


FIGURE 3.2 The principal gains for example 3.3

Table 3.1 MATLAB commands to compute Infinity-norm for Example 3.3

```
% Define the closed loop system:
p1=4; p2=0.1; k1=10; k2=0.01; k3=10; k4=150; a=0.4; b=4;
Acl=[-p1  0  -k1*k4   k1
     k2 -p2   0      0
 0  1    0      0
 0  0  k4*(b-a) -b ];
Bcl=[ k1*k4    0   -k1*k4
       0    k2*k3    0
       0      0      0
    k4*(a-b)  0   k4*(b-a)];
Ccl=[0 0 -1  0
    0 0 -k4 1];
Dcl=[1  0  0
    k4 0 -k4];
% Compute the principal gains of the closed loop system.
w=logspace(-2,3,200);
Gcl=pck(Acl,Bcl,Ccl,Dcl);
fr=frsp(Gcl,w);
pgains=vsvd(fr);
% Plot the principle gains.
figure(1)
set(0,'DefaultAxesFontName','times')
set(0,'DefaultAxesFontSize',16)
set(0,'DefaultTextFontName','times')
clf
vplot('liv,lm',pgains)
xlabel('Frequency (rad/sec)')
ylabel('Magnitude')
axis([.01 1000 .1 1000])
grid
% The infinity-norm is the maximum over frequency of the maximum
% principal gain.
[nr,nc]=size(pgains);
Hinf=max(pgains(1:nr-1,1))
% An alternative algorithm for computing upper and lower bounds on
% the infinity-norm is given in the Mu-Synthesis and Analysis Toolbox.
temp=hinfnorm(Gcl);
% The infinity-norm is given as the average of the bounds.
Hinf=(temp(1)+temp(2))/2
```

4 Robustness

The analysis of robustness requires that the discrepancy between the mathematical model of the plant and the actual plant be quantified. Since a perfect mathematical model of the plant is not available, this discrepancy cannot be uniquely defined. Instead, a set of mathematical model is defined which includes the actual plant dynamics. This set is specified by a nominal plant and a set of perturbations termed admissible perturbations. The admissible perturbations are typically assumed to be bounded, where the bound is dependent on the uncertainty in the model.

A controller that function adequately for all admissible perturbations is termed robust. A control system is said to be robustly stable if it is stable for all admissible perturbations. A control system is said to perform robustly if it satisfies the performance specification for all admissible perturbations.

4.1 Internal Stability of Feedback Systems

The internal stability of the feedback systems shown in Figure 4.1a and 4.1b is determined by considering the closed-loop system in Figure 4.1c. This feedback system is internally stable provided that all four of the transfer functions between each input and each error,

$$\begin{bmatrix} \mathbf{G}_{e_1u_1}(s) & \vdots & \mathbf{G}_{e_1u_2}(s) \\ \cdots & & \cdots \\ \mathbf{G}_{e_2u_1}(s) & \vdots & \mathbf{G}_{e_2u_2}(s) \end{bmatrix} = \begin{bmatrix} \{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}^{-1} & \vdots & -\{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}^{-1}\mathbf{K}(s) \\ \cdots & & \cdots \\ \{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}^{-1}\mathbf{G}(s) & \vdots & \{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}^{-1} \end{bmatrix} \tag{4.1}$$

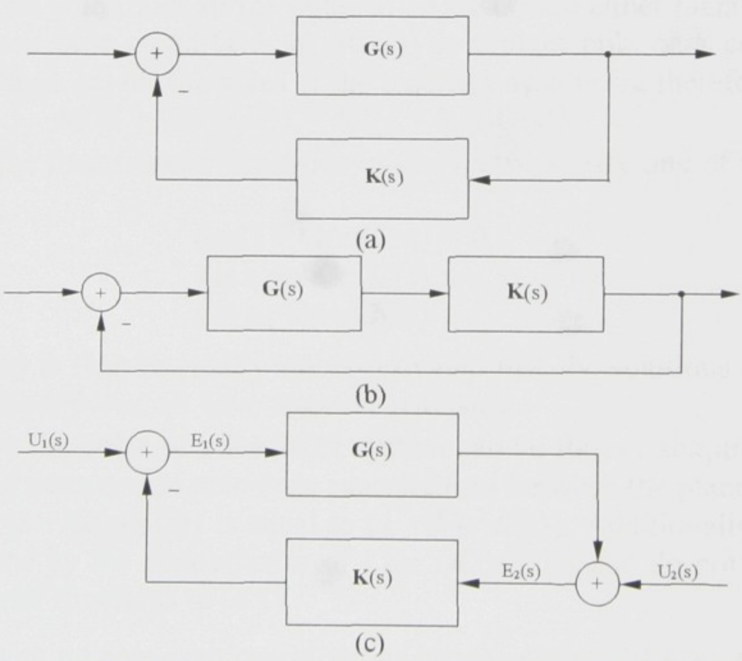


FIGURE 4.1 Feedback system block diagrams: (a) standard feedback; (b) unity feedback; (c) general feedback used to evaluate internal stability

are stable. Expanding the matrix inverses in these transfer functions yields

$$\begin{bmatrix} \mathbf{G}_{e_1u_1}(s) & \vdots & \mathbf{G}_{e_1u_2}(s) \\ \cdots & & \cdots \\ \mathbf{G}_{e_2u_1}(s) & \vdots & \mathbf{G}_{e_2u_2}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{\text{adj}\{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}}{\det\{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}} & \vdots & -\dfrac{\text{adj}\{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}\,\mathbf{K}(s)}{\det\{\mathbf{I}+\mathbf{K}(s)\mathbf{G}(s)\}} \\ \cdots & & \cdots \\ \dfrac{\text{adj}\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}\mathbf{G}(s)}{\det\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}} & \vdots & \dfrac{\text{adj}\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}}{\det\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}} \end{bmatrix} \tag{4.2}$$

where adj(•) denotes the adjugate and det(•) denotes the determinant. Note that the division above is valid since $\det\{\mathbf{I} + \mathbf{K}(s)\mathbf{G}(s)\}$ is a scalar quantity. The poles of these four transfer functions (denoted simply as the poles of the feedback system) must satisfy one of the following conditions:

$$\det\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\} = 0, \tag{4.2a}$$

$$\det\{\mathbf{I} + \mathbf{K}(s)\mathbf{G}(s)\} = 0, \tag{4.2b}$$

$$s \text{ is a pole of adj}\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\}, \tag{4.2c}$$

$$s \text{ is a pole of adj}\{\mathbf{I} + \mathbf{K}(s)\mathbf{G}(s)\}, \tag{4.2d}$$

$$s \text{ is a pole of } \mathbf{G}(s), \tag{4.2e}$$

$$s \text{ is a pole of } \mathbf{K}(s). \tag{4.2f}$$

These conditions for the poles of the feedback system can be condensed by utilizing some properties of the adjugate and the determinant operations. The adjugate of a matrix takes sums of the products of individual terms in the matrix. This is equivalent to taking series and parallel combinations of these transfer functions, operations that do not create new poles. The poles of the $\text{adj}\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\}$ are therefore a subset of the poles of $\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\}$. Each pole of $\text{adj}\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\}$ must be a pole of the plant $\mathbf{G}(s)$ or a pole of the controller $\mathbf{K}(s)$ since the identity matrix has no poles, and all the poles of $\mathbf{G}(s)\mathbf{K}(s)$ are either plant poles or controller poles. Similarly, each pole of $\text{adj}\{\mathbf{I} + \mathbf{K}(s)\mathbf{G}(s)\}$ is a plant pole or a controller pole. The conditions (4.2c) and (4.2d) for the poles of the feedback system are therefore superfluous.

$$\det\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\} = \det\{\mathbf{I} + \mathbf{K}(s)\mathbf{G}(s)\}. \tag{4.3}$$

Thus, the poles of the four transfer functions in (4.1) must satisfy one of the following three conditions:

$$\det\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\} = 0, \tag{4.3a}$$

$$s \text{ is a pole of } \mathbf{G}(s), \tag{4.3b}$$

$$s \text{ is a pole of } \mathbf{K}(s). \tag{4.3c}$$

The feedback system is then internally stable provided that the solutions of (4.3a), the plant poles, and the controller poles, all have negative real parts.

These conditions for the poles of a feedback system can be further simplified for the care of SISO systems where there are no pole-zero cancellations between the plant and the controller. In this case, the $\det\{\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)\}$ is equal to $\{1 + G(s)K(s)\}$. Additionally, the poles of $G(s)$ and $K(s)$ are canceled by the denominator of $\{1 + G(s)K(s)\}$ and do not appear as poles of any of the four transfer functions in (4.1).

For SISO systems with no pole-zero cancellations between $G(s)$ and $K(s)$, the poles of all four of the transfer functions in (4.1) are the solutions of

$$1 + G(s)K(s) = 0. \tag{4.4}$$

Further, the systems in Figure 4.1a, b, and c are internally stable if and only if all the solutions of (4.4) have negative real parts.

This result states that internal stability of a SISO system with no pole-zero cancellations between the plant and the controller depends only on the solution of (4.4).

An important special case for MIMO systems is when both the plant and the controller are stable. The MIMO feedback system, with a stable plant and a stable controller is internally stable if all the solutions of (4.3a) have negative real parts. This test for internal stability is both necessary and sufficient, as demonstrated below. If the system is internally stable, then

$$\det\left[\mathbf{G}_{e_2u_2}(s) = \right]\det\left[\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}^{-1}\right] = \frac{1}{\det\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\}} \tag{4.5}$$

is bounded for all s with non-negative real parts. Therefore, the determinant in this equation is nonzero for all non-negative values of s.

A MIMO feedback system, as in Figure 4.1a, b, or c with both **G**(s) and **K**(s) stable, is internally stable if and only if all of the solutions of

$$\det\{\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)\} = 0 \tag{4.6}$$

have negative real parts.
This result provides a convenient test for stability of a MIMO feedback system with a stable plant and a stable controller [Burel-1999].

4.2 Unstructured Uncertainty

In this section, uncertainty is modeled as a perturbation to the nominal plant. This perturbation is a bounded transfer function, where *bounded* is defined in terms of the system ∞-norm. This type of plant uncertainty is termed unstructured since no detailed model of the perturbation (the unknown transfer function) is employed.

4.2.1 Unstructured Uncertainty Models

An unstructured perturbation can be connected to the plant in a number of ways, each generating a unique set of possible plant models. Five Basic connections of the perturbation to the nominal plant model are presented: additive perturbation, input-multiplicative perturbation, output-multiplicative perturbation, input feedback perturbation, and output feedback perturbation. An additive unstructured uncertainty models the actual plant as equal to the nominal plant plus a perturbation:

$$\mathbf{G}(s) = \mathbf{G}_0(s) + \mathbf{\Delta}_a(s) \tag{4.7}$$

where $\mathbf{\Delta}_a(s)$ denotes the additive perturbation. An input-multiplicative uncertainty models the actual plant as the nominal plant plus a series combination of the perturbation and the nominal plant (the perturbation appears on the input to the nominal plant):

$$\mathbf{G}(s) = \mathbf{G}_0(s)[\mathbf{I} + \mathbf{\Delta}_i(s)], \tag{4.8}$$

where $\mathbf{\Delta}_i(s)$ denotes the input-multiplicative perturbation. An output-multiplicative uncertainty models the actual plant as the nominal plant plus a series combination of the nominal plant and the perturbation (the perturbation appears on the output to the nominal plant):

$$\mathbf{G}(s) = [\mathbf{I} + \mathbf{\Delta}_o(s)]\mathbf{G}_0(s), \tag{4.9}$$

where $\mathbf{\Delta}_o(s)$ denotes the output-multiplicative perturbation. An input feedback uncertainty models the actual plant as the nominal plant in series with the perturbation in a feedback loop (the feedback loop appears on the input to the nominal plant):

$$\mathbf{G}(s) = \mathbf{G}_0(s)[\mathbf{I} + \mathbf{\Delta}_{fi}(s)]^{-1} \tag{4.10}$$

where $\mathbf{\Delta}_{fi}(s)$ denotes the input feedback perturbation. An output feedback uncertainty models the actual plant as the nominal plant in series with the perturbation in a feedback loop (the feedback loop appears on the output to the nominal plant):

$$\mathbf{G}(s) = [\mathbf{I} + \mathbf{\Delta}_{f0}(s)]^{-1}\, \mathbf{G}_0(s), \tag{4.11}$$

where $\mathbf{\Delta}_{fo}(s)$ denotes the output feedback perturbation. Block diagrams of these five uncertainty models, appearing in a feedback system, are given in Figure 4.2. Note that for SISO systems, blocks can be interchanged without affecting the system response, and there is no difference between the input and output uncertainty models, but these models can yield substantially different results for MIMO systems. The actual plant equals the nominal plant in all cases when the perturbation equals zero.

The uncertainty models are used to represent various types of uncertainty in the

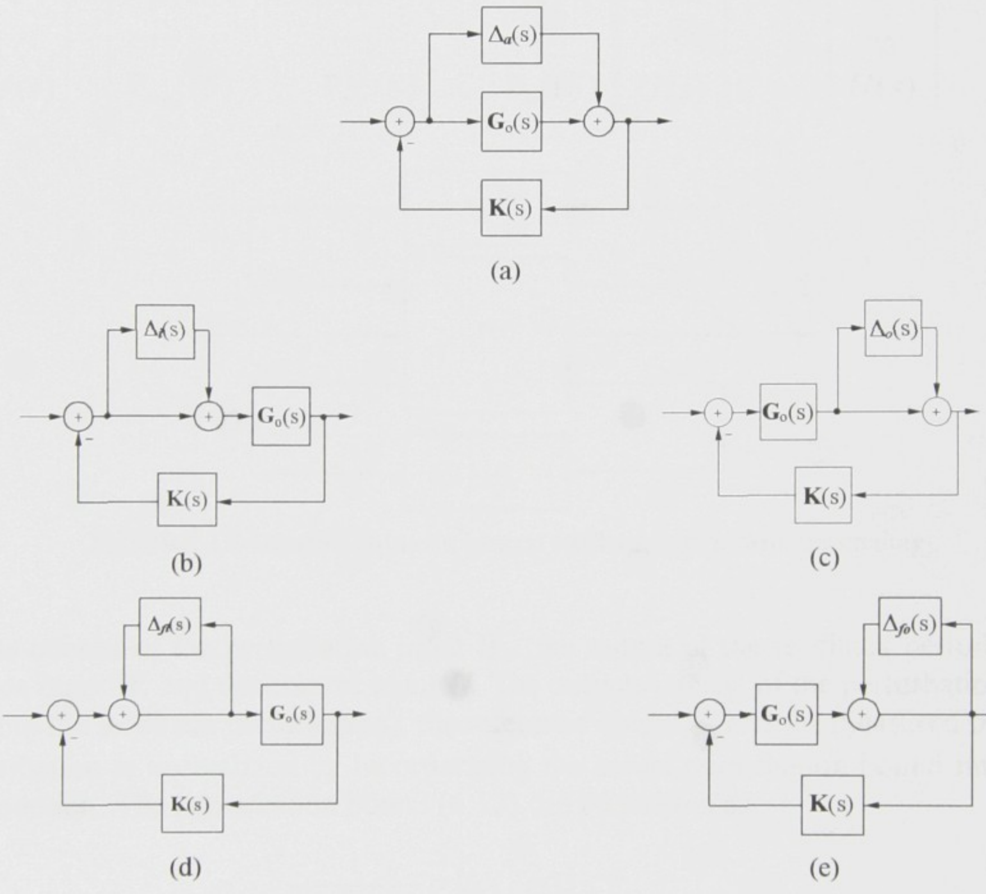


FIGURE 4.2 Unstructured uncertainties in the plant model: (a) additive uncertainty; (b) input-multiplicative uncertainty; (c) output-multiplicative uncertainty; (d) input feedback uncertainty; (e)output feedback uncertainty;

plant. The additive perturbation represents unknown dynamics operating in parallel with the plant. The multiplicative perturbations represent unknown dynamics operating in series with the plant. The feedback perturbations are used primarily to represent uncertainty in the gain and phase of the plant (or the control loop if a feedback control is applied to the plant).

Stability robustness or performance robustness can be evaluated when the perturbations in these models are bounded:

$$\bar{\sigma}\{\mathbf{\Delta}'(j\omega)\} \leq \Delta_{\max}(j\omega) \tag{4.12}$$

where $\bar{\sigma}(\bullet)$ is the maximum singular value, and $\mathbf{\Delta}'$ can be any of the perturbations described above. The bound given for the perturbation is in general frequency-dependent, allowing the specification of plant uncertainty to vary over frequency. In many applications the plant model is accurate at low frequencies but less accurate at high frequencies. The frequency-dependent bound allows this information to be incorporated into robustness analysis. Note that the perturbation transfer function is always stable since it possesses a bounded gain.

The unstructured uncertainty models presented above can all be analyzed in a similar manner by placing them within a common framework (called the standard form for robustness analysis or simply the standard form). This common framework has the perturbation normalized and in a feedback loop, as shown in Figure 4.3. The plant $\mathbf{P}(s)$ has three inputs and three outputs (in general, each of these inputs and outputs can be a vector):

$$\begin{bmatrix} Y_d(s) \\ \cdots\cdots \\ Y(s) \\ \cdots\cdots \\ M(s) \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{y_d w_d}(S) & \vdots & \mathbf{P}_{y_d w}(S) & \vdots & \mathbf{P}_{y_d u}(S) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{P}_{y w_d}(S) & \vdots & \mathbf{P}_{yw}(S) & \vdots & \mathbf{P}_{yu}(S) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{P}_{m w_d}(S) & \vdots & \mathbf{P}_{mw}(S) & \vdots & \mathbf{P}_{mu}(S) \end{bmatrix} \begin{bmatrix} W_d(s) \\ \cdots\cdots \\ W(s) \\ \cdots\cdots \\ U(s) \end{bmatrix} = \mathbf{P}(s) \begin{bmatrix} W_d(s) \\ \cdots\cdots \\ W(s) \\ \cdots\cdots \\ U(s) \end{bmatrix} \tag{4.13}$$

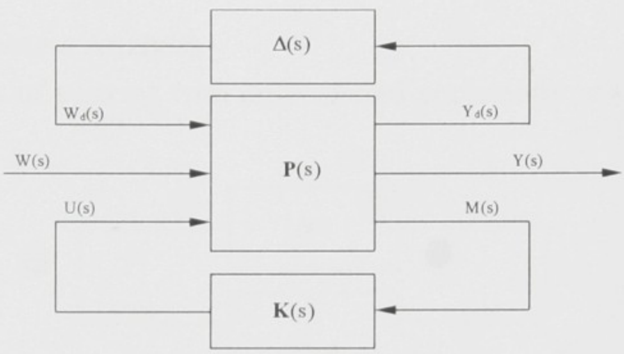


FIGURE 4.3 Standard form of a general feedback system with uncertainty

The inputs consist of the perturbation input W_d (the output of the feedback perturbation), the disturbance input W, and the control input U. The outputs consist of the perturbation output Y_d (the input to the feedback perturbation), the reference output Y, and the measured output M.

The perturbation is normalized by incorporating the actual perturbation bound into the plant transfer function. The perturbation bound (4.12) can be written as

$$\frac{1}{\Delta_{\max}(j\varpi)}\bar{\sigma}\{\mathbf{\Delta}'(j\varpi)\} \le 1 \tag{4.14}$$

and the normalized perturbation defined as follows:

$$\mathbf{\Delta}(j\omega) = \frac{1}{\Delta_{\max}(j\omega)}\mathbf{\Delta}'(j\omega) \tag{4.15}$$

The maximum singular value of the normalized perturbation is then

$$\bar{\sigma}\{\mathbf{\Delta}(j\omega)\} = \frac{1}{\Delta_{\max}(j\omega)}\bar{\sigma}\{\mathbf{\Delta}'(j\omega)\} \le 1 \tag{4.16}$$

Taking the maximum over all frequencies, the set of perturbations $\mathbf{\Delta}(j\omega)$ that satisfies this bound is defined by

$$\left\|\{\mathbf{\Delta}(j\omega)\}\right\|_{\infty} \le 1 \tag{4.17}$$

The normalized perturbation is incorporated into the system model by making the following substitution:

$$\mathbf{\Delta}'(j\omega) = \Delta_{\max}(j\omega)\mathbf{\Delta}(j\omega) \tag{4.18}$$

The use of the normalized perturbation simplifies robustness analysis by both normalizing the perturbation's magnitude and, more important, removing the frequency dependence of the bound. To simplify the notation, the term *perturbation* may be used to refer to either a normalized or an unnormalized perturbation. This practice is followed when the normalization of the perturbation is unimportant, or when the normalization of the perturbation is specified by the bound.

The perturbation bound $\Delta_{\max}(j\omega)$ is, in general, a scalar transfer function that is not necessarily rational (a ratio of polynomials in $j\omega$). The normalized plant may then be nonrational. Many design methods and analysis tools (especially computer-aided design tools) require that the plant be modeled as a rational transfer function. This can be accomplished by using a rational transfer function approximation of the perturbation bound. The rational approximation should have a magnitude that closely matches the perturbation bound [Zhou , Dolye, Glover-1995], [Dolye, Francis, Tannenbaum-1990].

4.2.2 Stability Robustness Analysis

The stability robustness of systems with unstructured uncertainty is addressed by analysis of the standard model.

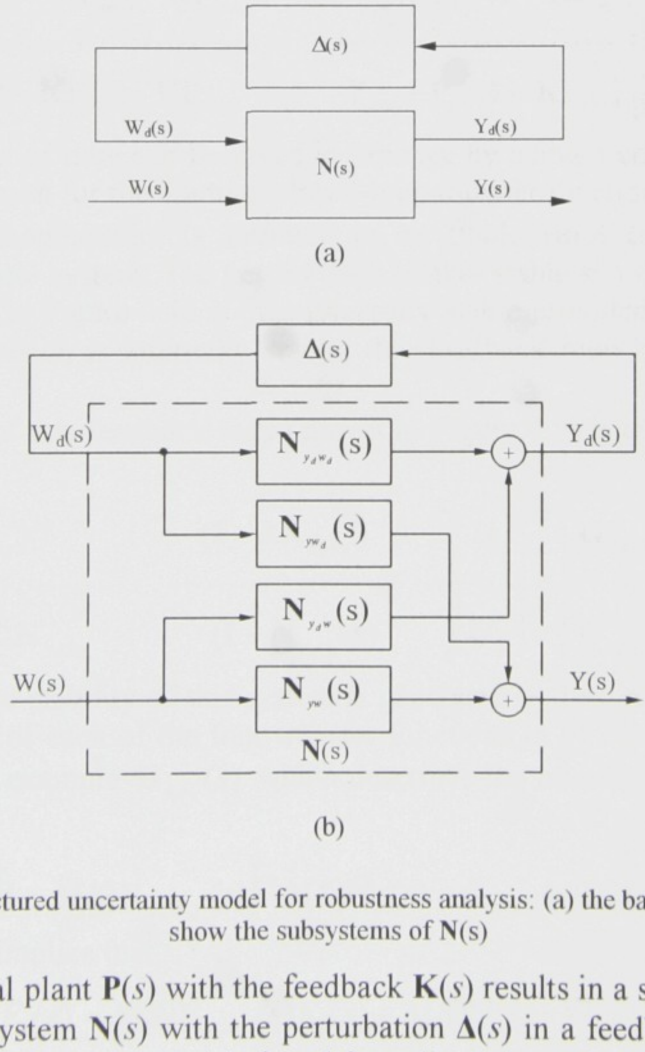


FIGURE 4.4 The unstructured uncertainty model for robustness analysis: (a) the basic model; (b) expanded to show the subsystems of **N**(s)

Combining the nominal plant **P**(s) with the feedback **K**(s) results in a system consisting of the nominal closed-loop system **N**(s) with the perturbation **Δ**(s) in a feedback loop, as shown in Figure 4.4a. The transfer function **N**(s) is found from (4.13) by noting that

$$U(s) = \mathbf{K}(s)M(s) = \mathbf{K}(s)\mathbf{P}_{mw_d}(s)W_d(s) + \mathbf{K}(s)\mathbf{P}_{mw}(s)W(s) + \mathbf{K}(s)\mathbf{P}_{mu}(s)U(s) \quad (4.19)$$

and solving for $U(s)$:

$$\begin{aligned} U(s) &= \left\{ \mathbf{I} - \mathbf{K}(s)\mathbf{P}_{mu}(s) \right\}^{-1} \mathbf{K}(s)\mathbf{P}_{mw_d}(s)W_d(s) \\ &+ \left\{ \mathbf{I} - \mathbf{K}(s)\mathbf{P}_{mu}(s) \right\}^{-1} \mathbf{K}(s)\mathbf{P}_{mw}(s)W(s). \end{aligned} \quad (4.20)$$

Substituting $U(s)$ into the equations for $Y_d(s)$ and $Y(s)$ in (4.13) yields the following closed-loop system:

$$\begin{bmatrix} Y_d \\ \cdots \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{y_d w_d} + \mathbf{P}_{y_d u}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw_d} & \vdots & \mathbf{P}_{y_d w} + \mathbf{P}_{y_d u}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{P}_{y w_d} + \mathbf{P}_{yu}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw_d} & \vdots & \mathbf{P}_{yw} + \mathbf{P}_{yu}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw} \end{bmatrix} \begin{bmatrix} W_d \\ \cdots \\ W \end{bmatrix}, \quad (4.21)$$

where all Laplace variables s have been dropped to simplify the notation. The nominal closed-loop transfer function $\mathbf{N}(s)$ is then

$$\begin{aligned} \mathbf{N} &= \begin{bmatrix} \mathbf{N}_{y_d w_d} & \vdots & \mathbf{N}_{y_d w} \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{N}_{y w_d} & \vdots & \mathbf{N}_{yw} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{P}_{y_d w_d} + \mathbf{P}_{y_d u}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw_d} & \vdots & \mathbf{P}_{y_d w} + \mathbf{P}_{y_d u}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{P}_{y w_d} + \mathbf{P}_{yu}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw_d} & \vdots & \mathbf{P}_{yw} + \mathbf{P}_{yu}\{\mathbf{I} - \mathbf{K}\mathbf{P}_{mu}\}^{-1}\mathbf{K}\mathbf{P}_{mw} \end{bmatrix} \end{aligned} \quad (4.22)$$

A considerable amount of time can be saved in practice by using a computer to evaluate this rather complex expression for the nominal closed-loop transfer function.

The nominal closed-loop system is assumed to be stable since the controller has been designed for the nominal system. The perturbation is also stable since it has a bounded gain. The combined system in Figure 4.4b is then internally stable provided that the feedback loop containing the perturbation is internally stable, this feedback loop being the only possible source of instability.

The internal stability of this feedback loop (shown in Figure 4.5) is evaluated by considering the following system:

$$\begin{bmatrix} E_1 \\ \cdots \\ E_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{\Delta}\mathbf{N}_{y_d w_d})^{-1} & \vdots & (\mathbf{I} - \mathbf{\Delta}\mathbf{N}_{y_d w_d})^{-1}\mathbf{\Delta} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ (\mathbf{I} - \mathbf{N}_{y_d w_d}\mathbf{\Delta})^{-1}\mathbf{N}_{y_d w_d} & \vdots & (\mathbf{I} - \mathbf{N}_{y_d w_d}\mathbf{\Delta})^{-1} \end{bmatrix} \begin{bmatrix} U_1 \\ \cdots \\ U_2 \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{e_1 u_1} & \vdots & \mathbf{G}_{e_1 u_2} \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{G}_{e_2 u_1} & \vdots & \mathbf{G}_{e_2 u_2} \end{bmatrix} \begin{bmatrix} U_1 \\ \cdots \\ U_2 \end{bmatrix} \quad (4.23)$$

A condition for internal stability of this system is generated by finding a bound on the signal 2-norm for the output of each of the four transfer functions in (4.23). Consider one of these transfer functions, for example $\mathbf{G}_{e_1 u_2}(s)$ which describes the relationship between $U_2(s)$ and $E_1(s)$:

$$E_1(s) = \left[\mathbf{I} - \mathbf{\Delta}(S)\mathbf{N}_{y_d w_d}(s)\right]^{-1} \mathbf{\Delta}(S)U_2(s) = \mathbf{G}_{e_1 u_2}(s)U_2(s) \quad (4.24)$$

This transfer function implies that

$$E_1(s) = \mathbf{\Delta}(S)\mathbf{N}_{y_d w_d}(s)E_1(s) + \mathbf{\Delta}(S)U_2(s) \quad (4.25)$$

Converting this expression into the time domain yields

$$e_1(t) = \boldsymbol{\delta}(t) \otimes \mathbf{n}_{y_d w_d}(t) \otimes e_1(t) + \boldsymbol{\delta}(t) \otimes u_2(t) \tag{4.26}$$

where $\mathbf{n}_{y_d w_d}(t)$ and $\delta(t)$ are the impulse responses of $\mathbf{N}_{y_d w_d}(s)$ and $\mathbf{\Delta}(s)$, respectively, and $\otimes$ denotes convolution. Taking the signal 2-norm of both sides of this expression and employing the triangle inequality yields

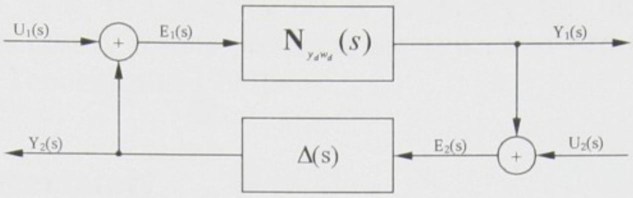


FIGURE 4.5 Internal stability of the uncertain system

$$\|e_1(t)\|_2 = \left\|\boldsymbol{\delta}(t) \otimes \mathbf{n}_{y_d w_d}(t) \otimes e_1(t) + \boldsymbol{\delta}(t) \otimes u_2(t)\right\|_2 \leq \\ \left\|\boldsymbol{\delta}(t) \otimes \mathbf{n}_{y_d w_d}(t) \otimes e_1(t)\right\|_2 + \left\|\boldsymbol{\delta}(t) \otimes u_2(t)\right\|_2. \tag{4.27}$$

The ∞-norm can be used to bound the terms on the right side of this inequality:

$$\|e_1(t)\|_2 = \left\|\mathbf{\Delta}\mathbf{N}_{y_d w_d}\right\|_\infty \|e_1(t)\|_2 + \|\mathbf{\Delta}\|_\infty \|u_2(t)\|_2. \tag{4.28}$$

Further, the ∞-norm of a product is less than or equal to the product of the ∞-norms:

$$\|e_1(t)\|_2 \leq \|\mathbf{\Delta}\|_\infty \left\|\mathbf{N}_{y_d w_d}\right\|_\infty \|e_1(t)\|_2 + \|\mathbf{\Delta}\|_\infty \|u_2(t)\|_2. \tag{4.29}$$

Solving for the signal 2-norm of $e_1(t)$ yields

$$\|e_1(t)\|_2 \leq \left(1 - \|\mathbf{\Delta}\|_\infty \left\|\mathbf{N}_{y_d w_d}\right\|_\infty\right)^{-1} \|\mathbf{\Delta}\|_\infty \|u_2(t)\|_2. \tag{4.30}$$

The system described by the transfer function $\mathbf{G}_{e_1 u_2}$ has a bounded gain and is therefore ℓ_2 stable and BIBO stable provided the inverse above is finite. Noting that $\|\mathbf{\Delta}\|_\infty \leq 1$, this inverse is finite if

$$\left\|\mathbf{N}_{y_d w_d}\right\|_\infty < 1. \tag{4.31}$$

A similar analysis can be applied to the other three transfer functions in (4.23). All of these transfer functions are stable provided condition (4.31) is satisfied. The following result is then obtained:

A general feedback system, as given in Figure 4.4, where the perturbation is bounded,

$$\|\mathbf{\Delta}\|_\infty \leq 1, \tag{4.32}$$

is internally stable for all possible perturbations provided the nominal closed-loop system is stable and

$$\left\|\mathbf{N}_{y_d w_d}\right\|_\infty = \sup_\omega \left\{ \bar{\sigma}\left[\mathbf{N}_{y_d w_d}(j\omega)\right] \right\} < 1. \tag{4.33}$$

This result is known as the small-gain theorem and provides a test for robust stability with respect to bounded perturbations. An alternative way of presenting the above results is to state that the system is internally stable if the ∞-norm of the loop dition for internal stability. In fact, this condition is also necessary for robust stability wtransfer function is less than 1.

The robust stability condition (4.33) is presented as a sufficient conith respect to unstructured perturbations; that is, a destabilizing perturbation can be found that satisfies the bound (4.32) whenever (4.33) does not hold. A necessary and sufficient stability robustness condition is subsequently derived for structured perturbations. The robustness condition (4.33) is a special case of this more general result, since an unstructured perturbation is a special case of a structured perturbation. The fact that (4.33) is necessary for interval stability follows immediately from the necessity results presented for structured perturbations [Zhou, Dolye-1998], [Dolye, Francis, Tannenbaum-1990].

4.3 Structured Uncertainty

Unstructured uncertainty is modeled by connecting an unknown but bounded perturbation to the plant. This type of uncertainty model requires very little information, that is, only the bound and how the perturbation is connected. In many applications, additional constraints on the set of admissible perturbations are available. These constraints add "structure" to the set of admissible perturbations, and the uncertainty is termed structured. Structured uncertainty is therefore a more general form of uncertainty than unstructured uncertainty.

Structured uncertainty arises when the plant is subject to multiple perturbations. Multiple perturbations occur when the plant contains a number of uncertain parameters, or when the plant contains multiple unstructured uncertainties. For example, the plant model may be well specified except for two uncertain time constants, which are modeled as a nominal value plus a perturbation. Another example of structured uncertainty is when the plant contains both an input multiplicative perturbation and an additive perturbation. Clearly, structured uncertainty is a very general way of modeling uncertainty.

4.3.1 The Structured Uncertainty Model

A plant subject to structured uncertainty can be placed in a standard form analogous to that used for unstructured uncertainty. The standard form of the structured uncertainty model has the individual perturbations normalized to 1 and placed in a feedback loop around the nominal plant. The standard form of the structured uncertainty model is shown in Figure 4.6. The structured perturbation $\mathbf{\Delta}(s)$ is a block diagonal transfer function:

$$\mathbf{\Delta}(s)=\begin{bmatrix} \mathbf{\Delta}_1(s) & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{\Delta}_2(s) & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{\Delta}_n(s) \end{bmatrix}, \tag{4.34}$$

where n is the number of perturbations and the blocks $\mathbf{\Delta}_i(s)\in \ell^{L_i * n_i}$ represent the individual perturbations applied to the plant. An individual block can represent an uncertainty in a parameter (a scalar perturbation) or an unstructured uncertainty. The set of all transfer function matrices with this block diagonal form is denoted $\overline{\mathbf{\Delta}}$. The structured perturbation is normalized so that its infinity norm is bounded by 1:

$$\|\mathbf{\Delta}\|_\infty \le 1, \tag{4.35}$$

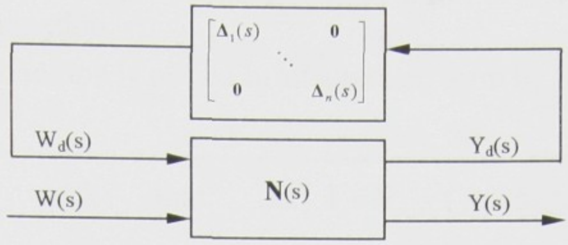


FIGURE 4.6 standard form of the structured uncertainty model

Additionally, all the blocks in the perturbation are scaled so that their infinity norms are bounded by 1:

$$\|\mathbf{\Delta}_1\|_\infty \le 1, \quad \|\mathbf{\Delta}_2\|_\infty \le 1, \quad \cdots \quad ,\|\mathbf{\Delta}_n\|_\infty \le 1, \tag{4.36}$$

Note that the bounds in (4.36) imply the bound in (4.35). The perturbation is normalized by incorporating the actual bound (which may be frequency-dependent) into the plant, as discussed in Section 4.2.1. The subset of $\bar{\mathbf{\Delta}}$ that satisfies the bounds in (4.34) is termed the set of admissible perturbations.

4.3.2 The Structured Singular Value and Stability Robustness

The stability of a system subject to a structured uncertainty is determined by analyzing the feedback system in Figure 4.6. The nominal closed-loop system is assumed to be stable. Any unstable poles of this system are therefore caused by closing the loop through the perturbation and are the solutions of

$$\det\left\{ \mathbf{I} + \mathbf{N}_{y_d w_d}(s)\mathbf{\Delta}(s) \right\} = 0. \tag{4.37}$$

Stability robustness may be evaluated by determining the "size" of the smallest perturbation that results in a pole-a solution of (4.37)-with a non-negative real part. A perturbation that results in such a pole is termed a destabilizing perturbation.

The locus of the solutions of (4.37) is a continuous function of $\mathbf{\Delta}(s)$. Therefore, the smallest destabilizing perturbation has one or more poles on the imaginary axis. The "size" of the smallest destabilizing perturbation is defined as follows:

$$\inf_{\omega}\left\{ \min_{\Delta(j\omega)\in\bar{\Delta}}\left\{ \bar{\sigma}[\mathbf{\Delta}(j\omega)] \quad \text{such} \quad \text{that} \quad \det\left\{ \mathbf{I} + \mathbf{N}_{y_d w_d}(j\omega)\mathbf{\Delta}(j\omega) \right\} = 0 \right\}\right\}. \tag{4.38}$$

Note that $\mathbf{\Delta}(j\omega)$in (4.38) is any perturbation (with the appropriate block structure) that places a pole at a specific point $j\omega$ on the imaginary axis. The maximum singular value is a measure of the size of this perturbation. The minimization over all appropriate perturbations results in the size of the smallest perturbation that places a pole at $j\omega$. The minimization over frequency then yields the size of the smallest perturbation that places a pole anywhere on the imaginary axis, that is, the size of the smallest destabilizing perturbation. A system in standard form is robustly stable if and only if the smallest destabilizing perturbation is greater than 1 (the infinity norm of the largest admissible perturbation):

$$\inf_{\omega}\left\{ \min_{\Delta(j\omega)\in\bar{\Delta}}\left\{ \bar{\sigma}[\mathbf{\Delta}(j\omega)] \quad \text{such} \quad \text{that} \quad \det\left\{ \mathbf{I} + \mathbf{N}_{y_d w_d}(j\omega)\mathbf{\Delta}(j\omega) \right\} = 0 \right\}\right\} > 1. \tag{4.39}$$

Unfortunately, finding the size of the smallest destabilizing perturbation using (4.38) is not a trivial matter. In fact, this problem is intractable in all but the simplest of cases. Therefore, bounds on the size of the smallest destabilizing perturbation are developed. Before generating these bounds, the stability robustness condition (4.39) is put in a form that is more useful for both application and computation.

The stability robustness condition is placed in an alternate form by inverting (4.39):

$$\sup_{\omega}\left\{ \frac{1}{\min\limits_{\Delta(j\omega)\in\bar{\Delta}} \left\{\bar{\sigma}[\boldsymbol{\Delta}(j\omega)] \quad \text{such} \quad \text{that} \quad \det\left\{ \mathbf{I}+\mathbf{N}_{y_d w_d}(j\omega)\boldsymbol{\Delta}(j\omega)\right\}=0\right\}} \right\}<1. \tag{4.40}$$

Note that the supremum of the ratio in this equation equals the inverse of the infimum of the denominator. This alternate stability robustness condition is very similar in form to the result obtained for unstructured uncertainty in (4.33). In addition, this form of the stability robustness condition can be simply generalized to include performance robustness.

The term within the brackets in (4.40) is called the structured singular value (SSV) and is formally defined as follows:

$$\mu_{\bar{\Delta}}(\mathbf{N}) = \frac{1}{\min\limits_{\boldsymbol{\Delta}\in\boldsymbol{\Delta}}\left[\bar{\sigma}(\boldsymbol{\Delta}) \quad \text{such} \quad \text{that} \quad \det(\mathbf{I}+\mathbf{N}\boldsymbol{\Delta})=0\right]}, \tag{4.41}$$

$$\mu_{\bar{\Delta}}(\mathbf{N}) = 0 \quad \text{if} \quad \det(\mathbf{I}+\mathbf{N}\boldsymbol{\Delta}) \neq 0 \quad \text{for all} \quad \boldsymbol{\Delta}\in\bar{\boldsymbol{\Delta}}. \tag{4.42}$$

The structured singular value is, in general, a real-valued function of a complex matrix **N**, which depends on the structure of the perturbations as defined by $\bar{\boldsymbol{\Delta}}$.

The stability robustness criteria for a system with unstructured uncertainty is summarized below.

A general feedback system, as given in Figure 4.6, is internally stable for all possible perturbations:

$$\boldsymbol{\Delta}(j\omega)\in\bar{\boldsymbol{\Delta}} \quad \text{and} \quad \left\|\boldsymbol{\Delta}(j\omega)\right\|_{\infty} \leq 1. \tag{4.43}$$

if and only if the nominal closed-loop system is internally stable and

$$\sup_{\omega}\left\{ \mu_{\bar{\Delta}}\left[\mathbf{N}_{y_d w_d}(j\omega)\right] \right\} < 1. \tag{4.44}$$

Note that satisfying this condition is both necessary and sufficient for robust stability. This test for stability robustness is very general and can be used in a wide range of applications, provided the structured singular value can be computed.

4.3.3 Bounds on the Structured Singular Value

The determination of robust stability is dependent on the computation of the structured singular value. The direct computation of the SSV by a search over all $\bar{\boldsymbol{\Delta}}$ is impractical since this set has an infinite number of elements. Quantized searches over $\bar{\boldsymbol{\Delta}}$ may yield unreliable results and are often hindered by the high dimensionality of $\bar{\boldsymbol{\Delta}}$. Therefore, bounds on the SSV, which can be generated using a moderate amount of computation, are presented. These bounds are often tight; that is, they provide good estimates of the SSV.

Upper Bounds The structured singular value is bounded from above by the maximum singular value:

$$\mu_{\bar{\Delta}}(\mathbf{N}) \leq \bar{\sigma}(\mathbf{N}). \tag{4.45}$$

To demonstrate this bound, first note that if $\mu_{\bar{\Delta}}(\mathbf{N}) = 0$, the bound is valid since $\bar{\sigma}(\mathbf{N}) \geq 0$. Now, when $\mu_{\bar{\Delta}}(\mathbf{N}) \neq 0$, the definition of $\mu_{\bar{\Delta}}(\mathbf{N})$ states that

$$\frac{1}{\mu_{\bar{\Delta}}(\mathbf{N})} = \min_{\Delta \in \bar{\Delta}} \left[\bar{\sigma}(\mathbf{\Delta}) \quad \text{such that} \quad \det(\mathbf{I} + \mathbf{N\Delta}) = 0 \right]. \tag{4.46}$$

Let $\tilde{\mathbf{\Delta}}$ be the value of $\mathbf{\Delta} \in \bar{\mathbf{\Delta}}$ that yields the minimum in (4.46). The fact that the determinant in (4.46) equals zero implies that for some nonzero vector v,

$$\begin{aligned} (\mathbf{I} + \mathbf{N}\tilde{\mathbf{\Delta}})v &= 0 \\ v &= -\mathbf{N}\tilde{\mathbf{\Delta}}v \end{aligned} \tag{4.47}$$

Taking the vector 2-norm of this equation, we have

$$\|v\|_2 = \left\|\mathbf{N}\tilde{\mathbf{\Delta}}v\right\|_2. \tag{4.48}$$

The gain of the matrix in this expression can be bounded by the maximum singular value:

$$\|v\|_2 \leq \bar{\sigma}(\mathbf{N}\tilde{\mathbf{\Delta}})\|v\|_2; \tag{4.49}$$

$$\|v\|_2 \leq \bar{\sigma}(\mathbf{N})\bar{\sigma}(\tilde{\mathbf{\Delta}})\|v\|_2. \tag{4.50}$$

Dividing by $\|v\|_2$ and $\bar{\sigma}(\tilde{\mathbf{\Delta}})$ yields the result given in (4.45):

$$\mu_{\bar{\Delta}}(\mathbf{N}) = \frac{1}{\bar{\sigma}(\tilde{\mathbf{\Delta}})} \leq \bar{\sigma}(\mathbf{N}). \tag{4.51}$$

In summary, the structured singular value of a matrix is bounded from above by the maximum singular value.

The bound in (4.45) is easy to compute but tends to be overly conservative; that is, the structured singular value can be appreciably less than the maximum singular value. Additional bounds can be generated by returning to the system interpretation of the SSV and considering the block diagrams in Figure 4.7.

The SSV of a transfer function $\mathbf{N}(s)$ is the inverse of the smallest perturbation that, when placed in the feedback loop, yields a closed-loop pole located at s. The closed-loop poles of this system are not changed by the inclusion (as shown in Figure 4.7b) of the diagonal scaling matrices $\mathbf{D}_L(s)$ and $\mathbf{D}_R(s)$ and their inverses:

$$\mathbf{D}_L(s) = \begin{bmatrix} d_1(s)\mathbf{I}_{l_1} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & d_2(s)\mathbf{I}_{l_2} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & d_n(s)\mathbf{I}_{l_n} \end{bmatrix} \tag{4.52}$$

$$\mathbf{D}_R(s) = \begin{bmatrix} d_1(s)\mathbf{I}_{n_1} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & d_2(s)\mathbf{I}_{n_2} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & d_n(s)\mathbf{I}_{n_n} \end{bmatrix} \tag{4.53}$$

The uncertainty blocks have the dimensions $\mathbf{\Delta}_i(s) \in \ell^{L_i * n_i}$, and the identity matrices have the dimensions $\mathbf{I}_{n_i}(s) \in \Re^{n_i * n_i}$. These dimensions match up with the perturbation blocks to yield

$$\mathbf{D}_L(s)\mathbf{\Delta}\mathbf{D}_R^{-1}(s)$$

$$= \begin{bmatrix} d_1(s)\mathbf{\Delta}_1(s)\frac{1}{d_1(s)} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & d_2(s)\mathbf{\Delta}_2(s)\frac{1}{d_2(s)} & \vdots & \cdots & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \vdots & d_n(s)\mathbf{\Delta}_n(s)\frac{1}{d_n(s)} \end{bmatrix} = \mathbf{\Delta}(s). \tag{4.54}$$

Therefore, the system in Figure 4.7c is equivalent to the system in Figure 4.7a and

$$\mu_{\bar{\Delta}}[\mathbf{N}(s)] = \mu_{\bar{\Delta}}\left[\mathbf{D}_R(s)\mathbf{N}(s)\mathbf{D}_L^{-1}(s)\right]. \tag{4.55}$$

An additional bound on the structured singular value,

$$\mu_{\bar{\Delta}}[\mathbf{N}(s)] = \mu_{\bar{\Delta}}\left[\mathbf{D}_R(s)\mathbf{N}(s)\mathbf{D}_L^{-1}(s)\right] \le \bar{\sigma}\left[\mathbf{D}_R(s)\mathbf{N}(s)\mathbf{D}_L^{-1}(s)\right], \tag{4.56}$$

is then obtained, since the maximum singular value is changed by inclusion of the diagonal scaling matrices. Since this result is valid for all diagonal scaling matrices (with the given block structure) and for all s, then

$$\mu_{\bar{\Delta}}(\mathbf{N}) \le \min_{\substack{(d_1, d_2, \ldots d_n) \\ d_i \in I}} \bar{\sigma}(\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1}), \tag{4.57}$$

The parameters d_i are called **D**-scales. The bound (4.57) is valid for all complex D -scales, and as a special case, for all real D -scales. For the case of complex perturbations, the phase shift of the perturbation is arbitrary, and any phase shift (including sign changes) imparted by the D –scales has no effect on the bound. Therefore, the minimization in (4.57) can be performed over the set of positive real D -scales without loss of generality:

$$\mu_{\bar{\Delta}}(\mathbf{N}) \le \min_{\substack{(d_1, d_2, \ldots d_n) \\ d_i \in (0, \infty)}} \bar{\sigma}(\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1}). \tag{4.58}$$

The expression $\bar{\sigma} = (\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1})$ is a convex function of $\{d_1, d_2, \ldots, d_{n-1}\}$ for $d_i \in (0, \infty)$. In this care, $\bar{\sigma} = (\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1})$ has a single global minimum, no local minima, and the minimum value can be computed reliably using any good gradient-decent algorithm.

Computing the upper bound for the SSV using a brute forte gradient-decent algorithm is often computationally quite intensive. This computational intensity is compounded by the fact that robustness analysis requires that the SSV be computed over a range of frequencies. These computational difficulties have led to the development of a number of algorithms for optimizing the upper bound for the SSV that exploit the structure inherent in this problem.

The upper bound in (4.58) has been found to lie Close to the true SSV in a number of applications. This bound is an equality for perturbations consisting of three or fewer complex blocks, typically within 5% of the true SSV for larger numbers of complex blocks, and rarely worse than within 15% of the true SSV. The exception is when some of the perturbations are real.

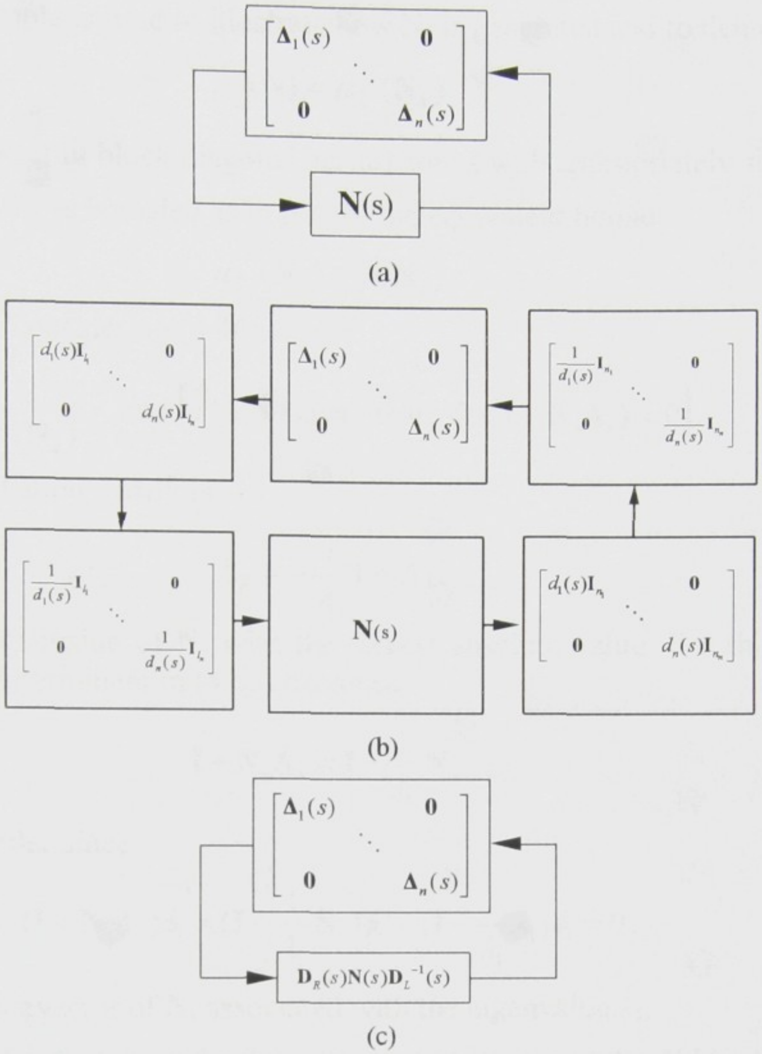


Figure 4.7 Diagonal scaling of the plant: (a) the feedback perturbation; (b) diagonal scaling added to the plant and perturbation; (c) diagonal scaling leaves the diagonal perturbation unchanged

The bound in (4.58) may be overly conservative for mixed real and complex perturbations, although even for these mixed perturbations, (4.58) often yields acceptable results.

A bound similar to (4.58) can be developed for the case where repeated perturbations are present. Repeated perturbations occur when multiple system parameters are dependent on a single external factor. For example, the Mach number (speed) of an aircraft influences both the lift produced by a given angle of attack and the torque produced by a given angle of attack. Therefore, the two parameters in the model relating lift and moment to angle of attack are related. These parameters can be modeled as nominal values plus scaled versions of the same perturbation.

Lower Bounds The structured singular value is bounded from below by the spectral radius of $\mathbf{N}_s$:

$$\mu_{\bar{\Delta}}(\mathbf{N}) \geq \rho(\mathbf{N}_s). \tag{4.59}$$

where $\mathbf{N}_s$ is generated from $\mathbf{N}$ by the addition of zero columns and/or rows. These zero columns and rows are added to make $\mathbf{N}_s$ square and to make the perturbation blocks square, and have no effect on the structured singular value. The spectral radius of a matrix is defined as the largest of the eigenvalue magnitudes:

$$\rho(\mathbf{N}) = \max_i |\lambda_i(\mathbf{N})|. \tag{4.60}$$

The following example is used to illustrate how $\mathbf{N}_s$ is generated and to demonstrate that

$$\mu_{\bar{\Delta}}(\mathbf{N}) = \mu_{\bar{\Delta}_s}(\mathbf{N}_s). \tag{4.61}$$

Note that $\bar{\mathbf{\Delta}}_s$ is the set of block diagonal perturbations with appropriately sized square blocks. To show that the SSV is bounded as in (4.59), the equivalent bound

$$\mu_{\bar{\Delta}_s}(\mathbf{N}_s) \ge \rho(\mathbf{N}_s). \tag{4.62}$$

is demonstrated by considering (4.46):

$$\frac{1}{\mu_{\bar{\Delta}_s}(\mathbf{N}_s)} = \min_{\Delta_s \in \bar{\Delta}_s} \left[\bar{\sigma}(\mathbf{\Delta}_s) \quad \text{such that} \quad \det(\mathbf{I} + \mathbf{N}_s \mathbf{\Delta}_s) = 0\right]. \tag{4.63}$$

A particular perturbation matrix is

$$\mathbf{\Delta}_s = -\frac{1}{\lambda_1}\mathbf{I} \in \bar{\mathbf{\Delta}}_s. \tag{4.64}$$

where λ_1 is the eigenvalue of $\mathbf{N}_s$ with the largest absolute value. For this perturbation, the matrix within the determinant in (4.63) becomes

$$\mathbf{I} + \mathbf{N}_s \mathbf{\Delta}_s = \mathbf{I} - \frac{1}{\lambda_1}\mathbf{N}_s. \tag{4.65}$$

This matrix is singular since

$$(\mathbf{I} + \mathbf{N}_s \mathbf{\Delta}_s)\phi_1 = (\mathbf{I} - \frac{1}{\lambda_1}\mathbf{N}_s)\phi_1 = (\mathbf{I} - \frac{1}{\lambda_1}\lambda_1)\phi_1 = 0. \tag{4.66}$$

where ϕ_1 is the eigenvector of $\mathbf{N}_s$ associated with the eigenvalue λ_1.

Therefore, the $\det(\mathbf{I} + \mathbf{N}_s\,\mathbf{\Delta}_s) = 0$ for the given perturbation, and the SSV is bounded:

$$\mu_{\bar{\Delta}_s}(\mathbf{N}_s) = \frac{1}{\min\limits_{\Delta_s \in \bar{\Delta}_s} \left[\bar{\sigma}(\mathbf{\Delta}_s) \quad \text{such that} \quad \det(\mathbf{I} + \mathbf{N}_s \mathbf{\Delta}_s) = 0\right]} \ge \frac{1}{\bar{\sigma}(-\frac{1}{\lambda_1}\mathbf{I})} = |\lambda_1| = \rho(\mathbf{N}_s). \tag{4.67}$$

The bound in (4.59) is easy to compute, but tends to be overly conservative; that is, the SSV can be appreciably greater than the spectral radius. Additional bounds can be generated by noting that

$$\mu_{\bar{\Delta}_s}[\mathbf{N}_s\mathbf{U}] = \mu_{\bar{\Delta}_s}[\mathbf{N}_s] = \mu_{\bar{\Delta}}[\mathbf{N}]. \tag{4.68}$$

where $\mathbf{U} \in \bar{\Delta}_s$, and $\mathbf{U}$ is a unitary matrix; that is,

$$\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}. \tag{4.69}$$

Combining (4.68) with the bound (4.59) yields a lower bound for the SSV:

$$\mu_{\bar{\Delta}_s}[\mathbf{N}] \ge \max_{\substack{\mathbf{U} \in \Delta_s \\ \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}}} \rho[\mathbf{N}_s\mathbf{U}]. \tag{4.70}$$

The lower bound in (4.70) is identically equal to the SSV. Unfortunately, the expression $\rho[\mathbf{N}_s\mathbf{U}]$ is not a convex function of the elements of $\mathbf{U}$, and may have multiple local maxima. Numerical optimization algorithms may fail to find the true maximum in (4.70). Therefore, it is best to treat (4.70) as a lower bound and not as an equality, since numerical algorithms may not yield the equality.

4.3.4 Additional Properties of the Structured Singular Value

The structured singular value is central to the analysis of robustness. A number of properties of the SSV have been generated in the literature. In this section, two of these properties are developed. These properties are selected because they add insight into the meaning of the SSV, and they are used in subsequent developments.

A consequence of the definition of the structured singular value is that

$$\mu_{\bar{\Delta}}[\mathbf{N}] < 1. \tag{4.71}$$

if and only if

$$\det(\mathbf{I} - \mathbf{N}\boldsymbol{\Delta}) > 0. \tag{4.72}$$

for all admissible perturbations:

$$\boldsymbol{\Delta} \in \bar{\boldsymbol{\Delta}} \quad \text{and} \quad \|\boldsymbol{\Delta}\|_{\infty} \leq 1. \tag{4.73}$$

The condition (4.72) results from the fact that the determinant is a continuous function of the perturbation. The determinant is positive (indeed, equal to 1) for the admissible perturbation **Δ=0.** If for some other admissible perturbation the determinant is non-positive, the determinant must equal zero for an admissible perturbation. The singular value of this perturbation is less than 1, implying that, $\mu_{\bar{\Delta}}(\mathbf{N})$ is greater than 1.

The SSV is bounded by the maximum singular value (4.45), in general. When the set of perturbations is the set of all complex matrices (i.e., $\bar{\boldsymbol{\Delta}}$ is the set of unstructured uncertainties), this bound becomes an equality. In addition, the bound is achieved when the perturbation is

$$\boldsymbol{\Delta} = -\frac{1}{\sigma} V_1 U_1{}^{\dagger}. \tag{4.74}$$

where σ_1, V_1, and U_1 are from the singular value decomposition of **N**:

$$\mathbf{N} = \sum_{i=1}^{p} \sigma_i U_i V_i{}^{\dagger}. \tag{4.75}$$

To demonstrate that the SSV equals the maximum singular value, note that

$$(\mathbf{I} + \mathbf{N}\boldsymbol{\Delta})U_1 = \left\{ \mathbf{I} - \sum_{i=1}^{p} \sigma_i U_i V_i{}^{\dagger} \frac{1}{\sigma_1} V_1 U_1{}^{\dagger} \right\} U_1 = 0. \tag{4.76}$$

Therefore, this matrix must be singular:

$$\det(\mathbf{I} + \mathbf{N}\boldsymbol{\Delta}) = 0. \tag{4.77}$$

The bound (4.45) on the SSV is then achieved for the given perturbation, and

$$\mu_{\bar{\Delta}}(\mathbf{N}) = \bar{\sigma}(\mathbf{N}). \tag{4.78}$$

when the perturbations are unstructured.

The result (4.78) shows that the maximum singular value of a matrix can be defined in a manner analogous to the structured singular value:

$$\bar{\sigma}(\mathbf{N}) = \frac{1}{\min\limits_{\Delta \in \mathcal{C}^{l \times n}} \left[\bar{\sigma}(\boldsymbol{\Delta}) \quad \text{such that} \quad \det(\mathbf{I} + \mathbf{N}\boldsymbol{\Delta}) = 0 \right]}. \tag{4.79}$$

The structured singular value is then a generalization of the maximum singular value. This generalization is accomplished by placing restrictions on the set over which the minimization is performed in (4.79). This relationship explains the similarity between the robust stability conditions for unstructured and structured uncertainty.

4.4 Performance Robustness Analysis Using the SSV

The stability of systems subject to gain, phase, unstructured, and structured perturbations was addressed in the previous sections of this chapter. Stability is typically required of feedback control systems, but stability alone does not insure suitability of the system. Suitability of the controller is dependent on both stability and the meeting of certain performance specifications.

Performance can be described in a number of ways. A particular method of specifying performance is to bound the ∞-norm of the closed loop transfer function. This form of performance specification is quite general and can be simply incorporated in SSV robustness analysis. The performance specification is given as

$$\left\|\mathbf{H}(s)\right\|_{\infty} < 1, \tag{4.80}$$

where $\mathbf{H}(s)$ is the perturbed closed-loop transfer function. This transfer function is obtained by closing the loop containing the perturbation in Figure 4.8a:

$$\mathbf{H}(s) = \mathbf{N}_{yw_d}(s)\left[\mathbf{I} - \mathbf{\Delta}(s)\mathbf{N}_{y_d w_d}(s)\right]^{-1}\mathbf{\Delta}(s)\mathbf{N}_{y_d w}(s) + \mathbf{N}_{yw}(s), \tag{4.81}$$

A bound of 1 can be used to specify any frequency-dependent bound on this gain by incorporating weighting functions into the plant. A system is said to possess robust performance if the system remains internally stable and the specification in (4.80) is satisfied for all admissible perturbations. Robust performance can also be defined using other performance criteria or cost functions. The ∞-norm cost function is typically used to specify performance robustness because fit yields a robustness test that is easily applied in practice.

The perturbed closed-loop transfer function is dependent on both the nominal closed-loop transfer function and the perturbation. The conditions for performance robustness can be precisely stated in terms of these transfer functions:

$$\sup_{\omega} \left\{ \mu_{\bar{\Delta}}\left[\mathbf{N}_{y_d w_d}(s)\right] \right\} < 1, \tag{4.82}$$

$$\left\|\mathbf{H}(s)\right\|_{\infty} = \left\|\mathbf{N}_{yw_d}(s)\left[\mathbf{I} - \mathbf{\Delta}(s)\mathbf{N}_{y_d w_d}(s)\right]^{-1}\mathbf{\Delta}(s)\mathbf{N}_{y_d w}(s) + \mathbf{N}_{yw}(s)\right\|_{\infty} < 1, \tag{4.83}$$

for all admissible $\mathbf{\Delta}$.

This robust performance problem can be converted into an equivalent robust stability problem by appending an uncertainty block to the system. This uncertainty block (known as a performance block) connects the performance output with the disturbance input, as shown in Figure 4.8. The system in Figure 4.8a meets the performance robustness objectives if and only if the system in Figure 4.8c is robustly stable. This fact can be intuitively understood by noting that for robust stability, the gain must be less than 1. This requirement is the same as the performance requirement. Therefore, requiring robust stability with respect to the perturbation $\mathbf{\Delta}_o(s)$, as shown in Figure 4.8b, is equivalent to the performance requirement. Adding this new perturbation to the original perturbations yields the block diagram in Figure 4.8c. A more mathematical demonstration of the connection between robust performance of the system in Figure 4.8a and robust stability of the system in Figure 4.8c is presented below.

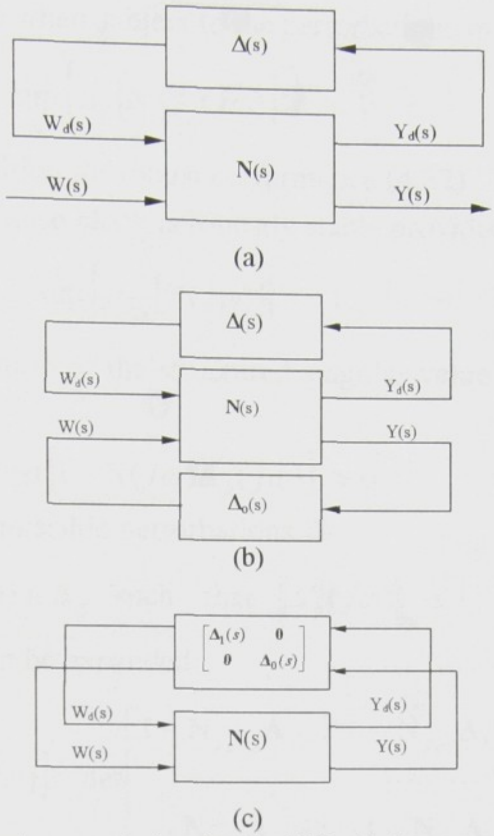


FIGURE 4.8 performance robustness analysis using the SSV: (a) the system with uncertainty; (b) adding a performance block; (c) structured uncertainty incorporating the performance block

Robust stability of the system with the performance block (Figure 4.8c) implies that the system is stable for all perturbations of the following kind:

$$\boldsymbol{\Delta}_p(s) = \begin{bmatrix} \boldsymbol{\Delta}(s) & \vdots & \mathbf{0} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{0} & \vdots & \boldsymbol{\Delta}_0(s) \end{bmatrix} \quad \text{such that} \quad \left\| \boldsymbol{\Delta}_p(s) \right\|_\infty \le 1, \tag{4.84}$$

The system is also stable for all perturbations of the following kind:

$$\boldsymbol{\Delta}_p(s) = \begin{bmatrix} \boldsymbol{\Delta}(s) & \vdots & \mathbf{0} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix} \quad \text{such that} \quad \left\| \boldsymbol{\Delta}(s) \right\|_\infty \le 1, \tag{4.85}$$

since this is a subset of the perturbations in (4.84). The feedback loop of the system in Figure 4.8c, with this set of perturbations, is identical to the feedback loop of the system in Figure 4.8a. This can be seen either from the block diagram or by looking at the closed-loop poles, which are the solutions of

$$\begin{aligned} &\det\left\{ \mathbf{I} - \begin{bmatrix} \mathbf{N}_{y_d w_d}(s) & \vdots & \mathbf{N}_{y_d w}(s) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{N}_{y w_d}(s) & \vdots & \mathbf{N}_{yw}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}(s) & \vdots & \mathbf{0} \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix} \right\} \\ &= \det\left\{ \mathbf{I} - \begin{bmatrix} \mathbf{N}_{y_d w_d}(s)\boldsymbol{\Delta}(s) & \vdots & \mathbf{0} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{N}_{y w_d}(s)\boldsymbol{\Delta}(s) & \vdots & \mathbf{0} \end{bmatrix} \right\} == \det\left\{ \mathbf{I} - \mathbf{N}_{y_d w_d}(s)\boldsymbol{\Delta}(s) \right\} = 0. \end{aligned} \tag{4.86}$$

The system is robustly stable when subject to the perturbations in (4.85) if and only if

$$\sup_{\omega}\left\{\mu_{\bar{\Delta}}\left[\mathbf{N}_{y_d w_d}(j\omega)\right]\right\} < 1, \tag{4.87}$$

which satisfies the first condition for robust performance (4.82).

The system with the performance block is robustly stable provided that

$$\sup_{\omega}\left\{\mu_{\bar{\Delta}_p}\left[\mathbf{N}(j\omega)\right]\right\} < 1. \tag{4.88}$$

A consequence of the definition of the structured singular value [see (4.72)] is that (4.88) is true if and only if

$$\det\left[\mathbf{I}-\mathbf{N}(j\omega)\mathbf{\Delta}_p(j\omega)\right] > 0, \tag{4.89}$$

for all frequencies and all admissible perturbations:

$$\mathbf{\Delta}_p(j\omega) \in \bar{\mathbf{\Delta}}_p \quad \text{such that} \quad \left\|\mathbf{\Delta}_p(j\omega)\right\|_\infty \le 1, \tag{4.90}$$

The determinant in (4.89) can be expanded:

$$\begin{aligned}
\det\left[\mathbf{I}-\mathbf{N}(j\omega)\mathbf{\Delta}_p(j\omega)\right] &= \det\begin{bmatrix} \mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta} & \vdots & -\mathbf{N}_{y_d w}\mathbf{\Delta}_0 \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ -\mathbf{N}_{y w_d}\mathbf{\Delta} & \vdots & \mathbf{I}-\mathbf{N}_{yw}\mathbf{\Delta}_0 \end{bmatrix} = \\
&= \det\left[\mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta}\right]\det\left\{\left[\mathbf{I}-\mathbf{N}_{yw}\mathbf{\Delta}_0\right]-\mathbf{N}_{yw_d}\mathbf{\Delta}\left[\mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta}\right]^{-1}\mathbf{N}_{y_d w}\mathbf{\Delta}_0\right\} = \\
&= \det(\mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta})\det\left[\mathbf{I}-\left\{\mathbf{N}_{yw}+\mathbf{N}_{yw_d}\mathbf{\Delta}(\mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta})^{-1}\mathbf{N}_{y_d w}\right\}\mathbf{\Delta}\right] > 0,
\end{aligned} \tag{4.91}$$

where the frequency designation has been dropped to simplify this expression. Since the system is robustly stable,

$$\det(\mathbf{I}-\mathbf{N}_{y_d w_d}\mathbf{\Delta}) > 0, \tag{4.92}$$

Equation (4.91) is then satisfied if and only if

$$\begin{aligned}
&\det\left[\mathbf{I}-\left\{\mathbf{N}_{yw}(j\omega)+\mathbf{N}_{yw_d}(j\omega)\mathbf{\Delta}(j\omega)\right.\right. \\
&\left.\left.\times(\mathbf{I}-\mathbf{N}_{y_d w_d}(j\omega)\mathbf{\Delta}(j\omega))^{-1}\mathbf{N}_{y_d w}(j\omega)\right\}\mathbf{\Delta}_0(j\omega)\right] > 0
\end{aligned} \tag{4.93}$$

for all frequencies and all admissible perturbations. The condition (4.93) holds both for all admissible $\mathbf{\Delta}$ and all $\mathbf{\Delta}_o$, such that

$$\left\|\mathbf{\Delta}_o(j\omega)\right\|_\infty \le 1, \tag{4.94}$$

Equation (4.93) is satisfied for all unstructured perturbations $\mathbf{\Delta}_o$, that satisfy (4.94) if and only if (by the small-gain theorem)

$$\left\|\mathbf{N}_{yw}(s)+\mathbf{N}_{yw_d}(s)\mathbf{\Delta}(s)\left[\mathbf{I}-\mathbf{N}_{y_d w_d}(s)\mathbf{\Delta}(s)\right]^{-1}\mathbf{N}_{y_d w}(s)\right\|_\infty < 1. \tag{4.95}$$

Applying the push-through theorem [see Appendix A3] to this result yields the condition for robust performance (4.83).

In summary, the structured singular value can be applied to evaluate robust performance. The only limitation on the applicability of this result is that the performance must be specified in terms of a bound on the closed-loop system ∞-norm. The following examples demonstrate the use of performance blocks and the SSV to evaluate robust performance [Zhou, Dolye-1998], [Zhou , Dolye, Glover-1995], [Burel-1999].

2.1 Examples

Example 4.1

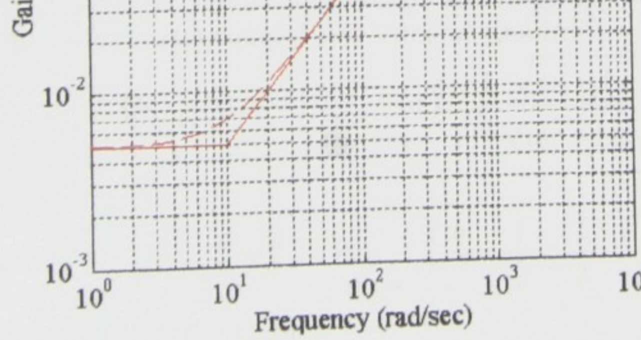
Consider the input multiplicative uncertainty model given in Figure 4.2b. The nominal plant model is fairly accurate (within about 0.5%) at frequencies below 10 rad/sec, but fairly inaccurate (within about 50%) at frequencies above 1000 rad/sec. The accuracy of the model should transition between these two extremes at intermediate frequencies. The frequency-dependent uncertainty bound is given in Figure 4.9a. This bound can be approximated by a first-order transfer function with a zero at 10 rad/sec and a pole at 1000 rad/sec:

$$\Delta'_{\max}(j\omega) = g\frac{(j\omega+10)}{(j\omega+1000)}$$

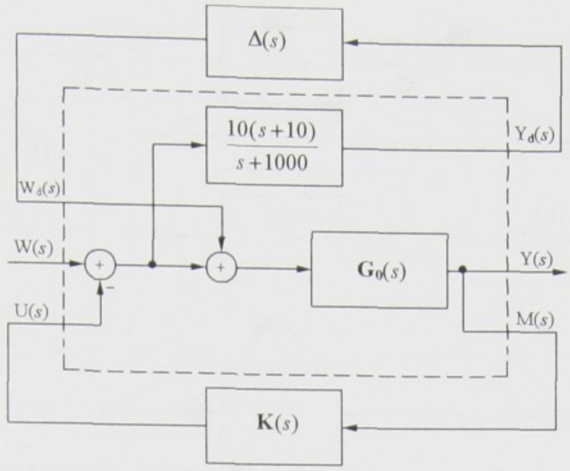
Note that the zero at 10 rad/sec generates an increase in the magnitude response with a slope of 20 dB/decade. This generates the desired rise in the uncertainty over the two-decade frequency range. The rise is then canceled by the pole at 1000 rad/sec. The transfer function gain is found by matching the bound at zero frequency:

$$g = 0.005\frac{(0+1000)}{(0+10)} = 0.5.$$

The magnitude response of the rational approximation to the uncertainty bound is plotted in Figure 4.9a. Assuming that this approximation is sufficiently accurate, the system can be placed in standard form by normalizing the perturbation and placing it in a feedback loop, as shown in Figure 4.9b.



(a)



(b)

FIGURE 2.3 The uncertainty model for example 4.1: (a) the bound on the perturbation; (b) the uncertainty model in standard form

Example 4.2

We are given the nominal plant

$$G_0(s) = \frac{10}{s-1}$$

with an unstructured additive uncertainty,

$$\|\Delta(s)\|_\infty \le 1$$

and a controller,

$$K(s) = \frac{1}{2}$$

The block diagram of this system in standard form is shown in Figure 4.10. Note that reference inputs and outputs are not specified for this example since they have no effect on robust stability analyses. From the block diagram, the transfer function from W_d to Y_d is computed with the loop closed through the controller:

$$N_{y_d w_d}(s) = \frac{-K(s)}{1+K(s)G_0(s)} = \frac{-\frac{1}{2}(s-1)}{s+4}$$

Note that the nominal system is stable. The maximum singular value of the scalar transfer function $N_{ydwd}(jw)$ equals the magnitude of this transfer function:

$$\bar{\sigma}\{N_{y_d w_d}(jw)\} = |N_{y_d w_d}(jw)| = \frac{1}{2}\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+16}}$$

The infinity norm is then

$$\|N_{y_d w_d}(jw)\|_\infty = \sup_\omega \frac{1}{2}\frac{\sqrt{\omega^2}+1}{\sqrt{\omega^2+16}} = \frac{1}{2}$$

Which implies the system is robustly stable.

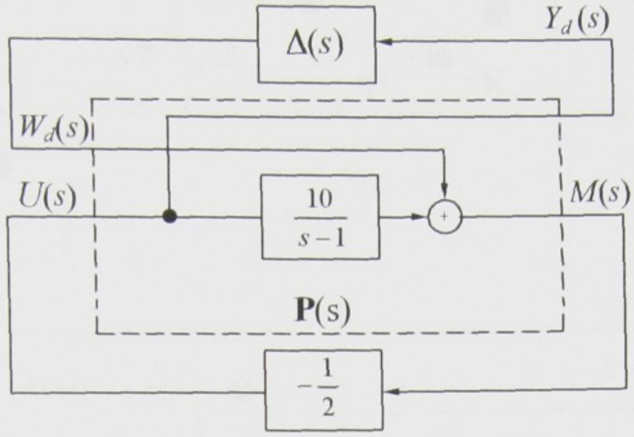


FIGURE 4.10 standard form for the system in Example 4.2

Example 4.3

The hover mode pitch and yaw dynamics of a vertical-takeoff-and-landing remotely piloted vehicle are given by the following state equations, Where these equations are partitioned as in (2.16) and (2.17). The control inputs are elevator deflection δ_q. The measured outputs are pitch angle p, yaw angle q, pitch rate, and yaw rate. For completeness, the disturbance inputs (commanded pitch angle p_c and commanded yaw angle q_c) and the reference outputs (pitch error and yaw error) are included in the model.

$$\begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1.6 \\ 0 & 0 & 0 & 1 \\ 0 & 1.8 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \vdots & 0 & 0 \\ 11 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ 0 & 12 & \vdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_p \\ \delta_q \\ \cdots \\ p_c \\ q_c \end{bmatrix} = \mathbf{A}_x + [\mathbf{B}_u \;\; \vdots \;\; 0] \begin{bmatrix} u \\ \cdots \\ w \end{bmatrix};$$

$$\begin{bmatrix} m \\ \cdots \\ y \end{bmatrix} = \begin{bmatrix} p \\ \dot{p} \\ q \\ \dot{q} \\ \cdots \\ e_p \\ e_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_p \\ \delta_q \\ \cdots \\ p_c \\ q_c \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{C}_m \\ \cdots \\ \mathbf{C}_y \end{bmatrix} x + \begin{bmatrix} 0 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \mathbf{D}_{yw} \end{bmatrix} \begin{bmatrix} u \\ \cdots \\ w \end{bmatrix}$$

The plant is subject to an unstructured input multiplicative uncertainty on the control:

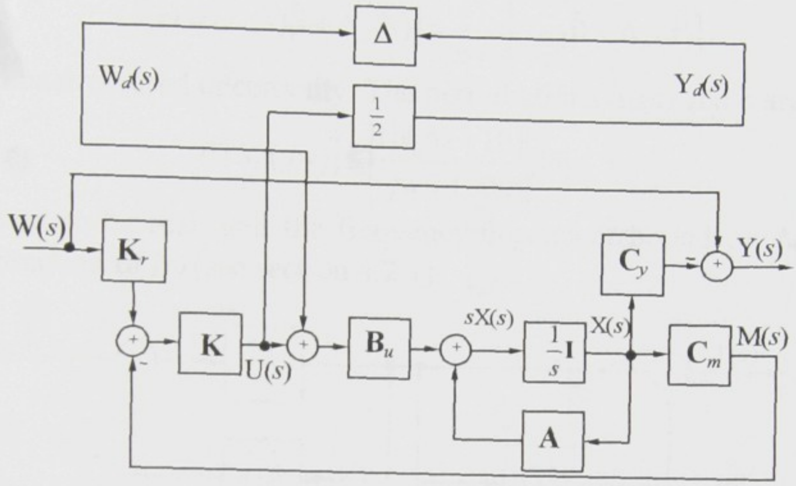
$$\|\Delta(s)\|_\infty \leq \frac{1}{2}$$

The controller is specified by the following equation:

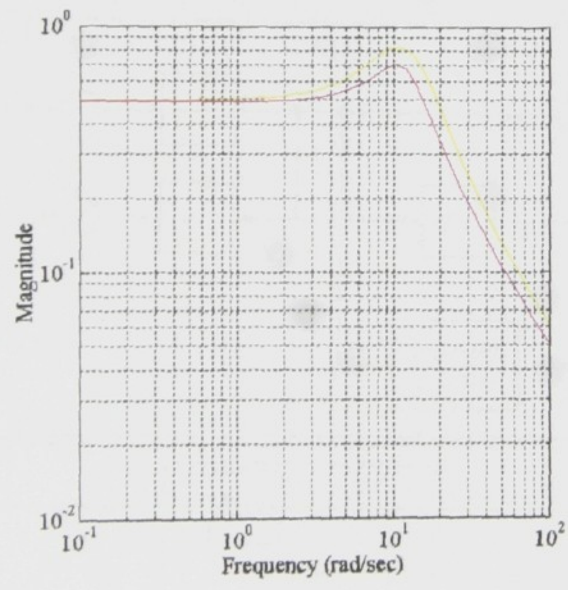
$$u(t) = -\mathbf{K}\mathbf{C}_m(x - \mathbf{K}_r w)$$

Where

$$\mathbf{K} = \begin{bmatrix} 15.92 & 1.08 & 0.91 & -0.18 \\ 1.00 & 0.25 & 10.76 & 0.84 \end{bmatrix}; \mathbf{K}_r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$



(a)



(a)

FIGURE. 4.11 Stability robustness analysis: (a) block diagram in standard form; (b) principal gains

Note that $\mathbf{C}_m$ is the identity matrix, so this is state feedback plus a feedforward control from the reference input. This controller is designed to place the poles at

$$\{-5+10j, -5-10j, -6+12j, -6-12j\}$$

And drive the pitch and yaw errors to zero. A block diagram of this system in standard form appears in Figure 4.11a. The transfer function from W_d to Y_d is found to be

$$\mathbf{N}_{y_d w_d}(s) = -\frac{1}{2}\mathbf{K}\mathbf{C}_m\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u.$$

The principal gains of this transfer function are computed numerically and plotted in Figure 4.11b. The infinity norm is the maximum value of the maximum principal gain,

$$\left\| \mathbf{N}_{y_d w_d}(s) \right\|_\infty = 0.84$$

Which implies that the system is robustly stable.

Example 4.4

Consider the transfer function below:

$$G(s) = G_0(s)[1+\Delta_0(s)] = \frac{1}{s+4+\delta}[1+\Delta_0(s)]$$

where $\Delta_0(s)$ is an unstructured uncertainty. The perturbations $\Delta_0(s)$ and δ are bounded:

$$\bar{\sigma}[\Delta_0(jw)] \le \left| \frac{10(jw+10)}{jw+1000} \right|; |\delta| \le 5$$

where δ is assumed to be real, and the frequency-dependent bound on $\Delta_0(s)$ is given as a rational transfer function of $j\omega$ (see section 4.2.1).

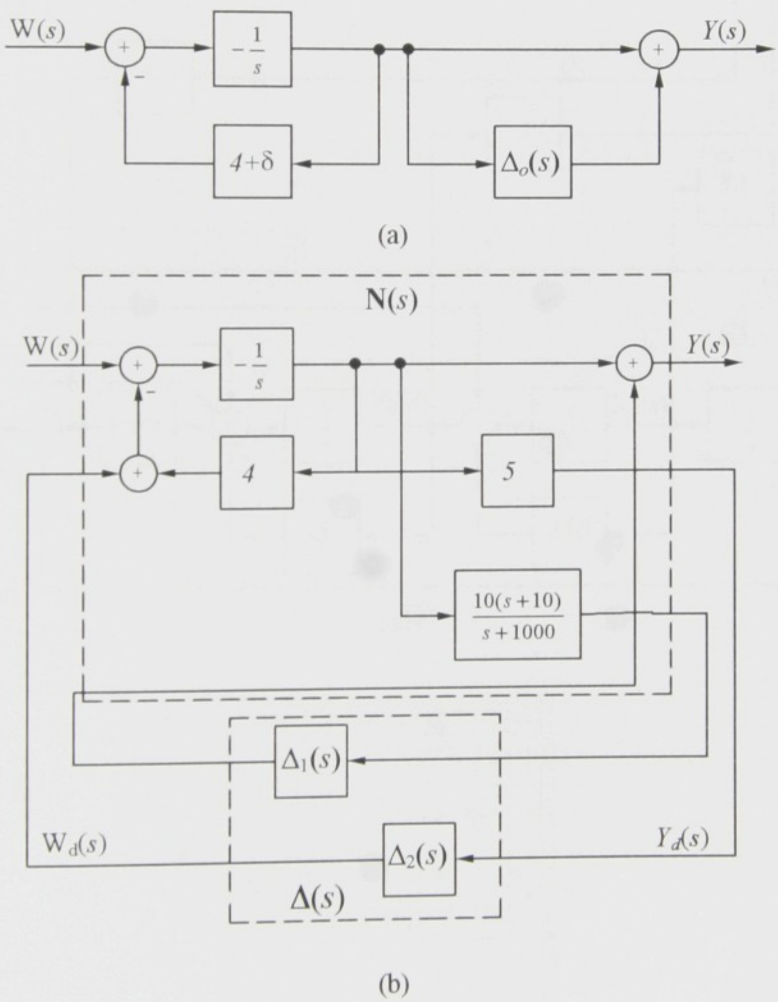


FIGURE 4.12 An example of modeling uncertainties as feedback perturbations: (a) the model with uncertainties; (b) the model with feedback perturbations

These perturbations can be placed in a feedback loop around the nominal plant, as shown in Figure 4.12, where

$$\Delta_0(s) = \frac{10(s+10)}{s+1000}\Delta_1(s); \delta = 5\Delta_2(s)$$

The resulting perturbation has the property that

$$\|\Delta_1\|_\infty \le 1; \|\Delta_2\|_\infty \le 1; \|\mathbf{\Delta}\|_\infty \le 1$$

$\Delta_1(s)$ is a complex perturbation, and $\Delta_2(s)$ is a real perturbation.

Example 4.5

Consider a motor with the following transfer function:

$$G(s) = \frac{g}{s(s+\tau_1)(s+\tau_2)}$$

where the forward path gain and each of the time constants are uncertain:

$$g \in [2.7\times10^6 \quad , \quad 3.3\times10^6] \; ; \; \tau_1 \in [50 \quad , \quad 60] \; ; \; \tau_2 \in [300 \quad , \quad 400]$$

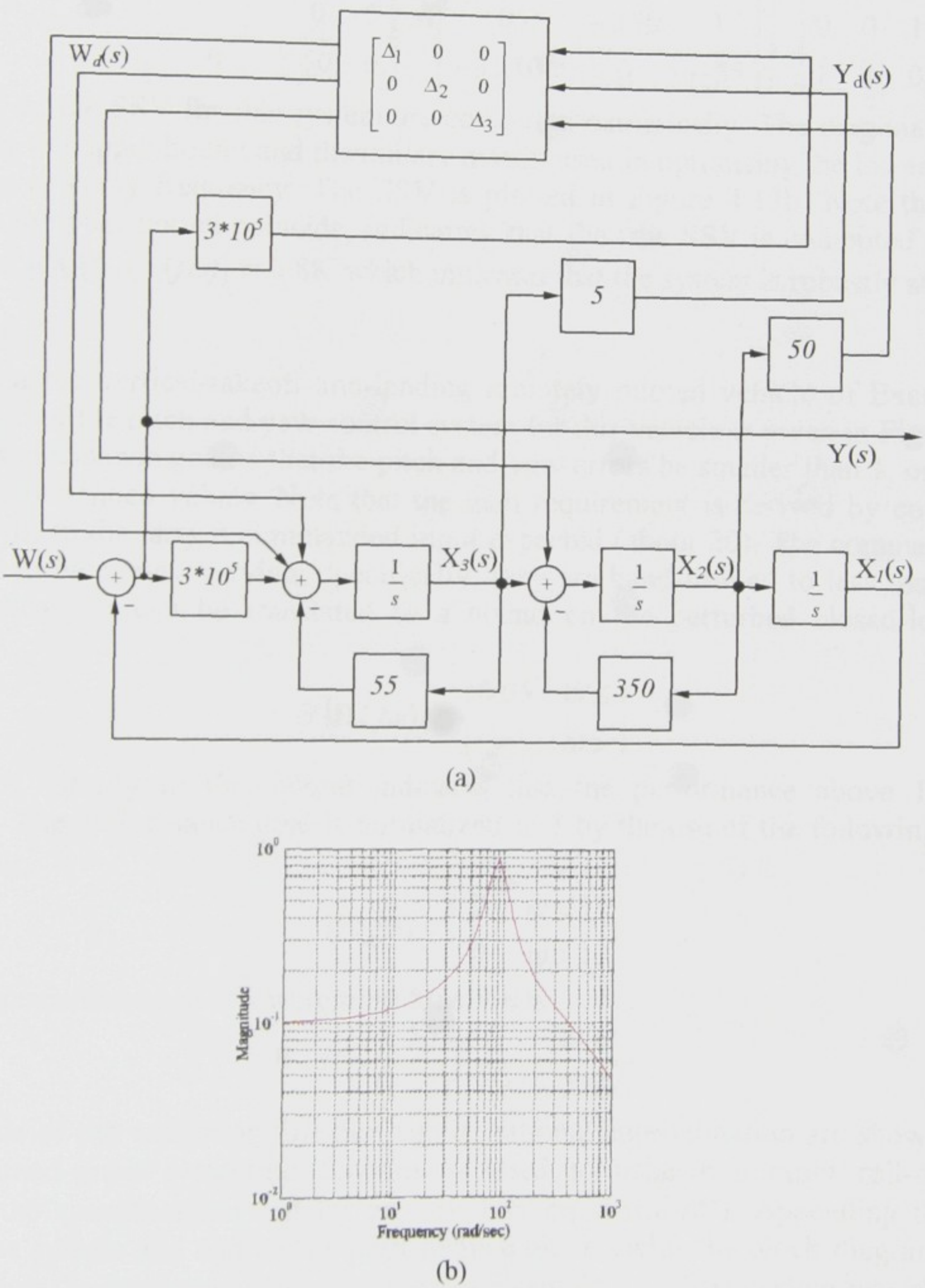


FIGURE 4.13 Stability robustness (a) block diagram; (b) The structured singular value

These parameters can be modelled as nominal values plus perturbations:

$$g = 3\times10^6 + 3\times10^5\Delta_1; \tau_1 = 55 + 5\Delta_2; \tau_2 = 350 + 50\Delta_3$$

where Δ_1,Δ_2, and Δ_3 are real perturbations bounded by 1. A unity feedback controller is designed for this plant:

$$K(s) = 1$$

A block diagram of the closed-loop system is given in Figure 4.13a. The closed-loop state model for this system is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -350 & 1 \\ -3\times10^6 & 0 & -55 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} w_d(t); y_d(t) = \begin{bmatrix} -3\times10^5 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 50 & 0 \end{bmatrix} x(t),$$

and the closed-loop transfer function from W_d to Y_d is

$$\mathbf{N}_{y_d w_d}(s) = \begin{bmatrix} -3\times10^5 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 50 & 0 \end{bmatrix} \left(s\mathbf{I} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -350 & 1 \\ -3\times10^6 & 0 & -55 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The bounds on the SSV for this system are computed numerically. The diagonal scales used in optimising the upper bound and the unitary matrix used in optimising the lower bound must be computer at every frequency. The SSV is plotted in Figure 4.13b. Note that the lower bound and the upper bound coincide, indicating that the true SSV is computed exactly. The supremum of $\mu_{\overline{\mathbf{\Delta}}}[\mathbf{N}_{ydwd}(j\omega)]$ is 0.88, which indicates that the system is robustly stable.

Example 4.6

We are given the vertical-takeoff-and-landing remotely piloted vehicle of Example 4.3. A block diagram of the pitch and yaw control system for this vehicle is given in Figure 4.11a. A reasonable performance goal is that the pitch and yaw errors be smaller than 1, or about 5 of the largest commanded values. Note that the gain requirement is derived by comparing the allowable error to the largest commanded input expected (about 20). The commanded values are assumed to be slowly varying; specifically, they are band-limited to less than 1 rad/sec. These specifications can be translated to a bound on the perturbed closed-loop transfer function:

$$\bar{\sigma}\{\mathbf{H}(jw)\} \le \begin{cases} 0.05 & \omega \le 1 \\ \infty & \omega > 1 \end{cases}$$

Note that the infinity in this bound indicates that the performance above 1 rad/sec is unimportant. The performance goal is normalized to 1 by the use of the following weighting function:

$$W(jw) = \begin{cases} 20 & \omega \le 1 \\ 0 & \omega > 1 \end{cases}$$

A rational approximation of this weighting function is

$$W(s) = \frac{20}{(s+1)^2} = \frac{20}{s^2 + 2s + 1}$$

The magnitude of the weighting function and its rational approximation are shown in Figure 4.14a. A second-order weighting function is used to achieve a rapid roll-off at high frequencies (the specifications call for an infinitely rapid roll-off). Appending this transfer function to the system and adding the performance block yields the block diagram in Figure 4.14b. Note that the feedback gains are given in Example 4.3. This system was shown to be robustly stable in Example 4.3, but no information was generated on performance. Nominal performance is evaluated by numerically computing the maximum singular value of the nominal closed-loop transfer function:

$$\mathbf{N}_{y_d w_d}(s) = W(s)\left\{\mathbf{I} - \mathbf{C}_y\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u\mathbf{K}\mathbf{K}_r\right\}$$

The principle gains for this transfer function are plotted in Figure 4.15.

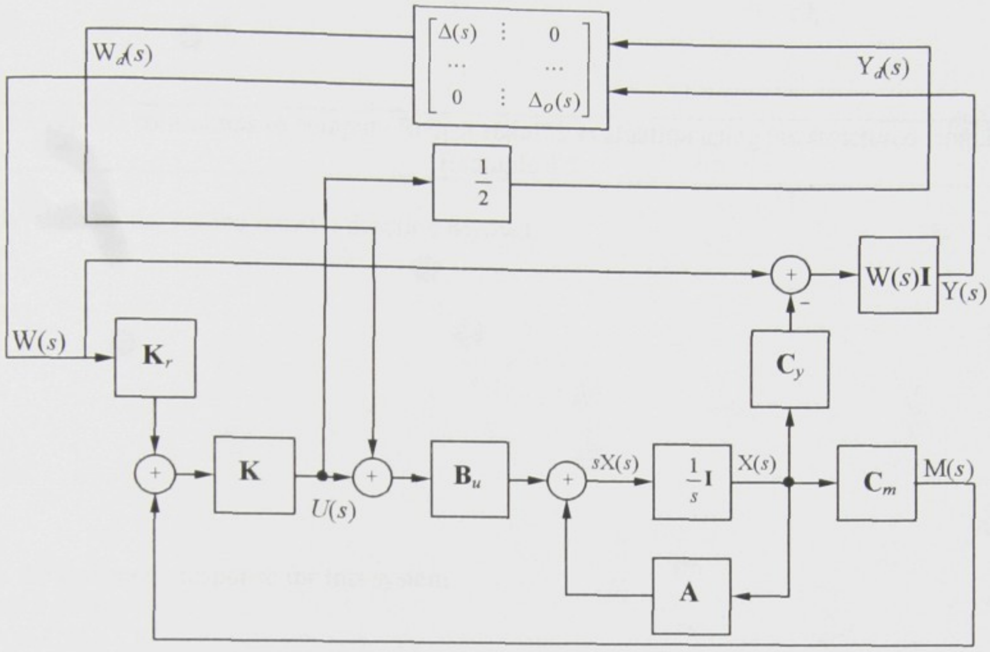


FIGURE 4.14 Robust performance analysis: block diagram of the system with weighting functions and the performance block

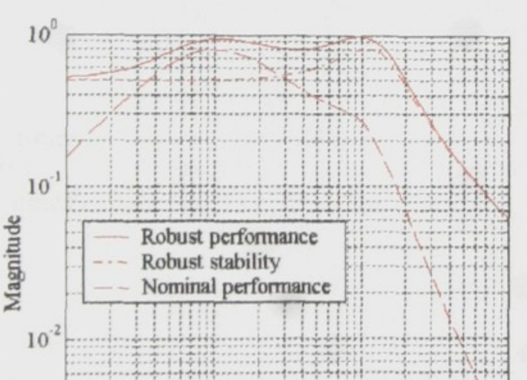


FIGURE 4.15 Robust performance analysis: The structured singular value of $\mathbf{N}(s)$

The infinity norm of the nominal system is 0.81, which indicates that the nominal system achieves the performance specifications. The total nominal closed-loop transfer function is

$$\mathbf{N}(s) = \begin{bmatrix} -\frac{1}{2}\mathbf{K}\mathbf{C}_m\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u & \vdots & -\frac{1}{2}\mathbf{K}\mathbf{C}_m\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u\mathbf{K}\mathbf{K}_r + \frac{1}{2}\mathbf{K}\mathbf{K}_r \\ \cdots & & \cdots \\ -W(s)\mathbf{C}_y\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u & \vdots & W(s)\left\{\mathbf{I} - \mathbf{C}_y\left(s\mathbf{I} - \mathbf{A} + \mathbf{B}_u\mathbf{K}\mathbf{C}_m\right)^{-1}\mathbf{B}_u\mathbf{K}\mathbf{K}_r\right\} \end{bmatrix}$$

Robust performance is evaluated by computing and plotting (see Figure 4.15) the structured singular value of $\mathbf{N}(s)$. The maximum structured singular value is 0.99, which indicates that the system is stable and meets the performance specifications for all admissible perturbations. For completeness, the maximum singular value of $\mathbf{N}_{y_d w_d}(s)$, which indicates robust stability, is also included in figure 4.15.

Table 4.1 MATLAB commands to compute Robust stability evaluation using the structured singular value for Example 4.5

```
clear,clc,
% Define the state model for the transfer function Mydwd.
A=[ 0  1  0
   0 -350  1
  -3e6  0  -55];
B=[0 0 0
  0 0 1
  1 1 0];
C=[-3e5 0  0
   0  0  5
   0 50 0];
D=zeros(3);
% Generate the frequency response for this system.
w=logspace(0,3,100);
M11=pck(A,B,C,D);
fr=frsp(M11,w);
% Compute the structured singular value from the frequency response.
%   Define the block structure.
blk=ones(3,2);
%   Compute the SSV.
muM=mu(fr,blk);
% Plot the results.
set(0,'DefaultAxesFontName','times')
set(0,'DefaultAxesFontSize',16)
set(0,'DefaultTextFontName','times')
figure(1)
clf
vplot('liv,lm',muM,'b.')
xlabel('Frequency (rad/sec)')
ylabel('Magnitude')
%grid
% Compute the maximum of the SSV.
maxmu=max(muM(:,1))
```

5 Controller Parameterization and Performance Specification

The optimization problems can also be posed using the system ∞-norm as a cost function. The ∞-norm is the worst-case gain of the system and therefore provides a good match to engineering specifications, which are typically given in terms of bounds on errors and controls.

The small-gain theorem states that, for unstructured perturbations, robust stability depends on the ∞-norm of the closed-lop system from the perturbation input to the perturbation output. The minimization of the closed-loop ∞-norm can, therefore also be used as a means of maximizing robustness.

Robust stability in the presence of structured perturbations and robust performance both depend on the supremum (over frequency) of the structured singular value from the perturbation input to the perturbation output. Note that a performance block is added to the perturbation (forming a structured perturbation) for robust performance analysis. The supremum of the SSV is bounded by the ∞-norm of the diagonal-scaled, closed-loop system. In fact, this bound is typically used in place of the SSV in robustness analysis. Therefore, it is reasonable to assume that minimization of the system ∞-norm plays a role in maximizing both robust stability (for structured perturbations) and robust performance (in general).

In summary, the optimization of the ∞-norm has applications in both maximizing performance and robustness. Control and estimation problems, with the goal of minimizing of the system ∞-norm, are termed H_∞ optimizations problems.[1] This chapter develops some basic results on the design of systems that minimize the system ∞-norm, in particular, full information control.

5.1 Differential Games

An H_∞ optimal controller minimizes the worst-case gain of the system. This problem can be thought of as a game with two participants: the designer, who is seeking a control that minimizes the gain; and nature, which is seeking a disturbance that maximizes the gain (the worst-case input). Games of this type are termed differential games when the dynamics of the game are described by differential equations. Some fundamental results from the field of differential game theory are presented in this section for application to H_∞ optimization problems.

A differential game is described by the game dynamics and the objective function.[2] The game dynamics are modeled by a generic state equation:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}_u u(t) + \mathbf{B}_w w(t), \tag{5.1}$$

where $u(t)$ is the control input, which is selected by the designer, and $w(t)$ is the disturbance input, which is selected by nature. The objective function is a real function (not necessarily positive definite) of the state, the control, and the disturbance:

$$J\{x(t), u(t), w(t)\}, \tag{5.2}$$

The solution of the deferential game consists of the optimal control trajectory $u^*(t)$ and the worst-case disturbance input $w^*(t)$. This solution is necessarily a saddle point of the objective function, which is defined by the following inequalities:

$$J\{x(u^*, w), u^*(t), w(t)\} \le J\{x(u^*, w^*), u^*(t), w^*(t)\} \le J\{x(u, w^*), u(t), w^*(t)\}. \tag{5.3}$$

The state is shown as a function of the inputs in this expression to emphasize that the state is completely specified by the inputs (provided the state initial condition is zero). A saddle point can also be defined as the argument of the mini-max problem:

$$\min_{u}[\max_{w} J\{x(u,v),u(t),w(t)\}] \tag{5.4}$$

Lagrange multipliers can be used to convert a constrained mini-max problem into an unconstrained mini-max problem of higher order. Appending the constraint equation (9.1) to the objective function yields the augmented objective function:

$$J_a(x,u,w,p) = J(x,u,w) + \int_0^{t_f} p^T(t)\{\mathbf{A}x(t) + \mathbf{B}_u u(t) + \mathbf{B}_w w(t) - \dot{x}(t)\}dt \tag{5.5}$$

A necessary condition for a saddle point is that the variation of this augmented objective function must be zero [Burel-1999].

5.2 Full Information Control

An optimal full information controller minimizes the infinity norm of the closed-loop system from the disturbance input to the reference output, assuming that all of the plant states and all of the disturbance inputs are available for feedback. In the stochastic regulator and in LQG control, the disturbance input is white noise. This disturbance input has an infinite variance, is totally unpredictable, and is not measurable. Using this input for feedback is then not practical. In H_∞ optimization problems, the disturbance input may be measured or predicted and is included as a possible feedback input. The term *full information* is used to reflect this change with respect to the state feedback controller, which is the solution of the linear quadratic regulator problem.

The H_∞ full information control problem is formally stated below. Let the plant be described by the following state model:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}_u u(t) + \mathbf{B}_w w(t) \tag{5.6a}$$

$$y(t) = \mathbf{C}_y x(t) + \mathbf{D}_{yu} u(t) \tag{5.6b}$$

where

$$\mathbf{D}_{yu}^T \mathbf{C}_y = \mathbf{0} \tag{5.7a}$$

$$\mathbf{D}_{yu}^T \mathbf{D}_{yu} = \mathbf{I} \tag{5.7b}$$

The plant is assumed to be observable from the reference output $y(t)$ and controllable from the control input $u(t)$. The zero in condition (5.7a) specifies that the output consists of two distinct components: linear combinations of the state and linear combinations of the control. This separation between the state and control is equivalent to the absence of cross terms between the state and control in the LQR cost function. The identity matrix in (5.7b) indicates that the coupling matrix between the control and the output is normalized; that is, the control is normalized so that all controls are equally weighted in the cost. This normalization is performed to simplify subsequent derivations, and so that the final results match those in the current literature. Normalization of this coupling matrix can always be accomplished provided $\mathbf{D}_{yu}$ has full rank.

The H_∞ full information control problem is to find a feedback controller, utilizing the state and the disturbance, that minimizes the closed-loop system ∞-norm:

$$J = \|\mathbf{G}_{yw}\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]} \neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} \tag{5.8}$$

A block diagram of the full information control system is given in Figure 9.1 where the controller is a linear system (not necessarily time-invariant) denoted $\mathbf{K}(\bullet)$. Note that defining the cost in terms of the ∞-norm implies that the initial state is zero:

$$x(0) = 0. \tag{5.9}$$

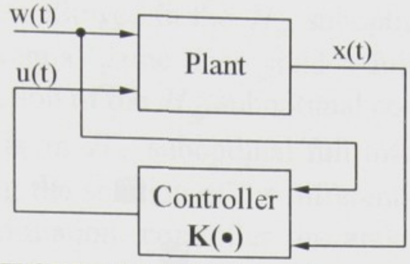


FIGURE 5.1 Full information control

The ∞-norm can be defined on either a finite or infinite ($t_f = \infty$) time interval. When the time interval is infinite, the controller must also internally stabilize the closed-loop system.

A suitable objective function is required in order to apply differential game theory to the solution of the full information control problem. The ∞-norm cost function (5.8) is not acceptable as an objective function, since this cost only depends on the controller; that is, the supremum makes this function independent of a particular disturbance input. Dropping the supremum from the cost function (5.8) yields a suitable objective function. While theoretically acceptable, the solution of the differential game with this objective function is intractable, in general.

A quadratic objective function, which yields tractable solutions of the deferential game, can be obtained by considering the bound on the closed-loop ∞-norm:

$$\left\|\mathbf{G}_{yw}\right\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} < \gamma \tag{5.10}$$

where γ is called the *performance bound.* A controller that satisfies this bound is called a suboptimal solution to the H_∞ full information control problem, or simply a suboptimal controller. The suboptimal controller must also satisfy the bound obtained by squaring (5.10):

$$\left\|\mathbf{G}_{yw}\right\|^2_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \left\{\frac{\|y(t)\|^2_{2,[0,t_f]}}{\|w(t)\|^2_{2,[0,t_f]}}\right\} < \gamma^2 \tag{5.11}$$

For the supremum to satisfy this strict inequality, the term within the curly brackets must be bounded away from γ^2; that is, for some ε,

$$\frac{\|y(t)\|^2_{2,[0,\ t_f]}}{\|w(t)\|^2_{2,[0,\ t_f]}} \le \gamma^2 - \varepsilon^2 \tag{5.12}$$

Multiplying both sides by the denominator, and grouping the resulting terms yields

$$\|y(t)\|^2_{2,[0,t_f]} - \gamma^2\|w(t)\|^2_{2,[0,t_f]} \le \varepsilon^2\|w(t)\|^2_{2,[0,t_f]} \tag{5.13}$$

Note that satisfaction of this inequality for all disturbance inputs and some e is equivalent to the bound on the closed-loop ∞-norm (5.10). The expression on the left of (5.13) can be used as an objective function:

$$J_\chi(x,u,w) = \|y(t)\|_2^2 - \gamma^2\|w(t)\|_2^2 \tag{5.14}$$

Differential game theory can then be applied to generate a control that minimizes this objective function in the presence of the worst-case disturbance. If the minimum of the objective function satisfies (5.13), then the bound on the closed-loop ∞-norm is achieved.

This differential game is posed as an open-loop mini-max problem; that is, the control input is not required to be generated by a feedback controller. Therefore, this problem allows more general control inputs than those allowed in the H_∞ suboptimal control problem. As will be seen, the solution of the deferential game does yield a full information feedback control, making it a valid candidate solution of the H_∞ suboptimal control problem. In fact, the saddle point of the differential game is an H_∞ suboptimal full information controller. This fact is demonstrated after first deriving the solution of the differential game, which is referred to as an H_∞ suboptimal full information controller in anticipation of demonstrating this equivalence.

General conditions can be given for the existence of H_∞ suboptimal controllers. The H_∞ optimal controller can then be approximated by decreasing the performance bound until a suboptimal controller no longer exists [Zhou, Dolye-1998], [Burel-1999].

5.2.1 The Hamiltonian Equations

The differential game specified by the objective function (5.14) and the game dynamics (5.7) yields a constrained mini-max problem. An unconstrained mini-max problem of higher dimension can be obtained by appending the constraint to the objective function:

$$\begin{aligned} J_{a\chi}(u,w,p) &= \int_0^{t_f} y^T(t)y(t) - \gamma^2 w^T(t) \\ &\quad + 2p^T(t)\{\mathbf{A}x(t) + \mathbf{B}_u u(t) + \mathbf{B}_w w(t) - \dot{x}(t)\}dt \end{aligned} \tag{5.15}$$

where the Lagrange multiplier term $p(t)$ is referred to as the *costate*. The factor of 2 multiplying the constraint equation is used to simplify the final results. Note that 2 times a constraint equation is also a constraint equation.

A necessary condition for a saddle point is that the variation of the augmented objective function equal zero. The increment of the augmented cost function is

$$\begin{aligned} &\Delta J_{a\gamma}(u,w,p,\delta,w,w,\delta,\,) = \\ &\Delta J_{a\gamma}(u+\delta u, w+\delta w, p+\delta p) - J_{a\gamma}(u,w,p) \\ &= \int_0^{t_f} (x+\delta x)^T \mathbf{C}_y^T \mathbf{C}_y (x+\delta x) + (u+\delta u)^T(u+\delta u) - \gamma^2 (w+\delta w)^T (w+\delta w) \\ &\quad + 2(p+\delta p)^T \{\mathbf{A}(x+\delta x) + \mathbf{B}_u(u+\delta u) + \mathbf{B}_w(w+\delta w) - (\dot{x}+\delta\dot{x})\}dt \\ &\quad - \int_0^{t_f} x^T \mathbf{C}_y^T \mathbf{C}_y x + u^T u - \gamma^2 w^T w + 2p^T\{\mathbf{A}x + \mathbf{B}_u u + \mathbf{B}_w w - \dot{x}\}dt \end{aligned} \tag{5.16}$$

where (5.6b) has been used to substitute for $y(t)$, the fact that $\mathbf{D}_{yu}^T \mathbf{D}_{yu} = \mathbf{I}$ has been used, and the time indexes have been omitted to simplify the notation. Expanding this expression and grouping terms, we have

$$\begin{aligned} \Delta J_{a\chi} &= \int_0^{t_f} \delta x^T \mathbf{C}_y^T \mathbf{C}_y \delta x + \delta u^T \delta u - \gamma^2 \delta w^T \delta w \\ &\quad + 2\delta p^T\{\mathbf{A}\delta x + \mathbf{B}_u \delta u + \mathbf{B}_w \delta w - \delta\dot{x}\} + 2u^T\delta u \\ &\quad + 2x^T \mathbf{C}_y^T \mathbf{C}_y \delta x - 2\gamma^2 w^T \delta w + 2\delta p^T\{\mathbf{A}x + \mathbf{B}_u u + \mathbf{B}_w w - \dot{x}\} \\ &\quad + 2p^T\{\mathbf{A}\delta x + \mathbf{B}_u \delta u + \mathbf{B}_w \delta w - \delta\dot{x}\}dt \end{aligned} \tag{5.17}$$

A necessary condition for the trajectory $x(t)$, $p(t)$, $u(t)$, and $w(t)$ to be a saddle point is that the variation of the objective function equal zero:

$$\delta J_{\alpha,\chi}(u,w,p,\delta u,\delta w,\delta p)=\int_0^{t_f} 2x^T\mathbf{C}_y^T\mathbf{C}_y\delta x+2u^T\delta u-2\gamma^2 w^T\delta w$$
$$+2\delta p^T\{\mathbf{A}x+\mathbf{B}_u u+\mathbf{B}_w w-\dot{x}\} \tag{5.18}$$
$$+2p^T\{\mathbf{A}\delta x+\mathbf{B}_u\delta u+\mathbf{B}_w\delta w-\delta\dot{x}\}dt=0$$

Integration by parts yields

$$\int_0^{t_f} p^T(t)\delta\dot{x}(t)dt=p^T(t_f)\delta x(t_f)-p^T(0)\delta x(0)-\int_0^{t_f}\dot{p}^T(t)\delta x(t)dt \tag{5.19}$$

which can be used to eliminate the variation of the state derivative in (5.18). Further, the state initial condition is fixed, so $\delta x(0)=0$. Substituting this result and (5.19) into (5.18), and regrouping terms yields the necessary condition for a saddle point:

$$\delta J_{\alpha\chi}(u,w,p,\delta u,\delta w,\delta p)=-2p^T(t_f)\delta\ x(t_f)+\int_0^{t_f}2x^T\mathbf{C}_y^T\mathbf{C}_y\delta\ x+2u^T\delta\ u-2\gamma^2w^T\delta\ w$$
$$+2\delta\ p^T\{\mathbf{A}x+\mathbf{B}_u u+\mathbf{B}_w w-\dot{x}\} \tag{5.20}$$
$$+2p^T\{\mathbf{A}\delta\ x+\mathbf{B}_u\delta\ u+\mathbf{B}_w\delta\ w-\delta\ \dot{x}\}dt=0.$$

Since the variations $\delta x(t_f)$, δx, δu, and δp are all arbitrary, this expression is only equal to zero if

$$\mathrm{p}(\mathrm{t_f})=0 \tag{5.21a}$$
$$\dot{p}(t)=-\mathbf{C}_y^T\mathbf{C}_y x(t)-\mathbf{A}^T p(t) \tag{5.21b}$$
$$u(t)=-\mathbf{B}_u^T p(t) \tag{5.21c}$$
$$w(t)=\gamma^{-2}\mathbf{B}_w^T p(t) \tag{5.21d}$$
$$\dot{x}(t)=\mathbf{A}x(t)+\mathbf{B}_u u(t)+\mathbf{B}_w w(t) \tag{5.21e}$$

Eliminating $u(t)$ and $w(t)$ from (5.21b) and (5.21e), and combining the resulting equations into a single state equation yields

$$\begin{bmatrix}\dot{x}(t)\\ \cdots\cdots\\ \dot{p}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \vdots & -\mathbf{B}_u\mathbf{B}_u^T+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\\ \cdots & & \cdots\cdots\cdots\\ -\mathbf{C}_y^T\mathbf{C}_y & \vdots & -\mathbf{A}^T\end{bmatrix}\begin{bmatrix}x(t)\\ \cdots\cdots\\ p(t)\end{bmatrix}=\mathcal{H}_\infty\begin{bmatrix}x(t)\\ \cdots\cdots\\ p(t)\end{bmatrix} \tag{5.22}$$

This is the Hamiltonian system for the full information H_∞ control problem, and the matrix $\mathcal{H}_\infty$ is called the Hamiltonian.

The H_∞ suboptimal control can be found by solving the Hamiltonian system subject to the final condition

$$p(t_f)=0. \tag{5.23}$$

The solution of the Hamiltonian system (5.22), at the final time, given an initial condition at time t, is

$$\begin{bmatrix}x(t_f)\\ \cdots\cdots\\ p(t_f)\end{bmatrix}=e^{\mathcal{H}_\infty(t_f-t)}\begin{bmatrix}x(t)\\ \cdots\cdots\\ p(t)\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{11}(t_f-t) & \vdots & \mathbf{\Phi}_{12}(t_f-t)\\ \cdots\cdots\cdots & & \cdots\cdots\cdots\\ \mathbf{\Phi}_{21}(t_f-t) & \vdots & \mathbf{\Phi}_{22}(t_f-t)\end{bmatrix}\cdot\begin{bmatrix}x(t)\\ \cdots\cdots\\ p(t)\end{bmatrix} \tag{5.24}$$

Substituting the final condition (5.21a) into this equation yields

$$\begin{bmatrix}x(t_f)\\ \cdots\cdots\\ 0\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{11}(t_f-t) & \vdots & \mathbf{\Phi}_{12}(t_f-t)\\ \cdots\cdots\cdots & & \cdots\cdots\cdots\\ \mathbf{\Phi}_{21}(t_f-t) & \vdots & \mathbf{\Phi}_{22}(t_f-t)\end{bmatrix}\cdot\begin{bmatrix}x(t)\\ \cdots\cdots\\ p(t)\end{bmatrix} \tag{5.25}$$

The lower block of this matrix equation can be solved for the costate in terms of the following state:

$$p(t) = -\{\mathbf{\Phi}_{22}(t_f - t)\}^{-1}\mathbf{\Phi}_{21}(t_f - t)x(t) = \mathbf{P}(t)x(t) \tag{5.26}$$

where $\mathbf{P}(t)$ is the matrix of proportionality between the costate and the state. This matrix of proportionality is fully specified by the state-transition matrix of the Hamiltonian system, provided the inverse in (5.26) exists at all times between the initial time and the final time.

The H_∞ suboptimal control is then found from (5.21c):

$$u(t) = -\mathbf{B}_u^T\mathbf{P}(t)x(t) = \mathbf{B}_u^T\{\mathbf{\Phi}_{22}(t_f - t)\}^{-1}\mathbf{\Phi}_{21}(t_f - t)x(t) = -\mathbf{K}(t)x(t) \tag{5.27}$$

where $\mathbf{K}(t)$ is termed the H_∞ suboptimal feedback gain matrix. While this control is generated from the necessary conditions for the solution of the differential game, it in fact specifies a solution of the deferential game (provided the inverse exists at all times). Further, this control is given by a linear, time-varying, state feedback law, making it a candidate solution of the H_∞ full information suboptimal control problem. Indeed, this feedback law is a solution of this suboptimal control problem, which will be showed that the matrix $\mathbf{P}(t)$ can be found as the solution of a Riccati equation.

An H_∞ suboptimal contrller fails over long time intervals, This gives a general property of H_∞ suboptimal control:

The feedback gain exists for arbitrary time intervals, and there is a solution to the H_∞ suboptimal control problem, only if the Hamiltonian matrix has no purely imaginary eigenvalues.

Note that this is an "only if" statement. The fact that the Hamiltonian matrix has no purely imaginary is not sufficient to guarantee existence of a solution to the steady-state H_∞ suboptimal control problem. Additional conditions for the existence of this steady-state control are presented in subsection 5.2.4 [Zhou , Dolye, Glover-1995], [Burel-1999].

5.2.2 The Riccati Equation

The Hamiltonian system for the H_∞ suboptimal control problem can also be related to a Riccati equation. This Riccati equation has only final conditions and can be solved backward in time using any numerical integration package.

The solution of the H_∞ suboptimal control problem can be reduced to finding the matrix $\mathbf{P}(t)$, since the feedback gain (5.27) only depends on this matrix and $\mathbf{B}_u$. A differential equation for $\mathbf{P}(t)$ can be generated by taking the derivative of (5.26):

$$\dot{p}(t) = \dot{\mathbf{P}}(t)x(t) + \mathbf{P}(t)\dot{x}(t) \tag{5.28}$$

Substituting for $\dot{\mathbf{P}}(t)$ and $\dot{x}(t)$, using the Hamiltonian system (5.22) yields

$$-\mathbf{C}_y^T\mathbf{C}_y x(t) - \mathbf{A}^T p(t) = \dot{\mathbf{P}}(t)x(t) + \mathbf{P}(t)\{\mathbf{A}x(t) - (\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)p(t)\} \tag{5.29}$$

Then, substituting for $p(t)$ using (5.26) and rearranging yields

$$\{\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) + \mathbf{C}_y^T\mathbf{C}_y - \mathbf{P}(t)\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}(t)\}x(t) = 0 \tag{5.30}$$

This equation is valid for any state $x(t)$; therefore, the matrix in curly brackets must equal zero:

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A} - \mathbf{A}^T\mathbf{P}(t) - \mathbf{C}_y^T\mathbf{C}_y + \mathbf{P}(t)(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}(t) \tag{5.31}$$

This equation is the Riccati differential equation for the H_∞ suboptimal control problem.

The Riccati solution $\mathbf{P}(t)$ is a symmetric matrix (if it exists), which can be found by solving (5.31) backward in time from the final condition:

$$\mathbf{P}(t_f) = \mathbf{0} \tag{5.32}$$

This final condition is obtained by letting $p(t_f) = 0$ in (5.26) and recognizing that the result is valid for any final state. The fact that the Riccati solution is symmetric is shown by noting that the derivative is symmetric whenever the Riccati solution is symmetric:

$$\begin{aligned} \dot{\mathbf{P}}^T(t) &= \{-\mathbf{P}(t)\mathbf{A} - \mathbf{A}^T\mathbf{P}(t) - \mathbf{C}_y^T\mathbf{C}_y + \mathbf{P}(t)(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}(t)\}^T \\ &= -\mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)\mathbf{A} - \mathbf{C}_y^T\mathbf{C}_y + \mathbf{P}(t)(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}(t) = \dot{\mathbf{P}}(t) \end{aligned} \tag{5.33}$$

Therefore, integrating this derivative backward in time from the final condition yields a symmetric Riccati solution [Burel-1999].

5.2.3 The Value of the Objective Function

A feedback controller and a worst-case disturbance result when applying Lagrange multipliers and variation theory to the differential game with dynamics and objective function. The feedback gain is found from the solution of either a Hamiltonian or a Riccati equation. All of these results are based on the variation of the objective function, which only specifies necessary conditions for the existence of a saddle point. In fact, the given control and worst-case disturbance form a saddle point of the differential game, and this control is an H_∞ suboptimal controller. These two results can be verified by cleverly rewriting the objective function.

To begin note that

$$\int_0^{t_f} \frac{dx^T(t)\mathbf{P}(t)x(t)}{dt}dt = x^T(t_f)\mathbf{P}(t_f)x(t_f) - x^T(0)\mathbf{P}(0)x(0) = 0, \tag{5.34}$$

Since $\mathbf{P}(t_f)$=0 and $x(0)$=0. This integral can be added to the objective function (to complete the square) without changing the value of this function:

$$\begin{aligned} J_\gamma &= \int_0^{t_f} x^T(t)\mathbf{C}_y^T\mathbf{C}_y x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) + \frac{dx^T(t)\mathbf{P}(t)x(t)}{dt}dt \\ &\int_0^{t_f} x^T(t)\mathbf{C}_y^T\mathbf{C}_y x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t) \\ &+ \dot{x}^T(t)\mathbf{P}x(t) + x^T(t)\dot{\mathbf{P}}(t)x(t) + x^T(t)\mathbf{P}(t)\dot{x}(t)dt. \end{aligned} \tag{5.35}$$

Substituting for the derivative of the state using (9,2a) and regrouping terms yields

$$\begin{aligned} J_\gamma &= \int_0^{t_f} x^T(\dot{\mathbf{P}} + \mathbf{C}_y^T\mathbf{C}_y + \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})x + u^T u - \gamma^2 w^T w + (\mathbf{B}_u u + \mathbf{B}_w w)^T\mathbf{P}x \\ &+ x^T\mathbf{P}(\mathbf{B}_u u + \mathbf{B}_w w)dt, \end{aligned} \tag{5.36}$$

where the time argument has been dropped to simplify the notation.

Substituting for $(\dot{\mathbf{P}} + \mathbf{C}_y^T\mathbf{C}_y + \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})$ Using the Riccati equation (5.31) gives

$$\begin{aligned} J_\gamma &= \int_0^{t_f} x^T\mathbf{P}(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}x + u^T u - \gamma^2 w^T w \\ &+ (\mathbf{B}_u u + \mathbf{B}_w w)^T\mathbf{P}x + x^T\mathbf{P}(\mathbf{B}_u u + \mathbf{B}_w w)dt. \end{aligned} \tag{5.37}$$

Regrouping again yields

$$\begin{aligned} J_\gamma &= \int_0^{t_f} (u + \mathbf{B}_u^T\mathbf{P}x)^T(u + \mathbf{B}_u^T\mathbf{P}x) - \gamma^2(w - \gamma^2\mathbf{B}_w^T\mathbf{P}x)^T(w - \gamma^2\mathbf{B}_w^T\mathbf{P}x)dt \\ &= \int_0^{t_f} \left\|u + \mathbf{B}_u^T\mathbf{P}x\right\|_E^2 - \gamma^2\left\|w - \gamma^2\mathbf{B}_w^T\mathbf{P}x\right\|_E^2 dt \\ &= \left\|u + \mathbf{B}_u^T\mathbf{P}x\right\|_{2,[0t_f]}^2 - \gamma^2\left\|w - \gamma^2\mathbf{B}_w^T\mathbf{P}x\right\|_{2,[0t_f]}^2, \end{aligned} \tag{5.38}$$

Where the norms in the final expression are signal norms. The objective function has a value of zero when the control and disturbance inputs are given by the necessary conditions:

$$u(t) = -\mathbf{B}_u^T \mathbf{P}(t)x(t); \tag{5.39}$$

$$w(t) = \gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t). \tag{5.40}$$

This point is a saddle point since any other disturbance input, with control (5.39), yields a decrease in the objective value, and any other control input, with disturbance (5.40), yields an increase in the objective value.

To verify that the controller is an H_∞ suboptimal controller, it must be shown (5.13), that

$$J_\gamma = \left\|u(t) + \mathbf{B}_u^T \mathbf{P}x(t)\right\|_{2,[0t_f]}^2 - \gamma^2 \left\|w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}x(t)\right\|_{2,[0t_f]}^2 \le -\varepsilon^2 \left\|w(t)\right\|_{2,[0t_f]}^2 \tag{5.41}$$

for some positive e. When using the controller reduees to

$$J_\gamma = -\gamma^2 \left\|w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}x(t)\right\|_{2,[0t_f]}^2 \le -\varepsilon^2 \left\|w(t)\right\|_{2,[0t_f]}^2 \tag{5.42}$$

Or equivalently,

$$\left\|w(t)\right\|_{2,[0t_f]}^2 \le \frac{\gamma^2}{\varepsilon^2} \left\|w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}x(t)\right\|_{2,[0t_f]}^2 \tag{5.43}$$

The fact that a positive e exists that satisfies this equation can be developed by noting that the disturbance input can be generated by a time-varying state model from the input $[w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t)]$:

$$\dot{x}(t) = [\mathbf{A} - \mathbf{B}_u \mathbf{B}_u^T \mathbf{P}(t) + \gamma^{-2}\mathbf{B}_w \mathbf{B}_w^T \mathbf{P}(t)]x(t) + \mathbf{B}_w[w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t)] \tag{5.44a}$$

$$w(t) = -\gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t) + [w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t)] \tag{5.44b}$$

Since all matrices in this state model are finite, the output is bounded:

$$\left\|w(t)\right\|_{2,[0,t_f]}^2 \le \left\|\mathbf{G}_{w\Delta w}\right\|_{2,[0,t_f]}^2 \left\|w(t) - \gamma^{-2}\mathbf{B}_w^T \mathbf{P}(t)x(t)\right\|_{2,[0,t_f]}^2, \tag{5.45}$$

Where $\mathbf{G}_{w\Delta w}$ denotes the state model (5.44). Comparing this result with (5.43), a positive ε that satisfies (5.43) is $\varepsilon = \gamma / \left\|\mathbf{G}_{w\Delta w}\right\|_{\infty,[0,t_f]}$ and the control law (5.39) is therefore a solution of the suboptimal H_∞ control problem.

The suboptimal control exists provided the Riccati equation (5.31) has a solution over the entire time interval from 0 to t_f. This control depends on the performance bound selected. Note that the Riccati equation (5.31) reduces to the LQR Riccati equation (5.44) when the performance bound approaches infinity. The LQR Riccati equation is guaranteed to have a solution, indicating that the H_∞ Riccati equation has a solution for sufficiently large performance bounds.

The control law (5.39) has been shown to satisfy the infinity norm bound (5.14), and to exist whenever the Riccati equation (5.31) has a solution. The folowing question immediately comes to mind: If this Riccati equation has no solution, is it possible to find a full information controller that satisfies the bound (5.14)? The answer to this question is no! The existence of a solution to the Riccati equation (5.31) is both necessary and sufficient for the existence of a solution to the H_∞ suboptimal control problem. Therefore, the H_∞ optimal solution can be approximated to an arbitrary degree of closeness by decreasing the bound until a Riccati solution no longer exists.

The H_∞ suboptimal controller presented is not unique. A family of controllers can be generated that all satisfy the bound (5.14), whenever a solution exists. The controller presented above is often referred to as the central, or minimum entropy, controller [Burel-1999].

5.2.4 Steady-State Full Information Control

The feedback gain in Example 5.1 approaches a steady-state value far from the final time. When operating over infinite time intervals, all finite times are infinitely far from the final time. The time-invariant controller that utilizes the steady-state gain is therefore a solution to the H_∞ suboptimal control problem when the time interval is infinite (provided this gain internally stabilizes the system). This time-invariant, full information control can also be utilized to simplify controller implementation when operating over finite time intervals. In this case, the steady-state controller may not satisfy the ∞-norm bound, but is probably reasonably close provided the time interval is long compared to the gain settling time. These observations, made concerning Example 5.1, can be applied to general H_∞ suboptimal control problems.

The existence of a steady-state solution to the H_∞ suboptimal control problem is not guaranteed for all values of the ∞-norm bound. A solution does exists for ∞-norm bounds sufficiently large, given that the plant is controllable from the control input and observable from the reference output (conditions included in the H_∞ suboptimal control problem specification). This fact can be deduced by noting that the H_∞ Riccati equation approaches the LQR Riccati equation as the performance bound approaches infinity. The above observability and controllability conditions are sufficient to guarantee the existence of a steady-state solution to the LQR and therefore also to guarantee the existence of a steady-state H_∞ suboptimal control for sufficiently large performance bounds.

For smaller bounds, the question immediately arises: Under what conditions does a steady-state solution to the H_∞ suboptimal control problem exist? The answer follows:

A suboptimal solution that internally stabilizes the closed-loop system and bounds the closed-loop ∞-norm (5.10) exists if and only if there exists a positive semidefinite solution of the algebraic Riccati equation,

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P} + \mathbf{C}_y^T\mathbf{C}_y = 0 \tag{5.46}$$

such that

$$\mathbf{A} - (\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P} \tag{5.47}$$

is stable, that is, all of the eigenvalues of this matrix have negative real parts. The suboptimal controller is then given as

$$u(t) = -\mathbf{B}_u^T\mathbf{P}x(t) = -\mathbf{K}x(t) \tag{5.48}$$

Note that condition (5.47) is included to guarantee the internal stability of the feedback system.

Equation (5.38) was used to show that the infinity norm is bounded as in (5.10), in the finite time case. The derivation of (5.38) requires the existence of the Riccati solution at all times during the interval of operation. While the above conditions are sufficient to guarantee the existence of the Riccati solution at all times, this fact is certainly not obvious. Simply talking the limit of (5.38) as the final time approaches infinity is therefore not sufficient to verify that the performance bound is achieved. An alternative method of demonstrating that (5.48) represents a solution of the H_∞ suboptimal control problem is presented below.

The fact that (5.48) is a solution of the H_∞ suboptimal control problem can be verified by forming the closed-loop state model, verifying that the closed-loop system is stable, and applying the bounded real lemma to verify that the gain satisfies the performance bound. The closed-loop state model is

$$\dot{x}(t) = (\mathbf{A} - \mathbf{B}_u\mathbf{B}_u^T\mathbf{P})x(t) + \mathbf{B}_w w(t) = \mathbf{A}_{cl}x(t) + \mathbf{B}_{cl}w(t) \tag{5.49}$$

$$y(t) = \begin{bmatrix} \mathbf{C}_y \\ \cdots\cdots\cdots \\ -\mathbf{D}\mathbf{B}_u^T\mathbf{P} \end{bmatrix} x(t) + \mathbf{0}w(t) = \mathbf{C}_{cl}x(t) + \mathbf{D}_{cl}w(t) \tag{5.50}$$

To show that this system is internally stable, add and subtract $(\mathbf{P}\mathbf{B}_u\mathbf{B}_u^T\mathbf{P})$ to the algebraic Riccati equation (5.46) and rearrange to give

$$\begin{aligned} &\mathbf{P}(\mathbf{A} - \mathbf{B}_u\mathbf{B}_u^T\mathbf{P}) + (\mathbf{A} - \mathbf{B}_u\mathbf{B}_u^T\mathbf{P})^T\mathbf{P} \\ &+\gamma^{-2}\mathbf{P}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P} + \mathbf{P}\mathbf{B}_u\mathbf{B}_u^T\mathbf{P} + \mathbf{C}_y^T\mathbf{C}_y = \mathbf{0} \end{aligned} \tag{5.51}$$

Note that this is a Lyapunov equation:

$$\mathbf{P}\mathbf{A}_{cl} + \mathbf{A}_{cl}^T\mathbf{P} + [\gamma^{-1}\mathbf{P}\mathbf{B}_w \vdots \mathbf{P}\mathbf{B}_u \vdots \mathbf{C}_y^T] \cdot \begin{bmatrix} \gamma^{-1}\mathbf{B}_w^T\mathbf{P} \\ \cdots\cdots\cdots\cdots \\ \mathbf{B}_u^T\mathbf{P} \\ \cdots\cdots\cdots\cdots \\ \mathbf{C}_y \end{bmatrix} = \mathbf{0} \tag{5.52}$$

The existence of a positive semidefinite solution $\mathbf{P}$ for this equation implies that

$$\lim_{t\to\infty} \begin{bmatrix} \gamma^{-1}\mathbf{B}_w^T\mathbf{P} \\ \cdots\cdots\cdots\cdots \\ \mathbf{B}_u^T\mathbf{P} \\ \cdots\cdots\cdots\cdots \\ \mathbf{C}_y \end{bmatrix} e^{\mathbf{A}_{cl}t} = \mathbf{0} \tag{5.53}$$

Since the plant is observable from $y(t)$ (the output associated with $\mathbf{C}_y$), and the controller contains no states, the closed-loop system is observable from $y(t)$. Therefore, the closed-loop system is internally stable, since any unstable modes, modes that do not decay to zero, will appear in the output.

The bounded real lemma is used to demonstrate that the closed-loop system ∞-norm satisfies the bound (5.10). This lemma states,

The infinity norm of the generic stable system (2.4) is bounded,

$$\|\mathbf{G}(s)\|_\infty < \gamma \tag{5.54}$$

if and only if

$$\bar{\sigma}(\mathbf{D}) < \gamma \tag{5.55}$$

and there exists a symmetric matrix $\mathbf{P}$ that satisfies the algebraic Riccati equation

$$\begin{aligned} &\mathbf{P}(\mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}) + (\mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C})^T\mathbf{P} \\ &+\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{C}^T(\mathbf{I} + \mathbf{D}\mathbf{R}^{-1}\mathbf{D}^T)\mathbf{C} = \mathbf{0} \end{aligned} \tag{5.56}$$

such that

$$\mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{5.57}$$

is stable, that is, has all eigenvalues with negative real parts, where $\mathbf{R} = \gamma^2\mathbf{I} + \mathbf{D}^T\mathbf{D}$.

To apply this result, it must be shown that the closed-loop state model satisfies the conditions given in (5.55) and (5.56) and (5.57).

Condition (5.55) requires that the input-to-output coupling matrix be bounded:

$$\bar{\sigma}(\mathbf{D}_{cl}) = \bar{\sigma}(\mathbf{0}) = 0 < \gamma \tag{5.58}$$

Further, conditions (5.46) and (5.47) are equivalent to conditions (5.56) and (5.57) when applied to the closed-loop system. To see this, substitute the matrices from the closed-loop state model into (5.56):

$$\begin{gathered}\mathbf{P}(\mathbf{A}-\mathbf{B}_u\mathbf{B}_u^T\mathbf{P})+(\mathbf{A}-\mathbf{B}_u\mathbf{B}_u^T\mathbf{P})^T\mathbf{P} \\ +\gamma^{-2}\mathbf{P}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}+[\mathbf{C}^T \vdots -\mathbf{P}\mathbf{B}_u\mathbf{D}^T]\begin{bmatrix}\mathbf{C} \\ \cdots\cdots\cdots \\ -\mathbf{D}\mathbf{B}_u^T\mathbf{P}\end{bmatrix}=\mathbf{0}\end{gathered} \tag{5.59}$$

Expanding the last term in this equation and remembering that $\mathbf{D}^T\mathbf{D}=\mathbf{I}$ yields (5.51), which is equivalent to (5.46), as demonstrated above. For the closed-loop system, the matrix in (5.57) reduces to (5.47):

$$\mathbf{A}_{cl}+\mathbf{B}_{cl}\mathbf{R}^{-1}\mathbf{D}_{cl}^T\mathbf{C}_{cl}+\mathbf{B}_{cl}\mathbf{R}^{-1}\mathbf{B}_{cl}^T\mathbf{P}=\mathbf{A}_{cl}-\mathbf{B}_u\mathbf{B}_u^T\mathbf{P}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P} \tag{5.60}$$

which is required to be stable. Summarizing, for the closed-loop system, (5.55) is satisfied, and there exists a solution to (5.56) such that (5.57) is stable. Together these results imply that the closed-loop system ∞-norm is bounded as given in (5.54) and equivalently in (5.10). Since the closed-loop system is stable, this implies that the control (5.48) is an H_∞ suboptimal controller.

The above presentation does not demonstrate that the conditions relating to (5.46) and (5.47) are necessary for the existence of an H_∞ suboptimal controller. But in fact, these conditions are both necessary and sufficient for the existence of such a controller. [Zhou, Dolye-1998], [Burel-1999].

5.2.5 Computation of the Steady-State H_∞ Full Information Control

The steady-state solution of the algebraic Riccati equation can be found from the eigensystem decomposition of the Hamiltonian matrix:

$$\mathbf{P}=\mathbf{\Psi}_{21}(\mathbf{\Psi}_{11})^{-1} \tag{5.61}$$

Where

$$\begin{bmatrix}\mathbf{\Psi}_{11} \\ \cdots\cdots \\ \mathbf{\Psi}_{21}\end{bmatrix} \tag{5.62}$$

is a matrix whose columns are the eigenvectors of the Hamiltonian associated with the stable eigenvalues. The derivation of this result is analogous to that presented for the LQR and is therefore omitted. The steady-state Riccati solution is then used to generate the steady-state full information controller as given in (5.48).

5.2.6 Existence Results in Terms of the Hamiltonian

The existence of an H_∞ suboptimal control can also be specified in terms of the Hamiltonian matrix:

A suboptimal solution that stabilizes the system and bounds the ∞-norm of the closed-loop transfer function (5.10) exists if and only if the Hamiltonian matrix,

$$\mathcal{H}_\infty=\begin{bmatrix}\mathbf{A} & \vdots & -\mathbf{B}_u\mathbf{B}_u^T+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\mathbf{C}_y^T\mathbf{C}_y & \vdots & -\mathbf{A}^T\end{bmatrix} \tag{5.63}$$

has no eigenvalues on the imaginary axis, $\mathbf{\Psi}_{11}$ is invertible, and

$$\mathbf{P} = \mathbf{\Psi}_{21}(\mathbf{\Psi}_{11})^{-1} \geq 0 \tag{5.64}$$

The suboptimal controller is then given as

$$u(t) = -\mathbf{B}_u^T \mathbf{\Psi}_{21}(\mathbf{\Psi}_{11})^{-1} x(t) = -\mathbf{K}x(t) \tag{5.65}$$

The fact that no H_∞ suboptimal control exists when the Hamiltonian has eigenvalues on the imaginary axis can be intuitively understood by considering (5.27). Purely imaginary eigenvalues give rise to undamped oscillations in the state-transition matrix of the Hamiltonian. These oscillations, in turn, yield oscillations in the time-varying Riccati solution, thus guaranteeing that a limit does not exist.

The above result is simply a restatement of the previous existence result for the H_∞ suboptimal control problem based on the relationship between the Hamiltonian and the steady-state Riccati solution. This formulation of the existence result is presented for completeness and to aid in understanding related results in the literature [Dolye, Francis, Tannenbaum-1990],[Burel-1999].

5.2.7 Generalizations

Several assumptions have been made in the development of the full information control to simplify the mathematics. Specifically, the control was assumed to be normalized so that the input-to-output coupling matrix between the control and the reference output satisfied

$$\mathbf{D}_{yu}^T \mathbf{D}_{yu} = \mathbf{I} \tag{5.66}$$

This condition can be achieved by proper definition (normalization) of the control input. Alternatively, the H_∞ suboptimal controller formulas could be modified to incorporate the use of a non-normalized control input. Most computer-aided design packages that allow the generation of H_∞ suboptimal controllers do not require normalization of the control. In particular, normalization is not required when using the μ-Analysis and Synthesis Toolbox of MATLAB.

The assumption that

$$\mathbf{D}_{yu}^T \mathbf{C}_y = \mathbf{0} \tag{5.67}$$

is also made in the derivation of the H_∞ suboptimal controller formulas. This assumption specifies that there are no cross terms between the control and the state in the cost function. These cross terms can be incorporated into the problem statement, yielding an increase in the complexity of the H_∞ suboptimal controller formulas. A software for finding the associated controllers can be found in the μ-Analysis and Synthesis Toolbox of MATLAB.

Terminal state weighting can be employed in the differential game used to generate the H_∞ full information control:

$$J_\chi(x,u,w) = \|y\|_2^2 - \gamma^2 \|w\|_2^2 + x^T(t_f)\mathbf{H}x(t_f) \tag{5.68}$$

The resulting controller (if it exists) is still an H_∞ full information suboptimal controller. This controller can be found by solving the Riccati equation with the final condition

$$\mathbf{P}(t_f) = \mathbf{H} \tag{5.69}$$

and using this Riccati solution to find the feedback gain. Note that the final condition can influence the existence of a Riccati equation solution, in general. The use of final state weighting has a similar effect on the H_∞ full information controller as on the LQR; that is, it forces the final state closer to zero.

The H_∞, suboptimal full information controller is not unique. The set of all controllers that satisfy the bound can be constructed by adding terms to the given state feedback controller. A second optimization may then be performed, if desired, over this set to yield controllers that have additional properties, for example, robustness. [Zhou, Dolye-1998], [Burel-1999].

5.3 H_∞ Estimation

The linear quadratic Gaussian control is generated by an optimal state feedback control law operating on estimates of the state. The states, in this case, are estimated using the Kalman filter. A similar structure exists for the H_∞ output feedback controller, as presented in the next chapter. Before discussing this structure in detail, the H_∞ estimation problem is posed and solved.

Two fundamental differences exist between the Kalman filter and the H_∞ optimal estimator (or filter). First, the H_∞ filter is optimal in terms of minimizing the ∞-norm of the gain between a set of disturbance inputs, and the estimation error. This performance criteria specifies that the worst-case gain be minimized. In contrast, the Kalman filter minimizes the mean square estimation error, or equivalently, minimizes the mean square gain between the disturbances and the estimation error.

The second difference stems from the fact that minimum mean square estimation commutes with linear operations; that is, the minimum mean square estimate of any linear combination of the state is simply the same linear combination of the optimal state estimate. The Kalman filter, which provides optimal estimates of the state, can therefore also be used to estimate any linear combination of the state. Minimal ∞-norm estimation does not possess this property, and the optimal H_∞ estimator depends on the plant output being estimated.

The specification of the H_∞ estimation problem requires a model of the plant and a cost function. The plant is modeled as follows:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}_u u(t) + \mathbf{B}_w w(t) \tag{5.70a}$$

$$m(t) = \mathbf{C}_m x(t) + \mathbf{D}_{mw} w(t) \tag{5.70b}$$

where $u(t)$ is a known input, and $w(t)$ is an unknown (but not necessarily random) disturbance input. The plant is assumed to be observable from the measurement and controllable from the disturbance input. The matrices $\mathbf{B}_w$ and $\mathbf{D}_{mu}$, are assumed to satisfy the following conditions:

$$\mathbf{D}_{mw}\mathbf{B}_w^T = \mathbf{0} \tag{5.71a}$$

$$\mathbf{D}_{mw}\mathbf{B}_{mw}^T = \mathbf{I} \tag{5.71b}$$

Condition (5.71a) specifies that the disturbances entering the plant via the state equation (similar to plant noise) and the disturbances entering the plant via the measurement equation (similar to measurement noise) must be distinct. This condition is akin to requiring the measurement and plant noises be independent in the Kalman filter setting. Condition (5.71b) specifies that the output equation must be scaled to normalize the input-to-output coupling matrix between the disturbance and the measurement. This normalization can always be accomplished provided $\mathbf{D}_{mw}$ has full rank. Assuming that the output equation is normalized is not necessary to the theory, but simplifies the subsequent derivations.

The $\mathbf{D}_m$ filter estimates linear combinations of the state

$$y(t) = \mathbf{C}_y x(t) \tag{5.72}$$

given the measured output of the plant. An optimal H_∞ estimator generates estimates that minimize the worst-case gain between the disturbance input and the estimation error

$e(t) = y(t) - \hat{y}(t)$

$$J = \left\| \mathbf{G}_{ew} \right\|_{\infty[0,t_f]} = \sup_{w \neq 0} \frac{\left\| y - \hat{y} \right\|_{2,[0,t_f]}}{\left\| w \right\|_{2,[0,t_f]}} \tag{5.73}$$

This infinity norm can be defined over either a finite or an infinite time interval. The estimator is required to be stable when operating over an infinite time interval. The plant and estimator are shown in Figure 5.2, where the estimator is a linear system (not necessarily time-invariant) denoted $\zeta(\bullet)$.

The H_∞ estimation problem is solved by utilizing the duality between estimation and control. Duality was first presented to explain the similarity between the equations for the linear quadratic regulator and the Kalman filter. The following section provides a more detailed treatment of duality based on the adjoint system[Dolye, Francis, Tannenbaum-1990],[Burel-1999].

5.3.1 The Adjoint System

The duality between the control and estimation problems can be explained by noting that these problems are related via the adjoint operation. The adjoint system is a modification of another system that has the same ∞-norm and 2-norm as the original system.

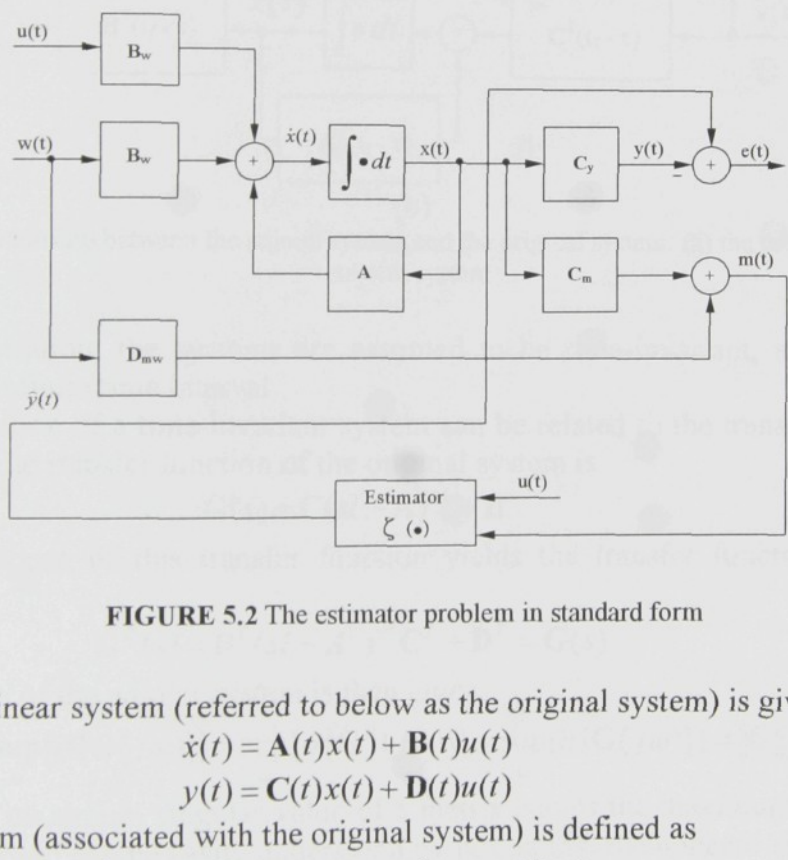


FIGURE 5.2 The estimator problem in standard form

A time-varying linear system (referred to below as the original system) is given:

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) \tag{5.74a}$$

$$y(t) = \mathbf{C}(t)x(t) + \mathbf{D}(t)u(t) \tag{5.74b}$$

The adjoint system (associated with the original system) is defined as

$$\dot{\tilde{x}}(\tau) = \mathbf{A}^T(t_f - \tau)\tilde{x}(\tau)\mathbf{C}^T(t_f - \tau)\tilde{y}(\tau) \tag{5.75a}$$

$$\tilde{u}(\tau) = \mathbf{B}^T(t_f - \tau)\tilde{x}(\tau)\mathbf{D}^T(t_f - \tau)\tilde{y}(\tau) \tag{5.75b}$$

Note that the input, output, and state of the adjoint system are distinct from the input, output, and state of the original system. Block diagrams of the original system and the adjoint system are shown in Figure 5.3. Comparing these block diagrams, note that the adjoint system is the "reverse" of the original system. In general, the adjoint of a collection of subsystems is the adjoint of each subsystem connected in reverse order.

Two important properties of the adjoint system are that it has the same 2-norm and the same ∞-norm as the original system. A simpler demonstration of the equivalence between the ∞-norms of the original system and the adjoint system is given below.

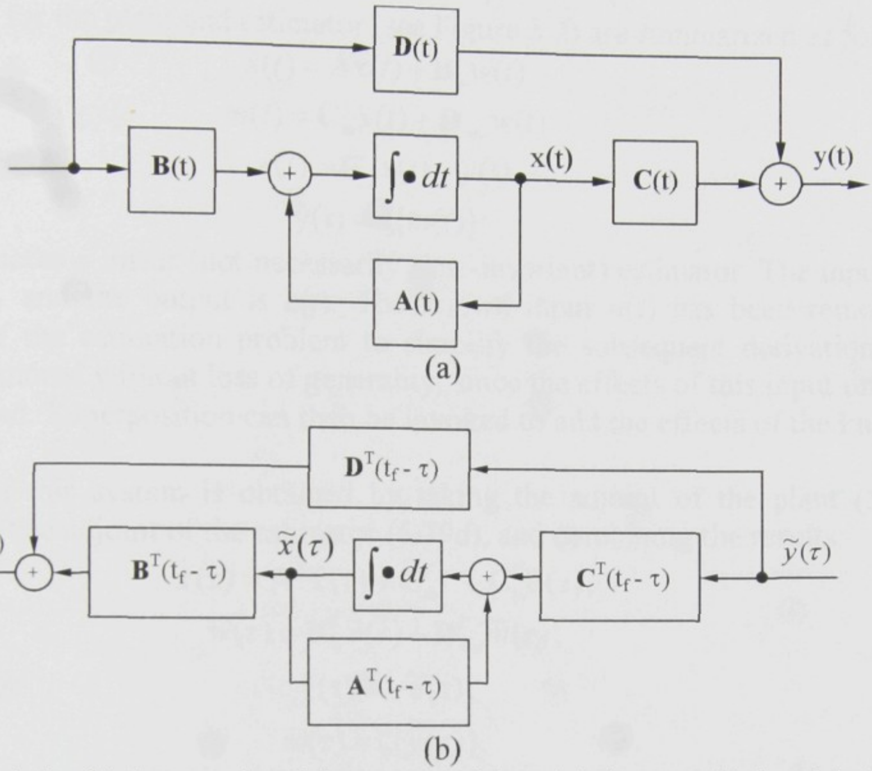


FIGURE 5.3 Relationship between the adjoint system and the original system: (a) the original system; (b) the adjoint system

For this demonstration, the systems are assumed to be time-invariant, and the ∞-norm is defined over an infinite time interval.

The transfer function of a time-invariant system can be related to the transfer function of the adjoint system. The transfer function of the original system is

$$\mathbf{G}(s) = \mathbf{C}(sI - \mathbf{A})^{-1} + \mathbf{D} \tag{5.76}$$

Taking the transpose of this transfer function yields the transfer function of the adjoint system:

$$\mathbf{G}^T(s) = \mathbf{B}^T(sI - A^T)^{-1}\mathbf{C}^T + \mathbf{D}^T = \widetilde{\mathbf{G}}(s) \tag{5.77}$$

The infinity norm of the adjoint system is then given:

$$\left\|\widetilde{\mathbf{G}}\right\|_\infty = \sup_\omega\{\overline{\sigma}[\widetilde{\mathbf{G}}(j\omega)]\} = \sup_\omega\{\overline{\sigma}[\mathbf{G}^T(j\omega)]\} = \sup_\omega\{\sigma[\mathbf{G}(j\omega)]\} = \left\|\mathbf{G}\right\|_\infty \tag{5.78}$$

The fact that the maximum singular value of a matrix equals the maximum singular value of the matrix transposed can be easily understood by noting that the nonzero singular values of a matrix $\mathbf{M}$ are the nonzero eigenvalues of $\mathbf{MM}^T$, which equal the nonzero eigenvalues of $\mathbf{M}^T\mathbf{M}$ [Zhou , Dolye, Glover-1995],[Burel-1999].

5.3.2 Finite-Time Estimation

The H_∞ estimation problem is solved by using the adjoint to convert the estimation problem into an equivalent control problem. The full information results generated in the previous section can be used to solve this equivalent control problem. The H_∞ estimator is then obtained by taking the adjoint of the resulting controller.

Paralleling the full information results, suboptimal estimators are generated that yield a given bound on the infinity norm from the disturbance input to the estimation error. General conditions can be given for the existence of these suboptimal estimators. The optimal estimator can then be approximated to an arbitrary degree of closeness by decreasing the bound until a solution no longer exists.

The equations for the plant and estimator (see Figure 5.2) are summarized as follows:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}_w w(t) \tag{5.79a}$$

$$m(t) = \mathbf{C}_m x(t) + \mathbf{D}_{mw} w(t) \tag{5.79b}$$

$$e(t) = \mathbf{C}_y x(t) - \hat{y}(t) \tag{5.79c}$$

$$\hat{y}(t) = \zeta\{m(t)\} \tag{5.79d}$$

where $\zeta(\bullet)$ denotes a linear (not necessarily time-invariant) estimator. The input to the model (5.79) is $w(t)$, and the output is $e(t)$. The known input $u(t)$ has been removed from this formulation of the estimation problem to simplify the subsequent derivations. The known input can be ignored without loss of generality, since the effects of this input on the output are easily computed. Superposition can then be invoked to add the effects of the known input into the estimator.

The adjoint of this system is obtained by taking the adjoint of the plant (5.79a) through (5.79c), taking the adjoint of the estimator (5.79d), and combining the results:

$$\dot{\widetilde{x}}(\tau) = \mathbf{A}^T \widetilde{x}(\tau) + \mathbf{C}_m^T \quad + \mathbf{C}_m^T \widetilde{e}(\tau); \tag{5.80a}$$

$$\widetilde{w}(\tau) = \mathbf{B}_w^T \widetilde{x}(\tau) + \mathbf{D}_{mw}^T \widetilde{m}(\tau); \tag{5.80b}$$

$$\dot{\hat{y}}(\tau) = -\widetilde{e}(\tau); \tag{5.80c}$$

$$\widetilde{m}(\tau) = \zeta\{\widetilde{\hat{y}}(\tau)\}. \tag{5.80d}$$

A block diagram of the adjoint system is shown in Figure 5.4.

The system in Figure 5.4 is a control system. For this system, the controller $\zeta\{\bullet\}$ can be selected to bound the cost:

$$J = \left\|\mathbf{G}_{\widetilde{w}\widetilde{e}}\right\|_{\infty,[0,t_f]} = \sup_{\|\widetilde{e}\|_{2[0,t_f]} \neq 0} \frac{\|\widetilde{w}\|_{2,[0,t_f]}}{\|\widetilde{e}\|_{2,[0,t_f]}} < \gamma \tag{5.81}$$

This suboptimal control problem is very similar to the full information control problem. Note that the plant model (5.80a) and (5.80b) has exactly the form of (5.6a) and (5.6b) when $\widetilde{m}(\tau)$ is the "control input." Further, the conditions (5.65) and (5.65) are equivalent to (5.7a) and (5.7b) for the adjoint plant:

$$(\mathbf{D}_{mw}^T)^T \mathbf{B}_w^T = \mathbf{0} \tag{5.82}$$

$$(\mathbf{D}_{mw}^T)^T \mathbf{B}_{mw}^T = \mathbf{I} \tag{5.83}$$

Lastly, the initial condition for the adjoint system can be assumed to equal zero, since the ∞-norm only depends on the forced response.

The control problem (5.80) and (5.81) differs from the full information control problem only in the fact that the measurement available for feedback (5.80c) is the disturbance input. In contrast, both the disturbance input and the state are assumed to be available in the full information control problem. A controller that utilizes only the disturbance input is referred to as a disturbance feedforward controller.

The existence of an H_∞ full information suboptimal control for the plant (5.80) is necessary for the existence of a solution to the disturbance feedforward control problem. This observation is based on the fact that a disturbance feedforward controller is a special case of a full information controller. Less obvious is the fact that the full information control problem is mathematically equivalent to the disturbance feedforward control problem.

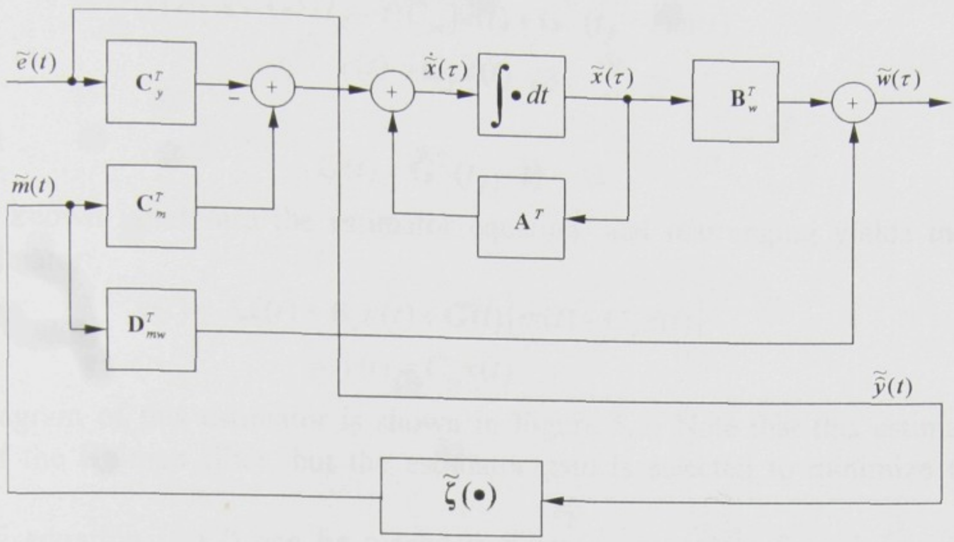


FIGURE 5.4 The adjoint of the estimator problem in standard form

The equivalence of the full information and disturbance feedforward control problems is demonstrated by noting that the state of a plant can be perfectly reconstructed from the initial condition and the inputs. Since the initial condition for the plant (5.80) is zero, the state can be perfectly reconstructed from the disturbance input and the control input. Since the controller always has knowledge of the control input, knowledge of the disturbance input is equivalent to full information.

The full information control results can be used to generate a suboptimal solution to the control problem in Figure 5.4. The estimator $\zeta(\bullet)$ can then be recovered from the resulting controller $\zeta(\bullet)$ by taking the adjoint. The resulting estimator is a suboptimal solution of the H_∞ estimation problem, since the ∞-norm of the closed-loop adjoint system equals the ∞-norm of the original system.

The Riccati Equation The disturbance feedforward controller is found by utilizing the full information results to yield a state feedback controller. The state used in this controller is reconstructed by applying the control and disturbance inputs to the adjoint model. The resulting controller is given:

$$\dot{\tilde{x}}(\tau) = \mathbf{A}^T \tilde{x}(\tau) + \mathbf{C}_m^T(\tau) + \mathbf{C}_y^T \tilde{e}(\tau) \tag{5.84a}$$

$$\tilde{m}(\tau) = -\tilde{\mathbf{G}}(\tau)\tilde{x}(\tau) \tag{5.84b}$$

or, equivalently,

$$\dot{\tilde{x}}(\tau) = [\mathbf{A}^T - \mathbf{C}_m^T \tilde{\mathbf{G}}(\tau)\tilde{x}(\tau) + \mathbf{C}_y^T \tilde{e}(\tau) \tag{5.85a}$$

$$\tilde{m}(\tau) = -\tilde{\mathbf{G}}(\tau)\tilde{x}(\tau) \tag{5.85b}$$

The state feedback gain in (5.85b) is equal to the full information control:

$$\tilde{\mathbf{G}}(\tau) = -\mathbf{C}_m \tilde{\mathbf{Q}}(\tau) \tag{5.86}$$

The matrix $\tilde{\mathbf{Q}}(\tau)$ is found by solving the following Riccati differential equation:

$$\dot{\tilde{\mathbf{Q}}}(\tau) = -\dot{\tilde{\mathbf{Q}}}(\tau)\mathbf{A}^T - \mathbf{A}\dot{\tilde{\mathbf{Q}}}(\tau) - \mathbf{B}_w \mathbf{B}_w^T + \dot{\tilde{\mathbf{Q}}}(\tau)(\mathbf{C}_m^T \mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T \mathbf{C}_y)\dot{\tilde{\mathbf{Q}}}(\tau) \tag{5.87}$$

backward in time from the final condition

$$\tilde{\mathbf{Q}}(t_f) = 0 \tag{5.88}$$

The H_∞ suboptimal estimator is the adjoint of the controller (5.85):

$$\dot{\hat{x}}(t) = [\mathbf{A} - \widetilde{\mathbf{G}}^T(t_f - t)\mathbf{C}_m]\hat{x}(t) + \widetilde{\mathbf{G}}^T(t_f - t)m(t) \tag{5.89a}$$

$$e(t) = \mathbf{C}_y \widetilde{x}(t) \tag{5.89b}$$

Defining,

$$\mathbf{G}(t) = \widetilde{\mathbf{G}}^T(t_f - t) \tag{5.90}$$

adding the known input into the estimator equation, and rearranging yields the H_∞ sub-optimal estimator:,

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \mathbf{B}_u u(t) + \mathbf{G}(t)\{m(t) - \mathbf{C}_y \hat{x}(t)\} \tag{5.91a}$$

$$\hat{y}(t) = \mathbf{C}_y \hat{x}(t) \tag{5.91b}$$

A block diagram of this estimator is shown in Figure 5.5. Note that this estimator has the structure of the Kalman filter, but the estimator gain is selected to minimize the ∞-norm criterion.

The Riccati equation (5.87) can be placed in a more convenient form by performing the change of variables $\tau = t_f - t$, and defining

$$\mathbf{Q}(t) = \widetilde{\mathbf{Q}}(t_f - \tau) \tag{5.92}$$

A Riccati equation for this new matrix is then

$$\dot{\mathbf{Q}}(t) = \mathbf{Q}(t)\mathbf{A}^T + \mathbf{A}\mathbf{Q}(t) + \mathbf{B}_w\mathbf{B}_w^T - \mathbf{Q}(t)(\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y)\mathbf{Q}(t)\,. \tag{5.93}$$

This Riccati equation is solved forward in time from the initial condition

$$\mathbf{Q}(0) = \mathbf{0} \tag{5.94}$$

The estimator gain can be written in terms of this new Riccati solution:

$$\mathbf{G}(t) = \mathbf{Q}(t)\mathbf{C}_m^T \tag{5.95}$$

Equations (5.91) through (5.95) completely specify the suboptimal H_∞ estimator. Note that the Riccati equation and therefore the estimator gain depend on the output being estimated, since this equation contains $\mathbf{C}_y$.

The Riccati equation (5.93) and (5.94) approaches the Kalman Riccati equation as the performance bound γ approaches infinity. A solution is therefore guaranteed to exist for sufficiently large performance bounds. For finite performance bounds, this equation may or may not have a solution. In general, solutions exist for performance bounds above a limiting value equal to the optimal value, where optimal is defined in terms o: minimizing the norm (5.73).

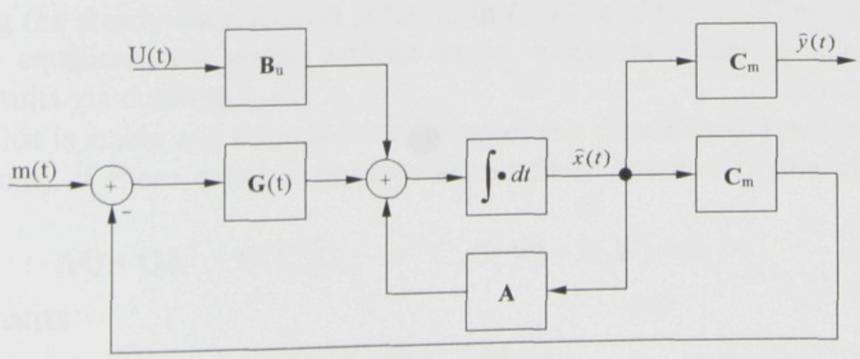


FIGURE 5.5 The H_∞ optimal estimator

The Hamiltonian Equations The suboptimal full information controller, and therefore the suboptimal estimator, can be generated by solving a Riccati equation or by solving the

Hamiltonian equations. The Hamiltonian equations for the estimator can be combined to form the Hamiltonian system:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \cdots\cdots \\ \dot{\tilde{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T & \vdots & -\mathbf{C}_m^T\mathbf{C}_m + \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\mathbf{B}_w\mathbf{B}_w^T & \vdots & -\mathbf{A} \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \cdots\cdots \\ \tilde{p}(t) \end{bmatrix} = \mathbf{y}_\infty \begin{bmatrix} \tilde{\mathrm{x}}(\mathrm{t}) \\ \cdots\cdots \\ \tilde{\mathrm{p}}(\mathrm{t}) \end{bmatrix} \tag{5.96}$$

where the matrix $\mathbf{y}_\infty$ is called the Hamiltonian.

The H_∞ suboptimal estimator gain can be found by solving the Hamiltonian system subject to the final condition

$$\tilde{\mathrm{p}}(\mathrm{t}_f) = 0 \tag{5.97}$$

This gain is then given as

$$\mathbf{G}(t) = -\mathbf{Q}(t)\mathbf{C}_m^T = -\mathbf{\Phi}_{21}^T(t)\{\mathbf{\Phi}_{22}(t)\}^{-T}\mathbf{C}_m^T = -\{\mathbf{\Phi}_{22}(t)\}^{-1}\mathbf{\Phi}_{21}^T(t)\mathbf{C}_m^T \tag{5.98}$$

where the terms $\mathbf{\Phi}_{ij}$ are blocks in the state-transition matrix of the Hamiltonian system:

$$e^{\mathbf{y}_\infty t} = \begin{bmatrix} \mathbf{\Phi}_{11}(t) & \vdots & \mathbf{\Phi}_{12}(t) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots \\ \mathbf{\Phi}_{21}(t) & \vdots & \mathbf{\Phi}_{22}(t) \end{bmatrix} \tag{5.99}$$

[Dolye, Francis, Tannenbaum-1990],[Burel-1999].

5.3.3 Steady-State Estimation

The H_∞ suboptimal estimator becomes time-invariant far from the initial time provided the bound is chosen sufficiently large that a steady-state solution to the Riccati equation exists. When operating over long time intervals, it is often desirable to use the steady-state estimator to simplify implementation. Mathematically, the steady-state estimator must be stable for the ∞-norm (5.73) to be defined over infinite time intervals. In practice, stable estimators are also desirable for operation over moderate to long time intervals, especially when initial errors and unmodeled dynamics are present. Thus the steady-state H_∞ suboptimal estimator problem is to find a stable, time-invariant estimator that satisfies an ∞-norm (defined over an infinite time interval) bound on the transfer function from the disturbance input to the estimation error.

Conditions for the existence of an H_∞ suboptimal estimator are summarized below. When these conditions are satisfied, the steady-state estimator is given by (5.91), with the gain found by using the steady-state Riccati solution in (5.93) and (5.94). These existence results and estimator equations are given without proof, since they can be related to the full information results via duality.

An estimator that is stable and bounds the ∞-norm of the closed-loop transfer function (5.73) exists if and only if there exists a positive semidefinite solution of the algebraic Riccati equation

$$\mathbf{AQ} + \mathbf{QA}^T - \mathbf{Q}(\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y)\mathbf{Q} + \mathbf{B}_w\mathbf{B}_w^T = 0 \tag{5.100}$$

such that the matrix

$$\mathbf{A} - \mathbf{Q}(\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y) \tag{5.101}$$

is stable.

The steady-state Riccati solution can be found from the eigensolution of the Hamiltonian:

$$\mathbf{Q} = \mathbf{\Psi}_{21}(\mathbf{\Psi}_{21})^{-1} \tag{5.102}$$

where

$$\begin{bmatrix} \mathbf{\Psi}_{11} \\ \cdots\cdots \\ \mathbf{\Psi}_{21} \end{bmatrix} \tag{5.103}$$

is a matrix whose columns are the eigenvectors of the Hamiltonian associated with the stable eigenvalues. As observed previously, the H_∞ optimal estimator can be approximated to an arbitrary degree of closeness by iteration of the bound and testing for the existence of suboptimal estimators [Burel-1999].

5.4 Examples

Example 5.1

Let the plant be modeled as follows:

$$\dot{x}(t) = x(t) + u(t) + 2w(t)$$

$$y(t) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and the objective function be

$$J_\gamma\{x(t), u(t), w(t)\} = \|y(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2$$

Note that state and control weightings are incorporated into the output equation, which is scaled to yield a unity weight on the control. The Hamiltonian system is

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} 1 & -1+4\gamma^{-2} \\ -100 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

which has the state-transition matrix

$$\mathbf{\Phi}(t) = e^{\mathcal{H}_\infty t} = \mathcal{H}^{-1}\{(s\mathbf{I}-\mathcal{H}_\infty)^{-1}\} = \mathcal{H}^{-1}\left\{\frac{1}{s^2-101+400\gamma^{-2}} \begin{bmatrix} s+1 & -1+4\gamma^{-2} \\ -100 & s-1 \end{bmatrix}\right\}$$

A saddle point of this differential game exists (and an H_∞ full information suboptimal controller exists) for a given performance bound provided $\mathbf{\Phi}_{22}(t_f,t)$ has an inverse throughout the time interval, that is, the scalar $\mathbf{\Phi}_{22}(t_f,t) \neq 0$. This element of the state transition matrix can be computed:

$$\mathbf{\Phi}_{22} = L^{-1}\left\{\frac{s-1}{s^2-101+400\gamma^{-2}}\right\} = \begin{cases} \dfrac{a-1}{2a}e^{at} + \dfrac{a+1}{2a}e^{-at} & \gamma > \sqrt{\dfrac{400}{101}} \\ 1-t & \gamma = \sqrt{\dfrac{400}{101}} \\ \sqrt{\dfrac{\omega^2+1}{\omega}}\sin(\omega t + \theta) & \gamma < \sqrt{\dfrac{400}{101}} \end{cases}$$

Where $a = \sqrt{101-400\gamma^{-2}}$, $\omega = \sqrt{-101+400\gamma^{-2}}$, and $\theta = -\tan^{-1}(\omega)$. Note that as γ gets smaller, a gets smaller, ω gets bigger, and θ gets more negative, When $\gamma < \sqrt{400/101}$, the eigenvalues of the Hamiltonian matrix are imaginary, and the gain does not exist for long time intervals, since $\mathbf{\Phi}_{22}(t)$ periodically crosses zero. Further, the fact that the feedback gain does not exist implies that there is no solution of the differential game (and no solution of the H_∞ suboptimal control problem) for the given bound. Less obvious from these expressions is that the time interval over which a feedback gain exists becomes monotonically larger as is increased, reaching infinity for $\gamma > 2$.

For the bound $\gamma = 2.03$ ($a = 2$), and the final time $t_f = 3$, the H_∞ suboptimal feedback gain is

$$\mathbf{K}(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-2(3-t)}}$$

which is plotted in Figure 5.6. Note that the gain exhibits a transient and then approaches a steady-state value far from the final time. In situations where the time interval is long compared to the settling time of this transient, it may be reasonable to use only the steady-state gain.

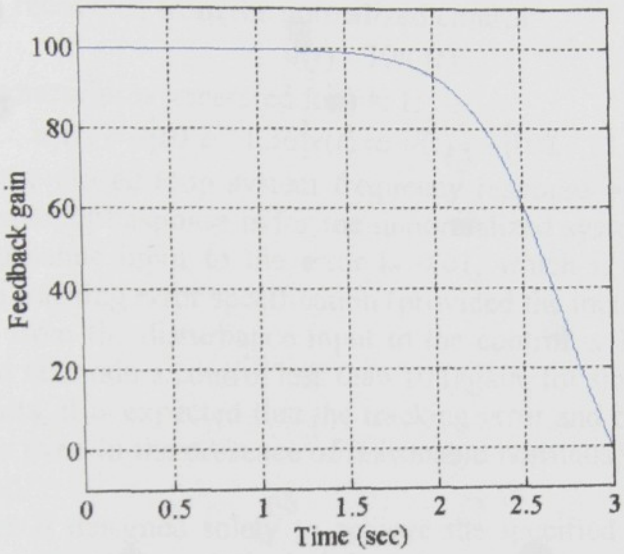


FIGURE 5.6 The feedback gain

The Riccati equation associated with the differential game (and H_∞ suboptimal control problem) in this example is:

$$\dot{P}(t) = -2P(t) + \left(1 - 4\gamma^{-2}\right)P^2(t) - 100$$

The final condition for this differential equation is $P(3) = 0$. The solution to this final value problem is

$$P(t) = K(t) = \frac{100e^{2(3-t)} - 100e^{-2(3-t)}}{e^{2(3-t)} + 3e^{-(3-t)}}$$

This solution can be verified by substitution into the Riccati equation and by evaluating the solution at the final time. Note that the feedback gain equals the Riccati solution, in this example, since $\mathbf{B}_u = 1$.

Example 5.2

An antenna is required to remain pointed at a satellite in the presence of disturbance torques caused by gravity, wind, squirrels climbing on the dish, and so on. A state model for this antenna is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t);$$

$$e(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

where $u(t)$is the control torque, $w(t)$is the disturbance torque, and $e(t)$is the tracking error of the antenna. The tracking error of less than 0.1 rad is desired, even in the presence of disturbance torques of up to 2 N-m. The control magnitude is also required to be less than 10

N-m. These specifications can be appended to the plant as weighting functions to yield the model in standard form:

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}x(t)+\begin{bmatrix}0\\ 10\end{bmatrix}u_1(t)+\begin{bmatrix}0\\ 2\end{bmatrix}w_1(t);$$

$$y(t)=\begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\end{bmatrix}u_1(t),$$

where $w_1(t)$ and $u_1(t)$ are the normalized disturbance and control inputs, respectively. In this form, $w_1(t)$ is less than 1 and both elements of $y(t)$ are required to be less than 1. The original control input can be recovered from the normalized control:

$$u(t)=10u_1(t)$$

A full information controller is generated for $\gamma = 1$:

$$u_1(t)=-[10.2 \quad 1.36]x(t)\Rightarrow u(t)=-[102 \quad 13.6]x(t)$$

The magnitude of the closed-loop system frequency response with this control is given in Figure 5.7. This frequency response is for the unnormalized system. Note that the maximum gain from the disturbance input to the error is 0.01, which is less than the gain of 0.05 required to meet the tracking error specification (provided the input is sinusoidal). In addition, the maximum gain from the disturbance input to the control is 1.26, which is less than the gain of 5 required to maintain a control less than 10 (again, for sinusoidal disturbance inputs). Based on these results, it is expected that the tracking error and control magnitude will meet the specified bounds even in the presence of reasonable nonsinusoidal disturbance inputs, but this is not guaranteed.

The given controller is designed solely to achieve the specified bounds on the closed-loop system gains. No specifications were included on the transient response of the control system. Therefore, the transient response should be checked to determine if it is acceptable. The closed-loop poles are $-7.3 \pm 7.0j$, which should yield a reasonably good transient response.

Smaller values of the performance bound (with $\gamma > 0.2$) can be used to generate additional controllers that meet the specifications. This limit on the performance bound can be obtained by decreasing the bound, solving (5.46), and checking to see if this solution is positive definite and (5.47) is stable. Alternatively, this bound can be generated by noting that the control and the disturbance enter the system at the same point. A full information controller,

$$u(t)=-w(t)$$

would then yield perfect tracking and satisfy the bound on the control input only for values of $\gamma > 0.2$.

Another H_∞ suboptimal controller is generated for $\gamma = 0.21$:

$$u_1(t)=-[32.8 \quad 7.39]x(t)\Rightarrow u(t)=-[328 \quad 73.9]x(t).$$

The closed-loop frequency response when using this controller is also shown in Figure 5.7. Comparing the two controllers, the use of the smaller performance bound increases the feedback gains, significantly decreases the maximum tracking error, and increases the bandwidth of the transfer function from the disturbance input to the control. This increase in bandwidth is also reflected in the closed-loop poles {-4.7, -70}. These poles are indicative of an overdamped, but probably acceptable, transient response.

The weights on the control and tracking errors can be adjusted individually to trade off tracking accuracy and control magnitude. The results of these changes are analogous to the results obtained when changing the weighting matrices in the LQR.

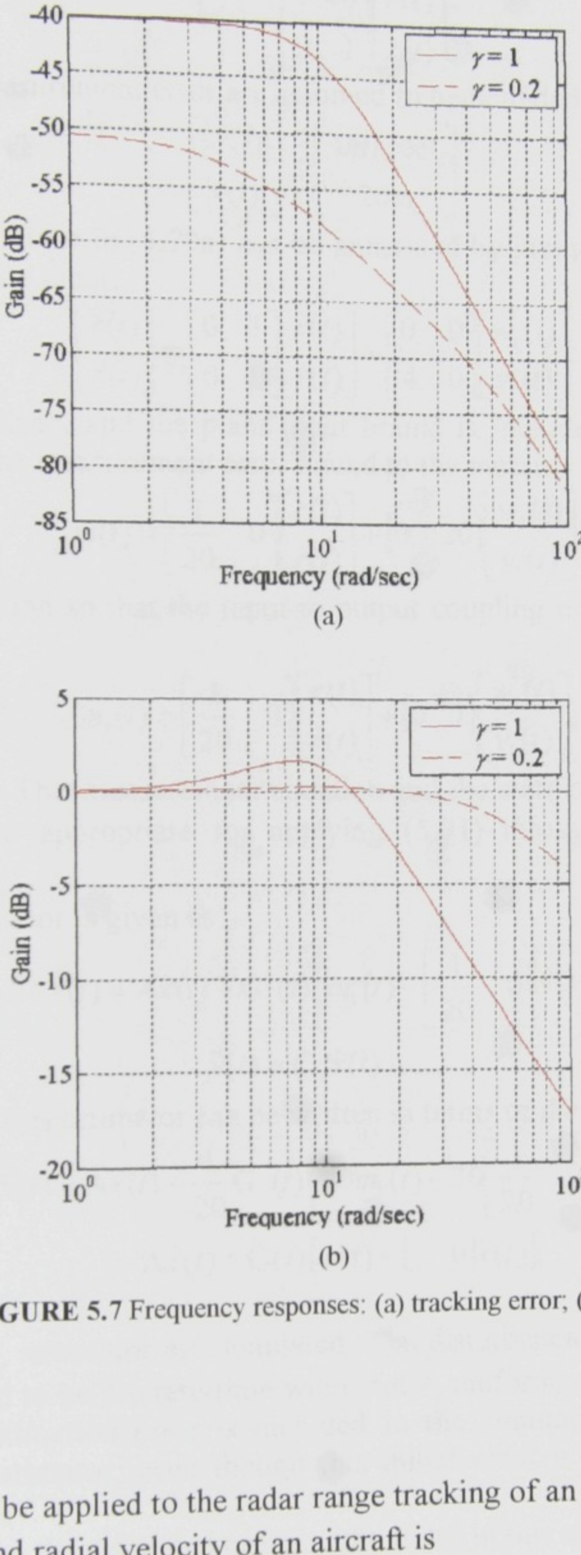


FIGURE 5.7 Frequency responses: (a) tracking error; (b) control

Example 5.3

An H_∞ estimator can be applied to the radar range tracking of an aircraft. A state equation for the actual range $r(t)$ and radial velocity of an aircraft is

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t).$$

The range measurements are given,

$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + v(t),$$

and the output to be estimated is the entire state:

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}.$$

The plant input and measurement error are assumed to be bounded:

$$\begin{bmatrix} |w(t)| \\ |v(t)| \end{bmatrix} \le \begin{bmatrix} 4m/\sec^2 \\ 20m \end{bmatrix}.$$

A state equation of the form in (5.70a) can be generated by including the measurement error as part of the disturbance input:

$$\begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix}.$$

The inputs are normalized, and the plant input bound is included as a weight in this state equation. Appending the measurement error bound to the measurement equation yields

$$m(t) = \begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 20 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix}.$$

Normalizing this equation so that the input-to output coupling matrix satisfies the constraint (5.71b) gives

$$m_1(t) = \begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ v_1(t) \end{bmatrix},$$

where $m(t) = 20\, m_1(t)$. This measurement equation and the state equation (including the input bound) are in a form appropriate for applying (5.91) through (5.95) for H_∞ estimator synthesis.

The resulting H_∞ estimator is given as

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \mathbf{G}_1(t)\left\{ m_1(t) - \begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \hat{x}(t) \right\};$$

$$\hat{y}(t) = \mathbf{C}_y \hat{x}(t)$$

The state equation for this estimator can be written in terms of the original measurement:

$$\dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \frac{1}{20}\mathbf{G}_1(t)\left\{ 20m_1(t) - 20\begin{bmatrix} \frac{1}{20} & 0 \end{bmatrix} \hat{x}(t) \right\}$$

$$= \mathbf{A}\hat{x}(t) + \mathbf{G}(t)\{m(t) - \begin{bmatrix} 1 & 0 \end{bmatrix}\hat{x}(t)\},$$

where $\mathbf{G}(t) = \mathbf{G}_1(t)/20$.

The plant and the H_∞ estimator are simulated. The disturbance input and the measurement error are both assumed to be discrete-time white noise, uniformly distributed within the given bounds. An initial estimation error is included in the simulation to display the transient performance of the estimator, even though this initial error is assumed to be zero during estimator development.

The estimator gains and the estimated outputs (the states in this case) are shown in Figure 5.8, where the estimator performance bound is: $\gamma = 22.5$. This particular performance bound is selected because it is roughly 10% larger than the optimal bound. The gains converge to steady-state values. Also, the estimates converge in slightly over 10 seconds for both position and velocity, and the estimator tracks the actual state.

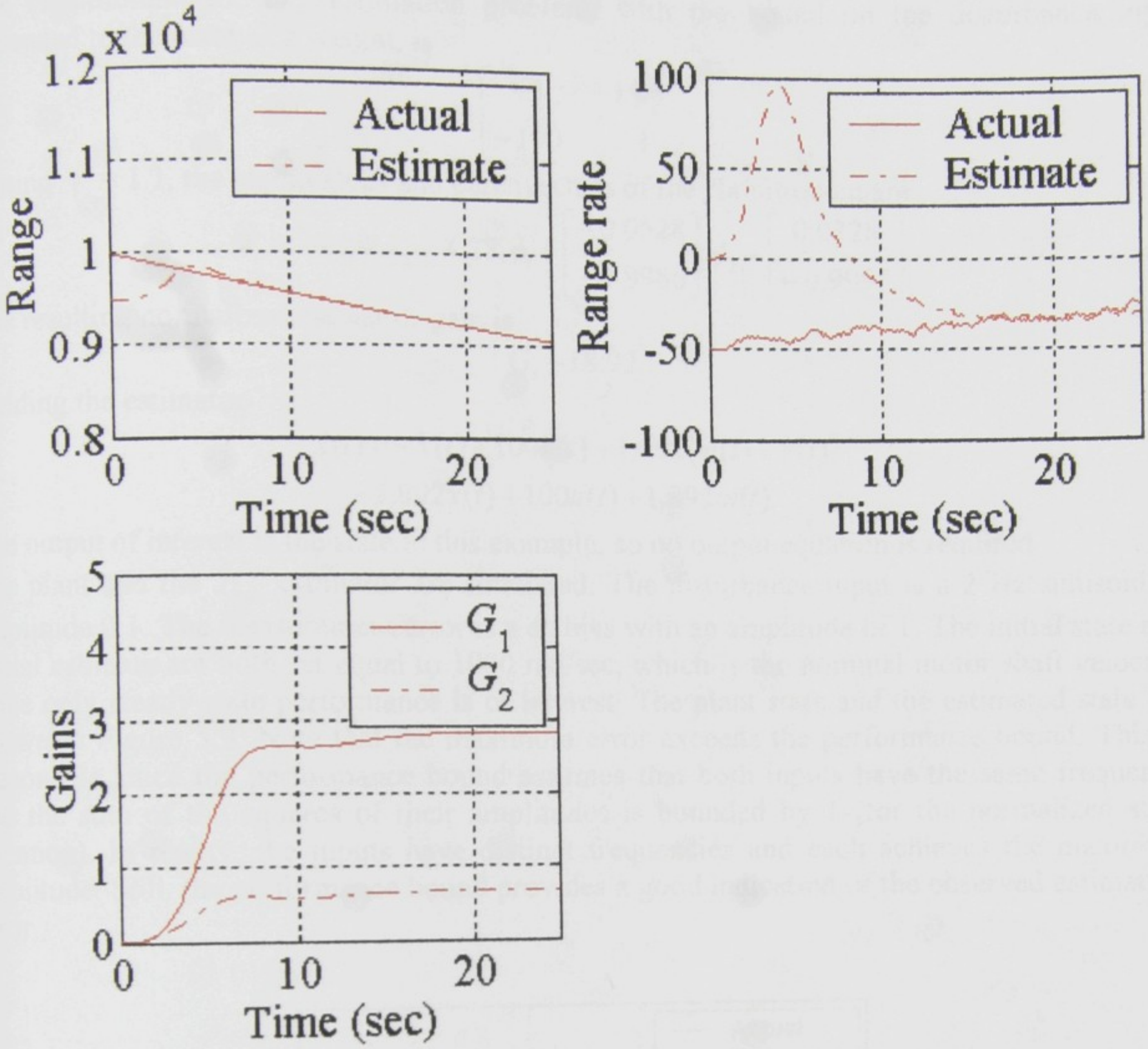


FIGURE 5.8 Baseline results for example 5.3

Example 5.4
An ac motor is described by the following state equation:

$$\dot{x}(t) = -x(t) + 100u(t) + 100w(t),$$

where x(t) is the rotational velocity of the shaft, $u(t)$ is the nominal input of 10 volts rms, and $w(t)$ is the error between the actual applied voltage and the nominal value. This error is assumed to be caused by modulation of the ac envelope due to harmonics on the power line. This modulation is assumed to be less than 1%, implying the disturbance input can be bounded:

$$|w(t)| \leq 0.1 \quad volts \quad rms.$$

An H_∞ estimator is used to estimate the true rotational velocity of the shaft from the noisy tachometer measurements:

$$m(t) = x(t) + v(t),$$

where $v(t)$ is the measurement error. This error consists of a possible dc bias and sinusoidal interference, and is bounded:

$$|v(t)| \leq 1 \quad volts \quad rms.$$

The output of interest is simply the state.

The Hamiltonian for this estimation problem, with the bound on the disturbance input appended to the plant as a weight, is

$$\mathbf{y}_\infty = \begin{bmatrix} -1 & -1+\gamma^{-2} \\ -100 & 1 \end{bmatrix}.$$

Letting γ = 1.1, the eigenvalues and eigenvectors of the Hamiltonian are

$$\lambda_1 = -4.28; \lambda_2 - 4.28; \phi_1 - \begin{bmatrix} -0.0528 \\ -0.9986 \end{bmatrix}; \phi_2 - \begin{bmatrix} 0.0328 \\ -0.9995 \end{bmatrix}.$$

The resulting normalized estimator gain is

$$G_1 - 18.92,$$

yielding the estimator

$$\begin{aligned} \dot{\hat{x}}(t) &= -\hat{x}(t) + 100u(t) + 1.892\{m(t) - \hat{x}(t)\} \\ &= -2.892\hat{x}(t) + 100u(t) + 1.892m(t). \end{aligned}$$

The output of interest is the state in this example, so no output equation is required.

The plant and the H_∞ estimator are simulated. The disturbance input is a 2 Hz sinusoid of amplitude 0.1. The measurement error is a dc bias with an amplitude of 1. The initial state and initial estimate are both set equal to 1000 rad/sec, which is the nominal motor shaft velocity, since only steady-state performance is of interest. The plant state and the estimated state are shown in Figure 5.9. Note that the maximum error exceeds the performance bound. This is reasonable since the performance bound assumes that both inputs have the same frequency and the sum of the squares of their amplitudes is bounded by 1 (for the normalized state equation). In reality, the inputs have distinct frequencies and each achieves the maximum amplitude. Still, the performance bound provides a good indication of the observed estimation error.

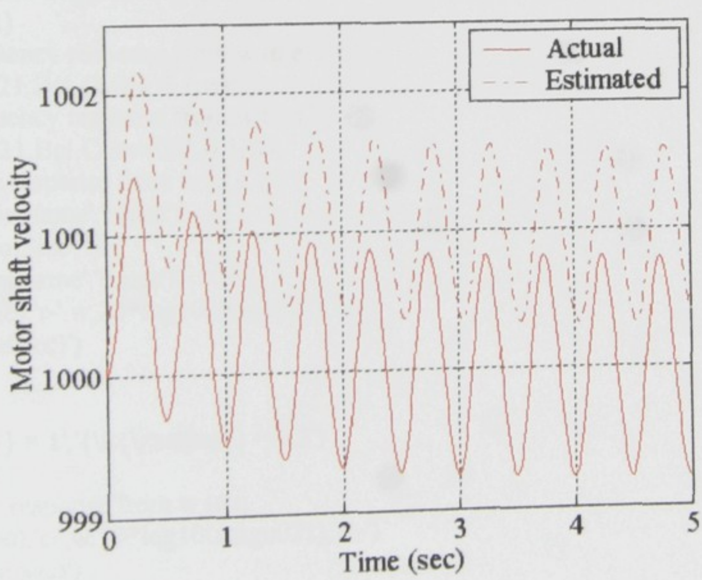


FIGURE 5.9 H_∞ estimation of motor shaft velocity

Table 5.1 MATLAB commands to compute H-infinity full information control of a DC motor for Example 5.2

```
clear
% Define the plant model.
A=[0 1, 0 -1];
Bu=[ 0,10];
Bw=[0, 2];
Cy=[10 0,0 0];
Dyu=[0,1];
Dyw=[0, 0];
Plant=pck(A,[Bw Bu],Cy,[Dyw Dyu]);
% Generate the full information control gain with gamma = 1.
K=hinffi(Plant,1,1,1,0.01);
K=-K(1:2)
% Generate the closed loop model.
Acl=A-Bu*K;
Bcl=[0;1];
Cclew=[1 0];
Ccluw=-10*K;
% Generate the poles of the closed loop system.
poles=eig(Acl)
% Generate the frequency response from w to e.
w=logspace(0,2);
mage=bode(Acl,Bcl,Cclew,0,1,w);
% Generate the frequency response from w to u.
magu=bode(Acl,Bcl,Ccluw,0,1,w);
% Generate a full information controller with gamma = 0.21
K021=hinffi(Plant,1,0.21,0.21,0.01);
K021=-K021(1:2)
% Generate closed loop model.
Acl021=A-Bu*K021;
Ccluw021=-10*K021;
% Generate the poles of the closed loop system.
poles021=eig(Acl021)
% Generate the frequency response from w to e.
mage021=bode(Acl021,Bcl,Cclew,0,1,w);
% Generate the frequency response from w to u.
magu021=bode(Acl021,Bcl,Ccluw021,0,1,w);
% Plot the frequency response from w to e.
set(0,'DefaultAxesFontName','times')
set(0,'DefaultAxesFontSize',16)
set(0,'DefaultTextFontName','times')
plot(w,20*log10(mage),'r-',w,20*log10(mage021),'r--')
xlabel('Frequency (rad/sec)')
ylabel('Gain (dB)')
grid
legend('{\it{\gamma}} = 1','{\it{\gamma}} = 0.2')
set(gca,'xscale','log')
% Plot the frequency response from w to u.
plot(w,20*log10(magu),'r-',w,20*log10(magu021),'r--')
xlabel('Frequency (rad/sec)')
ylabel('Gain (dB)')
grid
legend('{\it{\gamma}} = 1','{\it{\gamma}} = 0.2')
set(gca,'xscale','log')
```

6 H_∞ Controller

The H_∞ output feedback controller (or simply the H_∞ controller) utilizes partial state measurements, corrupted by disturbances, to generate the control. This controller can be synthesized by combining an H_∞ full information controller with an H_∞ estimator. At first glance it appears appropriate to estimate the state from the measurements and apply the full information feedback gain, as was done in the case of the LQG optimal controller. This turns out to be not exactly correct, since unlike the Kalman filter, the H_∞ estimator gain depends on what linear combination of the states is being estimated. Instead of estimating the state, it is more appropriate to estimate the desired control input. In addition, the worst-case disturbance input that appears in the full information mini-max problem must be included in the H_∞ estimator equations. The H_∞ output feedback controller therefore has a structure like that of the LQG controller, which consists of an estimator and a full information controller. But this structure technically violates the separation principle as presented for the LQG controller, since the estimator now depends on the full information controller design.

The suboptimal H_∞ control problem is defined by the plant and the cost function. The plant is given by the following state model:

$$\dot{x}(t) = \mathbf{A}x(t) + [\mathbf{B}_u \vdots \mathbf{B}_w] \cdot \begin{bmatrix} u(t) \\ \cdots\cdots \\ w(t) \end{bmatrix}, \tag{6.1a}$$

$$\begin{bmatrix} m(t) \\ \cdots\cdots \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_m \\ \cdots\cdots \\ \mathbf{C}_y \end{bmatrix} x(t) + \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{D}_{mw} \\ \cdots & & \cdots \\ \mathbf{D}_{yu} & \vdots & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ \cdots\cdots \\ w(t) \end{bmatrix}, \tag{6.1b}$$

The matrices $\mathbf{B}_w$ and $\mathbf{D}_{mw}$ are assumed to satisfy the following conditions:

$$\mathbf{D}_{mw}\mathbf{B}_w^T = \mathbf{0}, \tag{6.2a}$$

$$\mathbf{D}_{mw}\mathbf{B}_{mw}^T = \mathbf{I}. \tag{6.2b}$$

These conditions require that the disturbances entering the plant and the measurement be distinct and that the output equation of the plant be scaled to normalize the measurement noise. The matrices $\mathbf{C}_y$ and $\mathbf{D}_{yu}$ are assumed to satisfy the following conditions:

$$\mathbf{B}_{yu}^T\mathbf{D}_y = \mathbf{0}, \tag{6.3a}$$

$$\mathbf{B}_{yu}^T\mathbf{D}_{yu} = \mathbf{I} \tag{6.3b}$$

These conditions require that the reference output consist of an output dependent only on the state and a distinct output dependent only on the control input. Further, the portion of the output that depends on the control is simply equal to the control input or an orthogonally transformed version of this input. The plant is assumed to be controllable from the control input and observable from the measured output. These conditions guarantee that the plant can be stabilized using output feedback, a necessity when operating over infinite time intervals and always desirable. Further, the plant is assumed to be controllable from the disturbance input and observable from the reference output. These conditions guarantee the existence of a steady-state H_∞ suboptimal output feedback for sufficiently large performance bounds.

The suboptimal H_∞, control problem is to find a feedback controller for the above plant such that the ∞-norm of the closed-loop system is bounded:

$$\left\|\mathbf{G}_{yw}\right\|_{\infty,[0,t_f]} = \sup_{\|w(t)\|_{2,[0,t_f]}\neq 0} \frac{\|y(t)\|_{2,[0,t_f]}}{\|w(t)\|_{2,[0,t_f]}} < \gamma . \tag{6.4}$$

The closed-loop system is also required to be internally stable when the final time is infinite. The solution of the optimal H_∞ control problem (minimizing the closed-loop ∞-norm) is discussed after first presenting a suboptimal solution [Scherer-2001],[Burel-1999].

6.1 Controller Structure

The suboptimal H_∞ controller can be synthesized by combining a full information controller and an output estimator. This structure for the suboptimal H_∞ controller is derived in this subsection, assuming that the suboptimal full information controller exists.

The ∞-norm bound (6.4), that defines the suboptimal H_∞ controller, is equivalent to requiring that the inequality

$$J_\chi = \|y(t)\|^2_{2,[0,t_f]} - \gamma^2 \|w(t)\|^2_{2,[0,t_f]} \le -\varepsilon^2 \|w(t)\|^2_{2,[0,t_f]} \tag{6.5}$$

be satisfied for some positive s and for all disturbance inputs. The expression on the left in this inequality is the objective function for the differential game used when solving the full information control problem. A general expression for this objective function with arbitrary inputs was derived in Subsection, 5.2.3. This expression is derived assuming that a solution of the Riccati equation (5.31) exists (i.e., that a suboptimal H_∞ full information controller exists).

A suboptimal full information controller exists whenever a suboptimal output feedback controller exists, since all possible outputs can be generated from the state and disturbance input. Therefore, the inequality (6.5) can be written in terms of the full information Riccati solution by substituting (5.38) into (6.5):

$$\begin{aligned} J_\chi &= \left\|u(t) + \mathbf{B}_u^T \mathbf{P}(t) x(t)\right\|^2_{2,[0,t_f]} - \gamma^2 \left\|w(t) - \gamma^{-2} \mathbf{B}_w^T \mathbf{P}(t) x(t)\right\|^2_{2,[0,t_f]} \\ &\le -\varepsilon^2 \|w(t)\|^2_{2,[0,t_f]} \end{aligned} \tag{6.6}$$

This inequality is equivalent to the bound on the ∞-norm:

$$\left\|\mathbf{G}_\Delta\right\|_\infty = \sup_{w(t)-\gamma^{-2}\mathbf{B}_w^T\mathbf{P}(t)x(t)\neq 0} \frac{\left\|u(t) + \mathbf{B}_u^T \mathbf{P}(t) x(t)\right\|_{2,[0,t_f]}}{\left\|w(t) - \gamma^{-2} \mathbf{B}_w^T \mathbf{P}(t) x(t)\right\|_{2,[0,t_f]}} < \gamma \tag{6.7}$$

This bound can be used in formulating a suboptimal H_∞ output estimation problem. The solution of this problem leads directly to a suboptimal H_∞ controller.

This output estimation problem is stated as follows: Estimate the full information control input,

$$u(t) = -\mathbf{B}_u^T \mathbf{P}(t) x(t) \tag{6.8}$$

given the measurement $m(t)$, such that the ∞-norm of the transfer function between the disturbance input $\Delta w(t) = w(t) - \gamma^{-2} \mathbf{B}_w^T \mathbf{P}(t) x(t)$, and the estimation error is bounded:

$$\begin{aligned}\|\mathbf{G}_{\Delta}\|_{\infty} &= \sup_{\Delta w(t)=0} \frac{\|y(t)-\hat{y}(t)\|_{2,[0,t_f]}}{\|w(t)-\gamma^{-2}\mathbf{B}_w^T\mathbf{P}(t)x(t)\|_{2,[0,t_f]}} \\ &= \sup_{\Delta w(t)=0} \frac{\|-\mathbf{B}_w^T\mathbf{P}(t)x(t)-u(t)\|_{2,[0,t_f]}}{\|w(t)-\gamma^{-2}\mathbf{B}_w^T\mathbf{P}(t)x(t)\|_{2,[0,t_f]}} < \gamma\end{aligned} \tag{6.9}$$

Note that this equation is equivalent to (6.7), since $\|-\mathrm{x}\| = \|\mathrm{x}\|$.

The state model of the "plant" in this estimation problem has $\Delta w(t)$ as the disturbance input and $m(t)$ as the measured output. The state equation for this model can be generated by adding plus and minus $\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)x(t)$ to the original state equation (6.1a):

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t)+\mathbf{B}_u u(t)+\mathbf{B}_w w(t)+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)x(t)-\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)x(t) \\ &= [\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)]x(t)+\mathbf{B}_u u(t)+\mathbf{B}_w\Delta w(t)\end{aligned} \tag{6.10}$$

The measurement equation for this model can be generated by subtracting $\gamma^{-2}\mathbf{D}_{mw}\mathbf{B}_w^T\mathbf{P}x$, which equals zero because $\mathbf{D}_{mw}\mathbf{B}_w^T = 0$, from the original measurement equation (6.1b):

$$\begin{aligned}m(t) &= \mathbf{C}_m x(t)+\mathbf{D}_{mw}w(t)-\gamma^{-2}\mathbf{B}_{mw}\mathbf{B}_w^T\mathbf{P}(t)x(t) \\ &= \mathbf{C}_m x(t)+\mathbf{D}_{mw}\Delta w(t)\end{aligned} \tag{6.11}$$

The plant model (6.10) and (6.11) has the form of the plant model (5.70), which appears in the H_∞ estimation problem statement.

The suboptimal H_∞, estimator for the problem specified by (6.7) through (6.10) and (6.11) generates estimates of the full information control. The inequality (6.7), which is equivalent to (6.4), is satisfied when applying these estimates as control inputs to the original plant. In this case, the estimator becomes an output feedback controller, since it generates control inputs from the measurements. Therefore, the estimator is a suboptimal H_∞ controller.

In summary, the suboptimal (or optimal) H_∞, output feedback control law is the H_∞ suboptimal (or optimal) estimate of the full information control. The controller is therefore designed in two stages: a full information controller is synthesized, and an output estimator is synthesized. The final output feedback controller is generated by combining these two components. This controller has a structure similar to the LQG controller, but technically violates the separation principle, since the estimator design depends on the full information controller design[Scherer-2001], [Dolye, Francis, Tannenbaum-1990],[Burel-1999].

6.2 Finite-Time Control

The suboptimal H_∞ output estimation problem specified by (6.7) through (6.10) and (6.11) has a solution if the Riccati equation

$$\begin{aligned}\dot{\mathbf{Q}}_m(t) = &-\mathbf{Q}_m(t)[\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)]^T-[\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)]\mathbf{Q}_m(t) \\ &-\mathbf{B}_w\mathbf{B}_w^T+\mathbf{Q}_m(t)[\mathbf{C}_m^T\mathbf{C}_m-\gamma^{-2}\mathbf{P}(t)\mathbf{B}_u\mathbf{B}_u^T\mathbf{P}(t)]\mathbf{Q}_m(t)\end{aligned} \tag{6.12}$$

has a solution, given the initial condition $\mathbf{Q}_m(0) = \mathbf{0}$. The suboptimal estimator is

$$\dot{\hat{x}}(t) = [\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t)]\hat{x}(t)+\mathbf{B}_u u(t)+\mathbf{G}(t)[m(t)-\mathbf{C}_m\hat{x}(t)] \tag{6.13}$$

$$\hat{u}(t) = -\mathbf{B}_u^T\mathbf{P}(t)\hat{x}(t) \tag{6.14}$$

and the gain is

$$\mathbf{G}(t) = \mathbf{Q}_m(t)\mathbf{C}_m^T \tag{6.15}$$

This estimator contains two known inputs: $u(t)$ which enters through $\mathbf{B}_u$; and $\gamma^{-2}\mathbf{B}_w^T\mathbf{P}(t)\hat{x}(t)$ which enters through $\mathbf{B}_w$. The second input is the worst-case disturbance encountered during full information controller optimization. This estimator can then be described as estimating the full information control in the presence of the worst case disturbance.

This estimator can be used as a feedback controller. Substituting for the gain (6.15) in the estimator and setting $u(t) = \hat{u}(t)$ yields the controller:

$$\dot{\hat{x}}(t) = [\mathbf{A} + \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t) - \mathbf{B}_u\mathbf{B}_u^T\mathbf{P}(t) - \mathbf{Q}_m(t)\mathbf{C}_m^T\mathbf{C}_m]\hat{x}(t) + \mathbf{Q}_m(t)\mathbf{C}_m^T m(t); \qquad (6.16a)$$

$$u(t) = -\mathbf{B}_u^T\mathbf{P}(t)\hat{x}(t) \qquad (6.16b)$$

Solutions to the Riccati equation (6.12) and the full information Riccati equation (5.31) are sufficient to guarantee the existence of this suboptimal H_∞ controller [Zhou , Dolye, Glover-1995],[Burel-1999].

6.2.1 An Alternative Estimator Riccati Equation

The generation of the H_∞ output feedback controller (as given above) proceeds by synthesizing the full information control and then estimating this control. This process results in solving the full information Riccati equation for the plant. In addition, an estimator Riccati equation for a modified plant is solved. This Riccati solution can be related to the Riccati solution associated with reference output estimation for the original plant. Developing this correspondence produces a symmetric (the symmetry is between the control and estimation Riccati equations) pair of Riccati equations that can be solved to generate the H_∞ output feedback controller. The resulting equations are in the form most frequently encountered in the research literature.

The solution of the estimator Riccati equation (6.12) for the modified plant (if it exists) can be related to the solution of the output estimation Riccati equation (5.93) and (5.94) with $\mathbf{Q}(0) = \mathbf{0}$:

$$\mathbf{Q}_m(t) = \mathbf{Q}(t)[\mathbf{I} - \gamma^{-2}\mathbf{P}(t)\mathbf{Q}(t)]^{-1} \qquad (6.17)$$

This fact is demonstrated after first presenting an additional condition that is necessary: to guarantee the existence of $\mathbf{Q}_m(t)$.

The existence of $\mathbf{Q}_m(t)$ in (6.17) hinges on the existence of the matrix inverse in this equation on the interval from 0 to t_f. This matrix is invertible if it is positive definite, that is, all the eigenvalues of $[\mathbf{I} - \gamma^{-2}\mathbf{P}(t)\mathbf{Q}(t)]$ are positive. The eigenvalues of this matrix can be related to the eigenvalues of $\mathbf{P}(t)\mathbf{Q}(t)$. If λ is an eigenvalue of $\mathbf{P}(t)\mathbf{Q}(t)$, then

$$1 - \gamma_{-2}\lambda(t) \qquad (6.18)$$

is an eigenvalue of $[\mathbf{I} - \gamma^{-2}\mathbf{P}(t)\mathbf{Q}(t)]$. Therefore, the inverse in (6.17) exists, and $\mathbf{Q}_m$=(t) exists, provided $\mathbf{P}(t)$ exists, $\mathbf{Q}(t)$ exists, and all of the eigenvalues of $\mathbf{P}(t)\mathbf{Q}(t)$ are less than γ^{-2}; that is,

$$\rho[\mathbf{P}(t)\mathbf{Q}(t)] < \gamma^2 \qquad (6.19)$$

where $\rho(\bullet)$ is the spectral radius.

It remains to show that $\mathbf{Q}_m$ (t) is a solution of the Riccati equation (6.12), with the initial condition $\mathbf{Q}_m(0) = \mathbf{0}$. The initial condition on $\mathbf{Q}_m(t)$ is easily verified:

$$\mathbf{Q}_m(0) = \mathbf{Q}(0)[\mathbf{I} - \gamma^{-2}\mathbf{P}(0)\mathbf{Q}(0)]^{-1} = \mathbf{0} \qquad (6.20)$$

since $\mathbf{Q}(0) = \mathbf{0}$. The fact that $\mathbf{Q}_m(t)$ is a solution of the Riccati equation (6.12) can be verified by considering the Hamiltonian system for the estimator (5.96):

$$\dot{x}(t) = \begin{bmatrix} \mathbf{A}^T & \vdots & -\mathbf{C}_m^T\mathbf{C}_m + \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots \\ -\mathbf{B}_w\mathbf{B}_w^T & \vdots & -\mathbf{A} \end{bmatrix} x(t). \qquad (6.21)$$

The generic state $x(t)$ is used in this equation to simplify the notation. Using the transformation matrix

$$\mathbf{T}(t)=\begin{bmatrix}\mathbf{I} & \vdots & -\gamma^{-2}\mathbf{P}(t)\\ \cdots & & \cdots\cdots\cdots\\ \mathbf{0} & \vdots & \mathbf{I}\end{bmatrix}, \tag{6.22}$$

a time-varying similarity transformation can be per formed on the Hamiltonian system:

$$\begin{aligned}\dot{\tilde{x}}(t)&=\left(\begin{bmatrix}\mathbf{I} & \vdots & \gamma^{-2}\mathbf{P}\\ \cdots & & \cdots\cdots\cdots\\ \mathbf{0} & \vdots & \mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{A}^T & \vdots & -\mathbf{C}_m^T\mathbf{C}_m+\gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y\\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\\ -\mathbf{B}_w\mathbf{B}_w^T & \vdots & -\mathbf{A}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{I} & \vdots & -\gamma^{-2}\mathbf{P}\\ \cdots & & \cdots\cdots\cdots\\ \mathbf{0} & \vdots & \mathbf{I}\end{bmatrix}\right.\\ &\quad\left.-\begin{bmatrix}\mathbf{I} & \vdots & -\gamma^{-2}\mathbf{P}\\ \cdots & & \cdots\cdots\cdots\\ \mathbf{0} & \vdots & \mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{0} & \vdots & -\gamma^{-2}\dot{\mathbf{P}}\\ \cdots & & \cdots\cdots\cdots\\ \mathbf{0} & \vdots & \mathbf{0}\end{bmatrix}\right)\tilde{x}\\ &=\begin{bmatrix}(\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P})^T & \vdots & \gamma^{-2}\dot{\mathbf{P}}+\mathbf{PA}+\mathbf{A}^T\mathbf{P}+\gamma^{-2}\mathbf{PB}_w\mathbf{B}_w^T\mathbf{P}+\mathbf{C}_y^T\mathbf{C}_y)-\mathbf{C}_m^T\mathbf{C}_m\\ \cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ -\mathbf{B}_w\mathbf{B}_w^T & \vdots & -(\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P})\end{bmatrix}\tilde{x}\\ &=\tilde{\mathbf{y}}_m\tilde{x}\end{aligned} \tag{6.23}$$

where time index has been deleted to simplify the notation. Adding and subtracting $\gamma^{-2}\mathbf{PB}_u\mathbf{B}_u^T\mathbf{P}$ in the (1, 2) block of the transformed Hamiltonian matrix yields

$$\begin{aligned}\tilde{\mathbf{y}}_{12}&=\gamma^{-2}\{\dot{\mathbf{P}}+\mathbf{PA}+\mathbf{A}^T\mathbf{P}(\mathbf{B}_u\mathbf{B}_u^T-\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}+\mathbf{C}_y^T\mathbf{C}_y\}\\ &\quad+\gamma^{-2}\mathbf{PB}_u\mathbf{B}_u^T\mathbf{P}-\mathbf{C}_m^T\mathbf{C}_m\\ &=\gamma^{-2}\mathbf{PB}_u\mathbf{B}_u^T\mathbf{P}-\mathbf{C}_m^T\mathbf{C}_m.\end{aligned} \tag{6.24}$$

This expression has been simplified by noting that the term in curly brackets is the difference between the two sides of the full information Riccati equation (5.31) and is therefore equal to zero. The transformed Hamiltonian system is then

$$\dot{\tilde{x}}(t)=\begin{bmatrix}(\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P})^T & \vdots & \gamma^{-2}\mathbf{PB}_u\mathbf{B}_u^T\mathbf{P}-\mathbf{C}_m^T\mathbf{C}_m\\ \cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\\ -\mathbf{B}_w\mathbf{B}_w^T & \vdots & -(\mathbf{A}+\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P})\end{bmatrix}\tilde{x}(t) \tag{6.25}$$

This equation is the Hamiltonian system associated with the Riccati equation (6.12). The solution of this Riccati equation can be given in terms of the state-transition matrix $\mathbf{\Phi}_m(t)$ of the Hamiltonian system (6.25):

$$\mathbf{Q}_m(t)=[\mathbf{\Phi}_{m22}(t)]^{-1}\mathbf{\Phi}_{m21}(t) \tag{6.26}$$

where

$$\mathbf{Q}_m(t)=\begin{bmatrix}\mathbf{\Phi}_{m11}(t) & \vdots & \mathbf{\Phi}_{m12}(t)\\ \cdots\cdots\cdots & & \cdots\cdots\cdots\\ \mathbf{\Phi}_{m21}(t) & \vdots & \mathbf{\Phi}_{m22}(t)\end{bmatrix} \tag{6.27}$$

Further, this state-transition matrix of (6.25) can be related to the state-transition matrix of (6.17):

$$\begin{bmatrix} \mathbf{\Phi}_{m11}(t) & \vdots & \mathbf{\Phi}_{m12}(t) \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{\Phi}_{m21}(t) & \vdots & \mathbf{\Phi}_{m22}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \vdots & -\gamma^{-2}\mathbf{P} \\ \cdots & & \cdots\cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{m11}(t) & \vdots & \mathbf{\Phi}_{m12}(t) \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{\Phi}_{m21}(t) & \vdots & \mathbf{\Phi}_{m22}(t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{\Phi}_{11} - \gamma^{-2}\mathbf{P}\mathbf{\Phi}_{21} & \vdots & \mathbf{\Phi}_{12} - \gamma^{-2}\mathbf{P}\mathbf{\Phi}_{22} \\ \cdots\cdots & & \cdots\cdots \\ \mathbf{\Phi}_{21} & \vdots & \mathbf{\Phi}_{22} \end{bmatrix} \tag{6.28}$$

since these systems are related by a time-varying similarity transform. Note that the time indexes have again been dropped to simplify the notation. The solution of the Riccati equation (6.12) is then

$$\begin{aligned} \mathbf{Q}_m &= \mathbf{\Phi}_{21}(\mathbf{\Phi}_{22} - \gamma^{-2}\mathbf{P}\mathbf{\Phi}_{21})^{-1} \\ &= \mathbf{\Phi}_{21}(\mathbf{\Phi}_{22})^{-1}[(\mathbf{I} - \gamma^{-2}\mathbf{P}\mathbf{\Phi}_{21}(\mathbf{\Phi}_{22})^{-1}]^{-1} = \mathbf{Q}(\mathbf{I} - \gamma^{-2}\mathbf{P}\mathbf{\Phi}_{21})^{-1} \end{aligned} \tag{6.29}$$

as given in (6.17) [Toivonen-2001],[Dolye, Francis, Tannenbaum-1990],[Burel-1999].

6.2.2 Summary

The solution of the finite-time, suboptimal H_∞ output feedback control problem can be given in terms of the Riccati solutions **P** and **Q:** A suboptimal H_∞ controller exists if and only if the following conditions are satisfied:

1. There is a solution of the Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A} + \mathbf{A}^T\mathbf{P}(t) - \mathbf{P}(t)(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}(t) + \mathbf{C}_y^T\mathbf{C}_y \tag{6.30}$$

on the interval from 0 to t_f, given $\mathbf{P}(t_f) = \mathbf{0}$.

2. There is a solution of the Riccati equation

$$\dot{\mathbf{Q}}(t) = \mathbf{A}\mathbf{Q}(t) + \mathbf{Q}(t)\mathbf{A}^T - \mathbf{Q}(t)[\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y]\mathbf{Q}(t) + \mathbf{B}_w\mathbf{B}_w^T \tag{6.31}$$

   on the interval from 0 to t_f, given $\mathbf{Q}(0) = \mathbf{0}$

3. On the interval from 0 to t_f,

$$\rho[\mathbf{P}(t)\mathbf{Q}(t)] < \gamma^2 \tag{6.32}$$

Note that the Hamiltonian formulations of the first two conditions are given in the sections on H_∞, full information control and H_∞ estimation, respectively.

A suboptimal controller that satisfies this bound is given:

$$\dot{x}_c(t) = \mathbf{A}_c(t)x_c(t) + \mathbf{B}_c(t)m(t) \tag{6.33a}$$

$$u(t) = \mathbf{C}_c(t)x_c(t) \tag{6.33b}$$

where the matrices $\mathbf{A}_c(t)$, $\mathbf{B}_c(t)$, and $C_c(t)$ are found using the solutions of the above; algebraic Riccati equations:

$$\begin{aligned} \mathbf{A}_c(t) = \mathbf{A} &+ \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T\mathbf{P}(t) - \mathbf{B}_u\mathbf{B}_u^T\mathbf{P}(t) \\ &- [\mathbf{I} - \gamma^{-2}\mathbf{Q}(t)\mathbf{P}(t)]^{-1}\mathbf{Q}(t)\mathbf{C}_m^T\mathbf{C}_m \end{aligned} \tag{6.33c}$$

$$\mathbf{B}_c(t) = [\mathbf{I} - \gamma^{-2}\mathbf{Q}(t)\mathbf{P}(t)]^{-1}\mathbf{Q}(t)\mathbf{C}_m^T \tag{6.33d}$$

$$\mathbf{C}_c(t) = -\mathbf{B}_u^T\mathbf{P}(t) \tag{6.33e}$$

The existence conditions for the suboptimal H_∞, controller imply the existence of both a suboptimal H_∞ full information controller and a suboptimal H_∞ filter for estimating the reference output. The implied existence of the full information controller can be understood by noting that full information includes information on all possible outputs. Therefore, output feedback is a special case of full information control. The existence of the H_∞ suboptimal filter for estimating the reference output can be understood by noting that the reference output cannot be controlled to a greater degree of accuracy than that at which the reference output can be estimated.

The optimal controller can be approximated to an arbitrary degree of accuracy by decreasing the performance bound until the conditions (6.30) through (6.32) are no longer satisfied. Note that a solution always exists for a sufficiently large performance bound, since the above Riccati equations reduce to those for the LQR, and the Kalman filter in the limit as this bound approaches infinity.

6.3 Steady-State Control

The estimator gains and the state feedback gains of the H_∞ controller typically approach steady-state values far from the initial time and the final time, respectively. In applications where the control system is designed to operate for time periods that are long, compared to the transient times of these gains, it is reasonable to ignore the transients and use the steady-state gains, exclusively. The use of the steady-state gains simplifies controller implementation and results in a time-invariant, closed-loop system. Time invariance of the closed-loop system allows the use of many robustness and performance analysis techniques that are not applicable to time-varying systems.

The steady-state H_∞ controller is the solution of the following suboptimal control problem: Find a linear, time-invariant controller system, described in the Laplace domain as follows:

$$u(s) = \mathbf{K}(s)m(s) \tag{6.34}$$

that internally stabilizes the closed-loop system and bounds the ∞-norm of the closed-loop system:

$$\left\|\mathbf{G}_{yw}\right\|_\infty < \gamma \tag{6.35}$$

The steady-state H_∞ suboptimal control can be obtained by combining the steady-state H_∞ full information controller and the steady-state H_∞ estimator of this control. The existence of this steady-state controller is predicated on the existence of both the full information controller and the estimator. As in the finite-time case, the existence of an estimator of the full information control can be related to the existence of an estimator of the reference output. Combining the existence results for the full information controller and the output estimator, we find as follows: A solution exists for the suboptimal H_∞ control problem if and only if the following conditions are satisfied:

1. There is a positive semidefinite solution of the algebraic Riccati equation
$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - \mathbf{P}(\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P} + \mathbf{C}_y^T\mathbf{C}_y = \mathbf{0} \tag{6.36}$$
such that $\{\mathbf{A} - (\mathbf{B}_u\mathbf{B}_u^T - \gamma^{-2}\mathbf{B}_w\mathbf{B}_w^T)\mathbf{P}\}$ is stable (i.e., has only eigenvalues with negative real parts).
2. There is a positive semidefinite solution of the algebraic Riccati equation
$$\mathbf{AQ} + \mathbf{QA}^T - \mathbf{Q}(\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{BC}_y^T\mathbf{C}_y)\mathbf{Q} + \mathbf{B}_w\mathbf{B}_w^T = \mathbf{0} \tag{6.37}$$
such that $\{\mathbf{A} - \mathbf{Q}(\mathbf{C}_m^T\mathbf{C}_m - \gamma^{-2}\mathbf{C}_y^T\mathbf{C}_y)\}$ is stable.
3. The spectral radius of the product of these Riccati solutions is bounded:
$$\rho(\mathbf{PQ}) < \gamma^2 \tag{6.38}$$

Note that Hamiltonian formulations for the first two conditions are given in Chapter 5 for H_∞ full information control and H_∞ estimation. A state model for the steady-state H_∞

suboptimal controller is obtained by using the algebraic Riccati solutions in (6.33)[Dolye, Francis, Tannenbaum-1990],[Burel-1999].

6.4 Application of H_∞ Control

H_∞ control can be used as an alternative to LQG optimal control. Both cost functions are reasonable for a wide range of problems, and in many applications the choice of a quadratic versus an ∞-norm cost function is arbitrary. In these applications, the LQG controller is typically selected, since controller optimization is simpler and yields a unique solution. But the LQG control system may have undesirable properties; that is, it may not be robust, it may have an undesirable frequency response, and so on. In these cases, it is reasonable to try an H_∞ optimal (or suboptimal) controller. The results obtained with the H_∞ controller may be better or worse than those obtained with the LQG controller, since the H_∞ controller has no magic robustness or frequency-domain properties. Still, it is worth trying when the LQG controller is not performing adequately.

H_∞ control is a natural for applications where the specifications are given in terms of bounds on the outputs (both output errors and controls). Requiring the outputs to remain below prescribed levels is typical of engineering design specifications. Output bounds are given by the closed-loop system ∞-norm provided the disturbance inputs are sinusoidal (or constant), all inputs are at the same frequency, the inputs are normalized, and the outputs are normalized.

The inputs should be normalized so that

$$|w_i(t)| = \frac{1}{\sqrt{\#\ \text{of}\ \mathit{inputs}}} \tag{6.39}$$

when designing to achieve a given output bound. This normalization assures that the sum of the squares of the contributions from all inputs remains below the specification. The outputs should also be normalized so that the specifications require a bound of 1 for each output. The desired output bounds are then achieved provided the closed-loop ∞-norm is less than 1. Note that these bounds are only guaranteed for sinusoidal disturbance inputs.

H_∞ control can be used to generate systems that meet output bound specifications when the inputs are sinusoidal, but with different frequencies. In this case, the ∞-norm provides a worst-case gain from each input to each output. The output is then bounded by the sum, over all inputs, of the output amplitude bounds. In this case, the inputs should be normalized as in (6.39), except the number of inputs is replaced by the number of inputs at a given frequency. The resulting bound is typically conservative since, at each frequency, the input "direction" is constrained, whereas the ∞-norm gives the bound on the gain for unconstrained input directions. H_∞ control can also be used when the inputs are not sinusoidal. For nonsinusoidal inputs, the closed-loop system ∞-norm provides an indication of the maximum size of the outputs, but does not provide a formal bound on the outputs.

The existence conditions for the suboptimal H_∞ controller are very useful when performing trade-offs between competing control objectives. These existence conditions can be used to determine when a given set of specifications are consistent with a reasonable design and when the system is overspecified [Zhou, Dolye-1998],[Burel-1999].

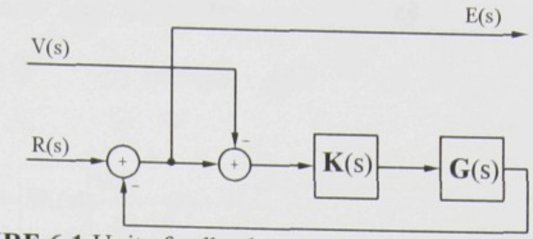


FIGURE 6.1 Unity feedback system with measurement noise

6.4.1 Performance Limitations

The specification of performance using the ∞-norm can easily result in unobtainable requirements. As an example, consider the generic SISO unity feedback system shown Figure 6.1. A controller is desired for this system so that

$$\|\mathbf{G}_{cl}\|_\infty = \max_\omega \{\bar{\sigma}[\mathbf{G}_{cl}(j\omega)]\} < 0.1, \tag{6.40}$$

where

$$\mathbf{G}_{cl}(s) = [G_{er}(s) \;\; G_{ev}(s)] = \left[\frac{1}{1+G(s)K(s)} \;\; \frac{G(s)K(s)}{1+G(s)K(s)} \right]. \tag{6.41}$$

While this specification seems quite reasonable, it is not achievable with any plant and controller unless the inputs are separated in frequency.

These two transfer functions are often called the sensitivity and complimentary sensitivity in classical control.

The minimum closed-loop system ∞-norm that can be achieved for this system is 0.707. This bound is a consequence of the relationship between the elements in the closed-loop transfer function:

$$G_{er}(s) + G_{ev}(s) = \frac{1}{1+G(s)K(s)} + \frac{G(s)K(s)}{1+G(s)K(s)} = 1 \tag{6.42}$$

To derive the ∞-norm bound, note that the singular value (there is only one) of the closed-loop transfer function is

$$\bar{\sigma}\{G_{cl}(j\omega)] = \sqrt{G_{cl}(j\omega)G_{cl}^T(j\omega)} = \sqrt{|G_{er}(j\omega)|^2 + |G_{ev}^T(j\omega)|^2} \tag{6.43}$$

Minimizing this singular value with respect to the constraint (6.42) yields

$$\min_{G_{er}+G_{ev}=1} \bar{\sigma}[G_{cl}(j\omega)] = \min_{G_{er}+G_{ev}=1} \sqrt{|G_{er}(j\omega)|^2 + |G_{ev}^T(j\omega)|^2} = \frac{1}{\sqrt{2}} \tag{6.44}$$

Since this is the minimum possible singular value, the closed-loop system ∞-norm is bounded:

$$\|G_{cl}\|_\infty = \max_\omega \{\bar{\sigma}[G_{cl}(j\omega)\}] \geq \frac{1}{\sqrt{2}} \tag{6.45}$$

The constraint (6.42) imposes a fundamental limitation on tracking performance in the presence of measurement error. This is also true for MIMO control systems, where (6.42) becomes

$$\mathbf{G}_{er}(s) + \mathbf{G}_{ev}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)]^{-1} + [\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s)]^{-1}\mathbf{G}(s)\mathbf{K}(s) = \mathbf{I} \tag{6.46}$$

This performance limitation can be intuitively understood by noting that large loop gain is required for good tracking performance. But large loop gain results in the plant tracking the measurement noise. Therefore, the controller cannot simultaneously reduce tracking errors due to both reference inputs and measurement errors unless these signals are separated in frequency.

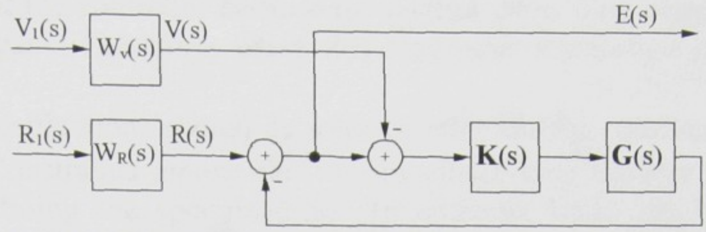


FIGURE 6.2 Weighting function for the unity feedback system with measurement noise

The specifications can be made more reasonable by separating the reference inputs and the measurement noise in frequency. For example, the reference input may be assumed to be slowly varying or, equivalently, to be a lowpass signal. The measurement error may be assumed to be rapidly varying (typical of noise signals) or, equivalently, to be a highpass signal. The transfer function from the reference input to the tracking error can then be made small over the frequency range allotted to the reference input. In addition, the transfer function from the measurement noise to the tackling error can be made small over the separate frequency band allotted to the measurement noise.

The ∞-norm can be used to specify performance over frequency bands by appending weighting functions to the plant inputs, as shown in Figure 6.2. A controller that yields an ∞-norm less than 0.1 is then possible, at least in theory, provided the weighting functions $W_R(j\omega)$ and $W_v(j\omega)$ are appropriately selected; that is, these weighting functions cannot contain significant overlap.

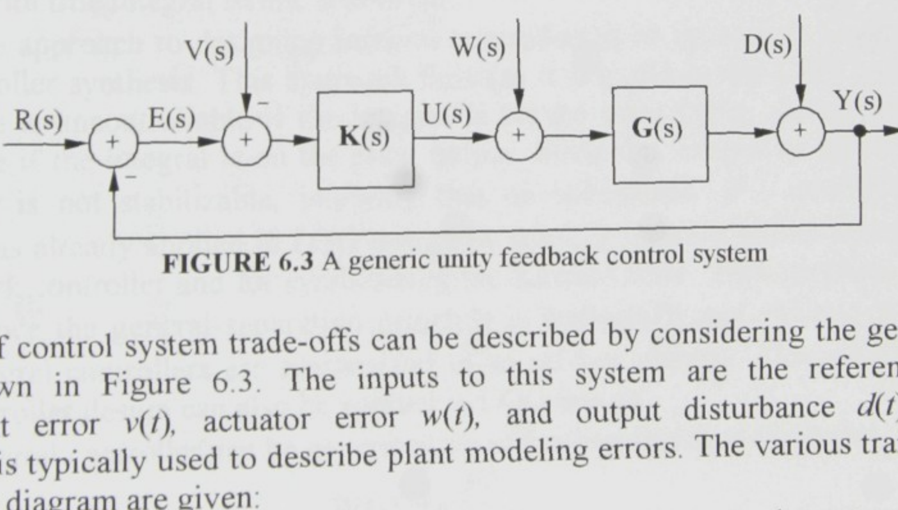


FIGURE 6.3 A generic unity feedback control system

A number of control system trade-offs can be described by considering the generic feedback system shown in Figure 6.3. The inputs to this system are the reference input $r(t)$, measurement error $v(t)$, actuator error $w(t)$, and output disturbance $d(t)$. The output disturbance is typically used to describe plant modeling errors. The various transfer functions in this block diagram are given:

$$E(s)=[\mathbf{I}+\mathbf{GK}]^{-1}[R(s)-D(s)]+[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{GK}V(s)-[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{G}W(s) \tag{6.47a}$$

$$Y(s)=[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{GK}[R(s)-V(s)]+[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{GK}W(s)+[\mathbf{I}+\mathbf{GK}]^{-1}D(s) \tag{6.47b}$$

$$U(s)=[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{K}[R(s)-D(s)-V(s)]-[\mathbf{I}+\mathbf{GK}]^{-1}\mathbf{KG}W(s) \tag{6.47c}$$

Note, for example, that

$$\mathbf{G}_{ed}(s)-\mathbf{G}_{ev}(s)=\mathbf{I} \tag{6.48}$$

Therefore, there is an inherent trade-off between making the system tolerant of output disturbances (i.e., providing good robustness to plant modeling errors), and making the system tolerant of measurement noise. Other trade-offs can be obtained by looking for additional constraints between the various transfer functions and by looking at the effects of large and/or small loop gains, controller gains, and plant gains on the various transfer functions.

Careful consideration of the trade-offs inherent in a control system design is also beneficial when iteration of the design is required. For example, changing the control bound in Example 6.2 had little effect of the controller because the control gains were being constrained by the measurement error gain. Therefore, using this parameter for design iteration is not very useful.

In summary, H_∞ specifications should be selected after careful consideration of the desired control objectives. Constraints among the various closed-loop transfer functions should be identified before defining the specifications. In addition, trade-offs between the various control objectives should be defined, and redundant control objectives should be identified. Using this information, the specifications should be frequency weighted to avoid conflicts caused by competing control objectives. It is typically a good idea to avoid allpass specifications when multiple disturbance inputs are included in the model. In addition, it is often a good idea to avoid redundant specifications whenever possible. Note that these redundant specifications can be added back into the model when evaluating performance, if desired. A good deal of art is involved in setting specifications for H_∞, controller design. This art is facilitated by a thorough understanding of the plant and by experience [Zhou , Dolye, Glover-1995].

6.4.2 Integral Control

Integral control is used to remove steady-state errors due to constant reference inputs and disturbances. Unfortunately, controllers with integral terms cannot be generated directly using the H_∞ synthesis theory presented. This limitation is typically not a problem since "nearly" integral controllers can be generated. In practice, these controllers can usually be replaced by controllers with true integral terms, if desired.

A reasonable approach to designing integral controllers is to append an integral to the plant before controller synthesis. This approach fails (as it also did in the LQG case) because the integral state is uncontrollable if the integral is on the plant input, and the integral state is unobservable if the integral is on the plant output. Since the integral is unstable, the closed-loop system is not stabilizable, implying that no suboptimal H_∞ controller exists. This limitation was already applied in LQG design by using a separate model for synthesizing the state feedback controller and for synthesizing the Kalman filter. This approach is not valid in this case, since the general separation principle is technically not valid in the H_∞ setting. Instead, integral controllers are synthesized in an *ad hoc* manner. This *ad hoc* approach to integral controller design can also be applied in LQG design.

A nearly integral controller can be generated by appending filters of the form

$$W(s) = \frac{1}{s+\varepsilon} \tag{6.49}$$

to the plant inputs (either reference or disturbance) where dc signals are to be rejected. For small ε, this filter has a large dc gain and approximates an integrator. The loop gain. The loop gain feedback system must therefore also be large in order to avoid a large de closed-loop gain. This large dc loop gain is obtained by placing a pole of the controller near the origin in the complex plane. When true integral action is desired, this controller pole can be replaced by a pole at the origin. This replacement typically has very little impact on the closed-loop poles and stability of the feedback system. But caution must be used when making this substitution, since there is no mathematical guarantee of performance or even stability[Scherer-2001],[Zhou, Dolye-1998].

6.4.3 Designing for Robustness

Stability robustness to an unstructured perturbation is guaranteed when the ∞-norm of the transfer function from perturbation input to the perturbation output is bounded by 1 (the perturbation bound is assumed to be normalized to 1). This transfer function can be include within the closed-loop transfer function by appending the perturbation input and output to the disturbance input and reference output, respectively (see Figure 6.4). Adding weights at the input and output of the perturbation (as in Figure 6.4) allows the designer to generate a family of controllers that trade off performance with robustness. Note that the original plant results when both of these weights equal zero,[Burel-1999],[Dolye, Francis, Tannenbaum-1990].

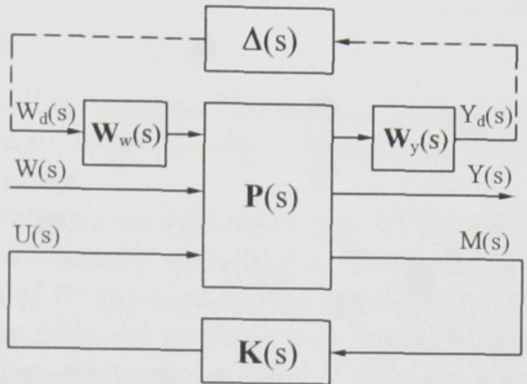


FIGURE 6.4 Block diagram for robust performance design

6.5 μ-Synthesis

Robust performance can be analysed using the structured singular value for systems containing both structured and unstructured perturbations. A system in standard form, with normalized performance criteria and perturbations (see Figure 6.5a), performs robustly if and only if

$$\sup_{\omega}\{\mu_{\bar{\Delta}}[\mathbf{N}(j\omega)]\} < 1 \tag{6.50}$$

where $\mathbf{N}(s)$ is the nominal closed-loop system formed by combining $\mathbf{P}(s)$ and $\mathbf{K}(s)$ and the block structure is implied by Figure 6.5b. Minimization of the cost function

$$J = \sup_{\omega}\{\mu_{\bar{\Delta}}[\mathbf{N}(j\omega)]\} \tag{6.51}$$

provides a means of obtaining robust performance, or demonstrating that robust performance is not possible, for the given specifications. The μ-synthesis controller design procedure addresses the minimization of the cost function (6.51).

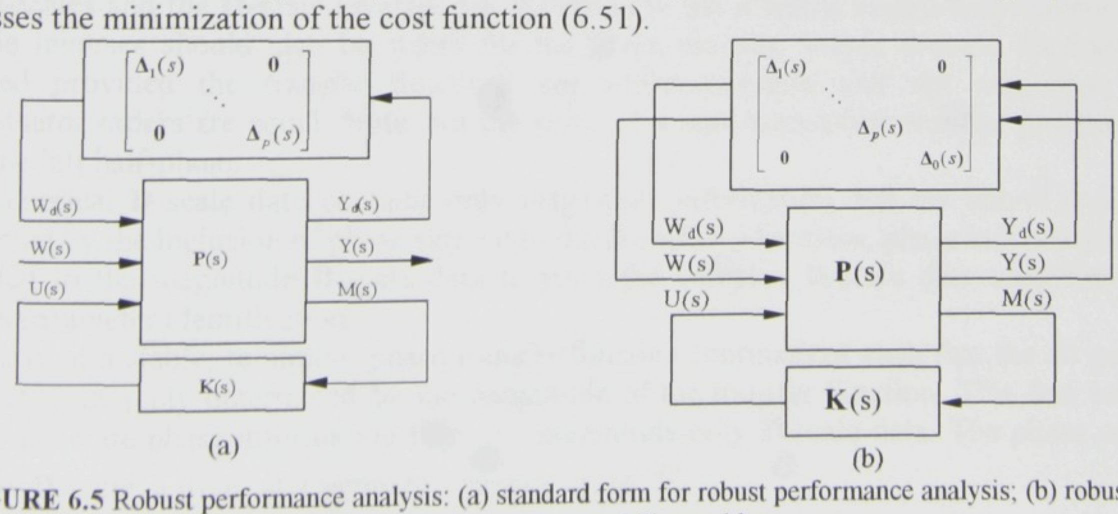


FIGURE 6.5 Robust performance analysis: (a) standard form for robust performance analysis; (b) robust performance as a robust stability problem

6.5.1 D-Scaling and the Structured Singular Value

The direct computation of the structured singular value is intractable in all but the simplest cases. Therefore, bounds on the SSV are typically used in place of the actual SSV during robust performance analysis. In particular,

$$\mu_{\bar{\Delta}}(\mathbf{N}) = \min_{\substack{\{d_1, d_2, \ldots, dn\} \\ d_i \in (0,\infty)}} \bar{\sigma}\left(\mathbf{D}_R \mathbf{N} \mathbf{D}_L^{-1}\right) \tag{6.52a}$$

provides a tight upper bound on the SSV and can be reliably computed. The matrices

$$\mathbf{D}_L = \begin{bmatrix} d_1\mathbf{I}_{t_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & d_2\mathbf{I}_{t_2} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & d_n\mathbf{I}_{t_n} \end{bmatrix}; \quad \mathbf{D}_R = \begin{bmatrix} d_1\mathbf{I}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & d_2\mathbf{I}_{n_2} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & d_n\mathbf{I}_{n_n} \end{bmatrix} \tag{6.52b}$$

are referred to as **D**-scaling matrices. Note that the identity matrices in (6.52b) are appropriately dimensioned to match the block structure of the perturbations. The scalar terms $\{d_i\}$ are referred to as **D** -scales.

Robust performance is evaluated by calculating the SSV bound for the nominal closed-loop transfer function $\mathbf{N}(j\omega)$. Frequency sampling is necessitated when generating this bound because the minimization of (6.52) is performed numerically. The resulting data are samples of the frequency-dependent SSV and the frequency-dependent **D**-scales.

The **D-K** iteration algorithm, discussed in detail in the next subsection, attempts to minimize the upper bound (6.52). This algorithm utilizes H_∞ optimization operating on the augmented plant $\mathbf{D}_R(s)\mathbf{N}(s)\mathbf{D}_L^{-1}(s)$. A state model for the augmented plant is required to apply the H_∞ optimization results given earlier in this chapter. Such a model can be obtained by appending $\mathbf{D}_R(s)$ and $\mathbf{D}_L^{-1}(s)$ to the nominal closed-loop transfer function $\mathbf{N}(s)$. This approach requires that the numerical **D**-scales, obtained when computing the SSV bound, be approximated by finite-order Laplace transfer functions.

Fitting Transfer Functions to the D-Scales Weighted least squares can be used to generate stable, minimum-phase transfer functions that approximate sampled, complex, frequency response data. Frequency response data can be approximated by either stable or unstable transfer functions. Stable transfer functions are preferred because instability of the **D**-scales needlessly complicates the controller design. In addition, the frequency response of an unstable system should always be viewed with caution because persistent transients typically overwhelm the sinusoidal response.

The **D**-scales and the **D**-scale inverses are appended to the nominal closed-loop system. The **D**-scale inverses should also be stable for the given reasons. Stable **D**-scale inverses are obtained provided the transfer functions are minimum-phase and the numerator and denominator orders are equal. Note that the zeros of a minimum-phase transfer function are all in the left half-plane.

The numerical **D**-scale data contains only magnitude information, but the bound (6.52) is unaffected by the inclusion of phase shifts into the **D**-scales. Therefore, phase information can be added to the magnitude **D**-scale data to yield the complex **D**-scale data used for least squares parameter identification.

The phase of a stable, minimum-phase transfer function (normalized such that the dc gain is positive) is uniquely determined by the magnitude of the transfer function. This fact can be used to generate phase information from the magnitude-only **D**-scale data. The phase of the complex **D**-scale $\tilde{d}_k(j\omega)$, at a sample frequency ω_t, is

$$\angle\tilde{d}_k(j\omega_t) = \frac{2\omega_t}{\pi}\int_0^{\infty} \frac{\ln\{d_k(\omega)\} - \ln\{d_k(\omega_l)\}}{\omega^2 - \omega_l^2} d\omega \tag{6.53}$$

where $d_k(\omega)$ is the magnitude of the **D**-scale. This integral can be evaluated numerically using the samples $d_k(\omega_l)$. The complex **D**-scale is then given as

$$\tilde{d}_k(j\omega_t) = d_k(\omega) e^{j\angle\tilde{d}_k(j\omega)} \tag{6.54}$$

Least squares can now be used to generate a transfer function model of order n_o,

$$G(s) = \frac{b_{n_0}s^{n_0} + \ldots + b_1 s + b_0}{s^{n_0} + a_{(n_0-1)}s^{(n_0-1)} + \ldots + a_1 s + a_0} \tag{6.55}$$

that approximates each complex **D**-scale. The order is assumed to be fixed during the subsequent least squares development. A discussion of order selection is provided at the end of this subsection.

The frequency response of the transfer function model should approximate the complex **D**-scale. At each sample frequency, this implies that

$$\tilde{d}_k(\omega_l) \approx \frac{b_{n_0}(j\omega_l)^{n_0} + \ldots + b_1(j\omega_l) + b_0}{(j\omega_l)^{n_0} + a_{(n_0-1)}(j\omega_l)^{(n_0-1)} + \ldots + a_1(j\omega_l) + a_0} \tag{6.56}$$

The error is then

$$\begin{aligned} e_k(\omega_l) = {} & \tilde{d}_k(\omega_l)(j\omega_l)^{n_0} + \tilde{d}_k(\omega_l)a_{(n_0-1)}(j\omega_l)^{(n_0-1)} \\ & + \ldots + \tilde{d}_k(\omega_l)a_1(j\omega_l) + \tilde{d}_k(\omega_l)a_0 \\ & - b_{n_0}(j\omega_l)^{n_0} - \ldots - b_1(j\omega_l) - b_0 \end{aligned} \tag{6.57}$$

The errors at all sample frequencies can be combined into a vector:

$$\mathrm{E}_k = [\mathrm{e}_k(\omega_1) \;\; \mathrm{e}_k(\omega_2) \;\; \ldots \mathrm{e}_k(\omega_{n_\omega})]^{\mathrm{T}} \tag{6.58}$$

where n_ω, is the number of frequencies. This error vector can be computed for a given transfer function model,

$$E_k = Y_k - \mathbf{M}_k\theta_k \tag{6.59}$$

where

$$\mathrm{Y}_k = [\tilde{\mathrm{d}}_k(\omega_1)(\mathrm{j}\omega_1)^{\mathrm{n}_0} \;\; \tilde{\mathrm{d}}_k(\omega_2)(\mathrm{j}\omega_2)^{\mathrm{n}_0} \ldots \tilde{\mathrm{d}}_k(\omega_{\mathrm{n}_\omega})(\mathrm{j}\omega_{\mathrm{n}_\omega})^{\mathrm{n}_0}]^{\mathrm{T}} \tag{6.60a}$$

$$\mathbf{M}_k = \begin{bmatrix} \tilde{d}_k(\omega_1)(j\omega_1)^{n_0} & \cdots & \tilde{d}_k(\omega_1) & (j\omega_1)^{n_0} & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \tilde{d}_k(\omega_{n_\omega})(j\omega_{n_\omega})^{(n_0-1)} & \cdots & \tilde{d}_k(\omega_{n_\omega}) & (j\omega_{n_\omega})^{(n_0-1)} & \cdots & 1 \end{bmatrix} \tag{6.60b}$$

$$\theta_k = [a_{(n_0-1)} \cdots a_0 \quad b_{n_0} \cdots b_0]^T \tag{6.60c}$$

The transfer function parameters, the elements of θ_k, are selected to minimize the weighted square error:

$$J_k = \sum_{l=1}^{n_\omega} W_l |e_k(\omega_l)|^2 = E_k^{\dagger}\mathbf{W}E_k = (Y_k - \mathbf{M}_k\theta_k)^{\dagger}\mathbf{W}(Y_k - \mathbf{M}_k\theta_k) \tag{6.61}$$

where **W** is a diagonal matrix with the elements $W_t \in [0, \infty)$. The weights W_t can be used to emphasize or deemphasize the fit at a given frequency or frequencies.

The transfer function parameters that minimize the weighted square error can be found by expanding the cost function:

$$\begin{aligned} J_k &= (Y_k - \mathbf{M}_k\theta_k)^{\dagger}\mathbf{W}(Y_k - \mathbf{M}_k\theta_k) \\ &= Y_k^{\dagger}\mathbf{W}Y_k - \theta_k^{\dagger}\mathbf{M}_k^{\dagger}\mathbf{W}Y_k - Y_k^{\dagger}\mathbf{W}\mathbf{M}_k\theta_k + \theta_k^{\dagger}\mathbf{M}_k^{\dagger}\mathbf{W}\mathbf{M}_k\theta_k \end{aligned} \tag{6.62}$$

Taking the derivative of this scalar with respect to the parameter vector θ_k yields

$$\frac{\partial J_k(\theta_k)}{\partial(\theta_k)} = -2Y_k^{\dagger}\mathbf{W}\mathbf{M}_k + 2\theta_k^{\dagger}\mathbf{M}_k^{\dagger}\mathbf{W}\mathbf{M}_k \tag{6.63}$$

Setting this derivative equal to zero and solving for θ_k yields the parameter vector that minimizes the weighted square error:

$$\widehat{\theta} = (\mathbf{M}_k^{\dagger}\mathbf{W}\mathbf{M}_k)^{-1}\mathbf{M}_k^{\dagger}\mathbf{W}Y_k \tag{6.64}$$

provided the inverse exists. This inverse typically exists when the number of frequencies is large compared to the number of parameters that are being identified (i.e., the model order is small compared to the number of sample frequencies used in computing the SSV).

Frequency weighting is included in the squared error criterion. The frequency weights can be selected to provide an accurate match at frequencies where the bound (6.52) is particularly sensitive to the **D**-scale. The sensitivity of the bound (6.52) to the **D**-scale can be computed by perturbing the **D**-scale and finding the change in the bound. The frequency weighting should then be set equal to this change. Applying this frequency weighting results in an approximation that is most accurate at frequencies where the sensitivity of the bound to the **D**-scale is high.

The order of the transfer function approximation is selected so that the weighted frequency error is small. This can be accomplished by plotting the magnitude frequency response of the approximations for several orders along with the numerical **D**-scale data. The user is then free to make a decision. Alternatively, an automated algorithm can be used that increases the order of the transfer function until the weighted frequency error is less than a given bound at all frequencies. Regardless of the method used to select order, the order should be maintained as small as is reasonable because high-order **D**-scales increase computation time and result in higher-order controllers when applying the **D-K** iteration algorithm[Matlab-1992-2001],[Zhou, Dolye-1998].

6.5.2 D-K Iteration

The μ-synthesis design methodology attempts to minimize the supremum of the closed-loop system's structured singular value as given by the cost function (6.51). Direct minimization of this cost function is typically not tractable. As an alternative, it is reasonable to minimize the upper bound on the SSV:

$$J = \sup_{\omega} \min_{\substack{\{d_1, d_2, \dots, dp\} \\ d_i \in (0,\infty)}} \bar{\sigma}[\mathbf{D}_R(j\omega)\mathbf{N}(j\omega)\mathbf{D}_L^{-1}(j\omega)], \tag{6.65}$$

where the **D**-scales are frequency-dependent. Note that robust performance is achieved when the supremum of the SSV is appropriately bounded. Therefore, achieving this bound guarantees robust performance, but the design may be conservative if the **D**-scaled bound (6.66) is not tight.

The optimization of the cost function (6.65) is still intractable in most cases, but an *ad hoc* algorithm known as **D-K** iteration has been found to work well in many applications. This algorithm is based on the observation that for a given set of **D**-scales, the cost function (6.65) is simply an ∞-norm optimization problem:

$$J_D = \sup_{\omega} \bar{\sigma}[\mathbf{D}_R(j\omega)\mathbf{N}(j\omega)\mathbf{D}_L^{-1}(j\omega)] = \left\|\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1}\right\|_{\infty}. \tag{6.66}$$

The solution to this problem given earlier in this chapter is valid provided that $\mathbf{D}_R\mathbf{N}\mathbf{D}_L^{-1}$ is described by a state model that satisfies the conditions (6.2). Unfortunately, the **D**-scales in (6.65) are not fixed, since they depend on the closed-loop model which, in turn, depends on the controller. **D-K** iteration seeks to overcome this problem by alternatively performing ∞-norm optimization and **D**-scale optimization.

The **D-K** iteration algorithm is summarized as follows.

1. Model the plant. The plant model should include disturbance inputs, control inputs, reference outputs, measured outputs, and perturbations. Append the performance block to the uncertainty matrix.
2. Generate a control system to minimize the ∞-norm of the transfer function from the augmented perturbation input to the augmented perturbation output.
3. Compute the structured singular values for the closed-loop system (with both uncertainty and performance blocks). Save the **D**-scales used in computing the structured singular value.
4. Fit a low-order transfer function to each frequency-dependent **D**-scale, as discussed in the previous subsection.
5. Append these transfer functions to the plant. The rational transfer function approximations for the **D**-scales and the inverse **D**-scales are appended to the nominal closed-loop system. This is typically accomplished by generating state models for the **D**-scales and the inverse **D**-scales, and appending these state models to the nominal closed-loop system.
6. For this augmented plant, generate a controller to minimize the ∞-norm of the transfer function from the augmented perturbation input to the augmented perturbation output.
7. Return to step 4, until the structured singular value of the closed-loop system fails to improve.

This algorithm has typically been found to converge to a minimum cost in a few iterations. Caution must be exercised in interpreting the meaning of this minimum, since the **D-K** iteration algorithm is not guaranteed to converge to the global minimum of the cost function (6.65). Further, this global minimum is not guaranteed to equal the global minimum of the cost function (6.51), except when the number of performance and perturbation blocks is less than or equal to 3[Matlab-1992-2001],[Zhou, Dolye-1998].

6.6 Examples

Example 6.1

A satellite tracking antenna, containing measurement errors and subject to wind torques, can be modelled as follows:

$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.001W_b \end{bmatrix} w_1(t),$$

where $\theta(t)$ is the pointing error of the antenna in degrees, $u(t)$ is the control torque, $w(t)$ is the normalized wind torque (disturbance input), and W_b is the bound on the wind torque. The normalized wind torque is assumed to be sinusoidal with a bounded amplitude $|w_1(t)| \le 1$, implying that the true wind torque is bounded: $|w(t)| = |W_b w_1(t)| \le W_b$. The pointing error is measured as

$$m(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + v(t),$$

where the measurement error is assumed to be sinusoidal with a bounded amplitude: $|v(t)| \le 1$ degree. The outputs of interest are the pointing error and the control input

$$y(t) = \begin{bmatrix} W_\theta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

The weight on the control input is selected to be 1 in order to satisfy condition (6.3b). A variable weight W_θ is added to the pointing error to allow running of the controller. Note that the bound on the disturbance torque can also be used as a tuning parameter for this controller. The steady-state controller is synthesized for $W_b = 70$ and $W_\theta = 13$. A performance bound $\gamma = 90$ (approximately 10% over the optimal) is used in this design. The system is simulated with a true disturbance input $w(t) = 70$. The measurement error is sampled white noise (the sampling time is 0.1), uniformly distributed between −1 to 1. The results are shown in Figure 6.6.

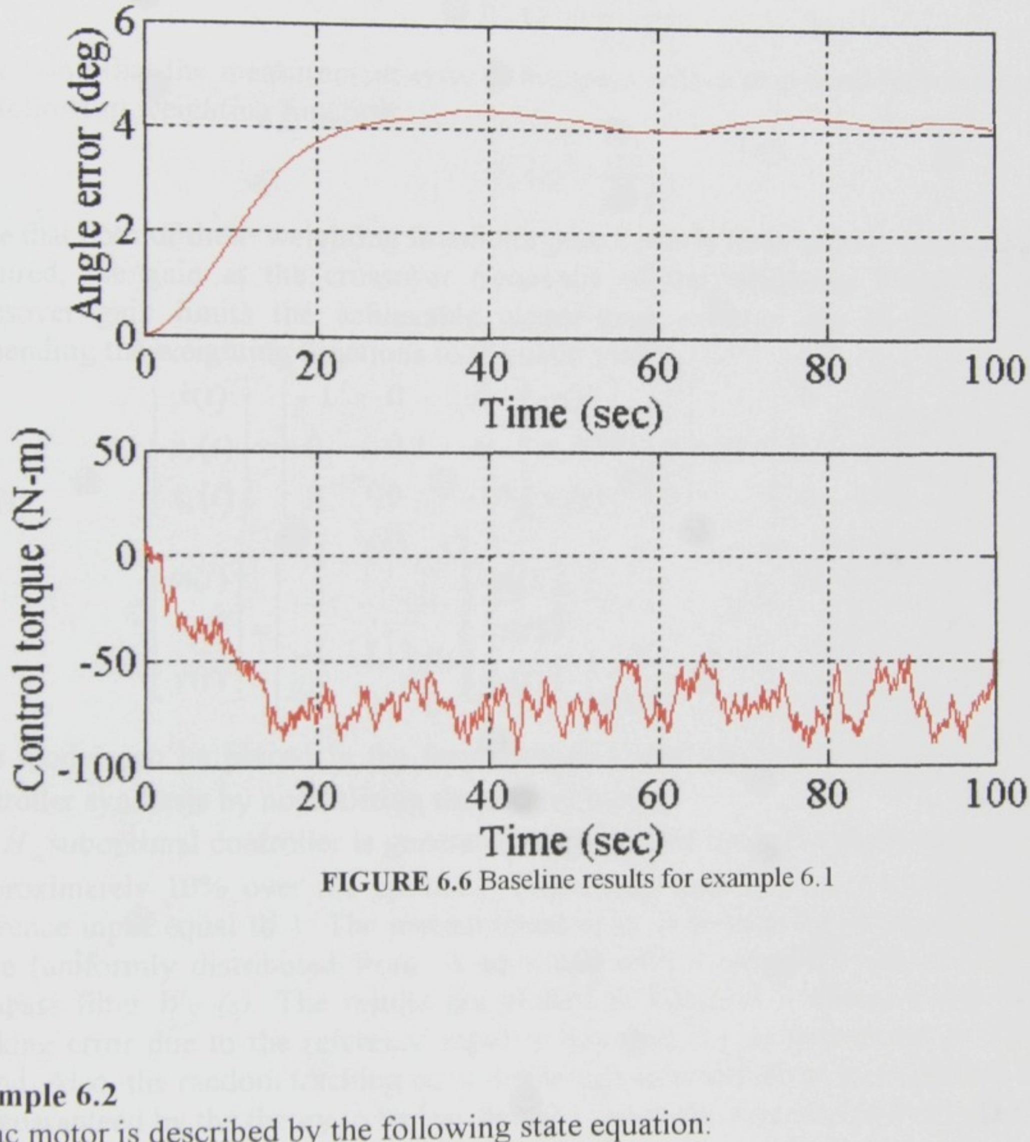


FIGURE 6.6 Baseline results for example 6.1

Example 6.2

An ac motor is described by the following state equation:

$$\dot{x}(t) = -x(t) + u(t)$$

where $x(t)$ is the rotational velocity of the shaft, and $u(t)$ is the control input. Noisy measurements of the rotational velocity error are available:

$$m(t) = r(t) - x(t) - v(t)$$

where $r(t)$ is the reference input.

The measurement noise and reference input are both assumed to be bounded by 1. The normalized reference output is then defined as

$$y(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t).$$

The control is included as a reference output to keep the control finite. But the control weight is small, which means that this term should not significantly affect the performance.

Performing H_∞ optimisation, we find that the minimum achievable ∞-norm for this system is 1. This bound is achievable in this example by using the controller $u(t)= 0$. Obviously, this bound on the error and this controller are not acceptable.

A more appropriate controller can be generated by H_∞ optimization, after adding frequency weighting to the reference input and measurement error, as shown in Figure 6.2. Specifying that the reference input has a bandwidth of 0.1 rad/sec yields the following weighting function:

$$W_R(s) = \frac{0.1}{s+0.1}.$$

Specifying that the measurement error is highpass with a stop band below 10 rad/sec, yields the following weighting function:

$$W_V(s) = \frac{s}{s+10}.$$

Note that both of these weighting functions gave a nearly unity gain in the band of interest. As required, the gain at the crossover frequency of the weighting functions is small. This crossover gain limits the achievable closed-loop ∞-norm due to the constraint (6.42). Appending the weighting functions to the plant yields

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \\ \dot{x}_v(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \\ x_v(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix};$$

$$\begin{bmatrix} m(t) \\ \cdots \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ \cdots & \cdots & \cdots \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_r(t) \\ x_v(t) \end{bmatrix} + \begin{bmatrix} 0 & \vdots & 0 & 1 \\ \cdots & & \cdots & \cdots \\ 0 & \vdots & 0 & 0 \\ 0.001 & \vdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ \cdots \\ w_1(t) \\ w_2(t) \end{bmatrix}.$$

This model can be placed in the form [see (6.1) and (6.2)] appropriate for H_∞ suboptimal controller synthesis by normalizing the control input.

An H_∞ suboptimal controller is generated for this plant using the performance bound $\gamma = 0.1$ (approximately 10% over the optimal). The closed-loop system is then simulated with the reference input equal to 1. The measurement error is formed by passing discrete-time white noise (uniformly distributed from –1 to 1 and with a sampling time of 0.01) through the highpass filter $W_V(s)$. The results are plotted in Figure 6.7. Observe that the steady-state tracking error due to the reference input is less than 0.1 as guaranteed by the performance bound. Also, the random tracking error due to measurement error is less than 0.1. This error is not guaranteed by the theory to be less than 0.1, since the measurement error is not sinusoidal. Still, the closed-loop ∞-norm provides a reasonable gauge of the output size.

The control input is considerably less than the guaranteed bound of 100, because the control input is not limited by the specified bound on the control. Instead, the control input is limited by the bound on the gain from the measurement error to the tracking error. This bound limits the control input, since large feedback gains result in a large coupling between the measurement noise and the tracking error.

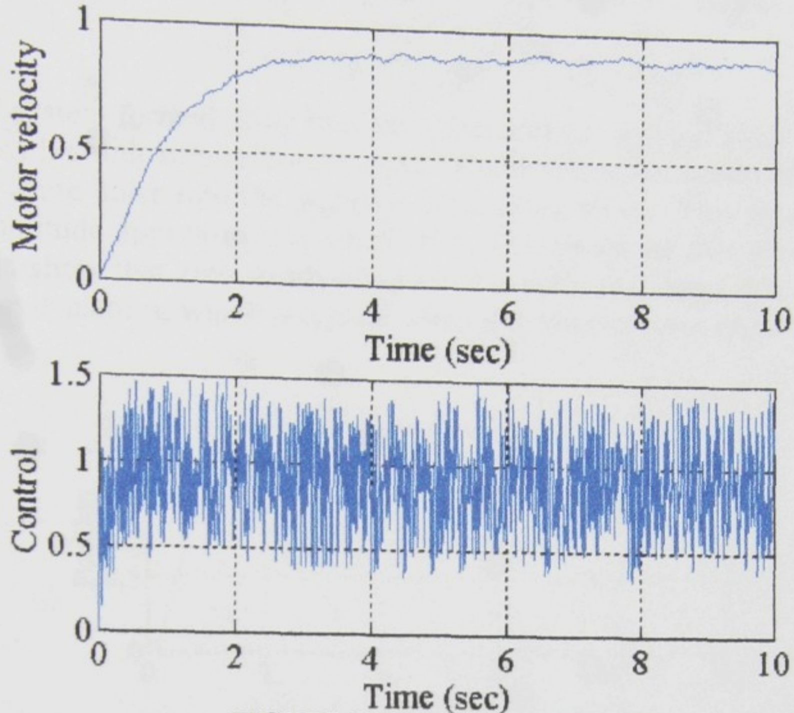


FIGURE 6.7 Results for example 6.2

Example 6.3

A valve is constructed with a spring on the "flapper" so that if power is removed the valve closes . The control input is a torque applied to the flapper. The state equation for this valve is

$$\dot{x}(t)=\begin{bmatrix}0 & 1\\ -10 & -1\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t)+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}r(t)\\ v(t)\end{bmatrix},$$

where $r(t)$ is the reference input and $v(t)$ is measurement error. The reference input is assumed to be generated by application of a normalized input to the filter (6.49). The measurement consists of the difference between the position of the valve and the desired position, plus a measurement error. The measurement noise is assumed to be the output of a highpass filter:

$$W_v(s)=\frac{s}{s+10}.$$

appending this filter and (6.49) to the plant yields the following state model:

$$\dot{x}(t)=\begin{bmatrix}0 & 1 & 0 & 0\\ -10 & -1 & 0 & 0\\ 0 & 0 & -\varepsilon & 0\\ 0 & 0 & 0 & -10\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{r}(t)\\ \dot{v}(t)\end{bmatrix}+\begin{bmatrix}0 & \vdots & 0 & 0\\ 1 & \vdots & 0 & 0\\ 0 & \vdots & 1 & 0\\ 0 & \vdots & 0 & 10\end{bmatrix}\begin{bmatrix}u(t)\\ \cdots\\ r_0(t)\\ v_0(t)\end{bmatrix};$$

$$\begin{bmatrix}m(t)\\ \cdots\\ y(t)\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1 & -1\\ \cdots & \cdots & \cdots & \cdots\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ r(t)\\ v(t)\end{bmatrix}+\begin{bmatrix}0 & \vdots & 0 & 1\\ \cdots & & \cdots & \cdots\\ 0 & \vdots & 0 & 0\\ 0.001 & \vdots & 0 & 0\end{bmatrix}\begin{bmatrix}u(t)\\ \cdots\\ r_0(t)\\ v_0(t)\end{bmatrix}.$$

A suboptimal H_∞ controller is generated with ε = 0.01 and γ = 1100 (approximately 10% over optimal). This controller is described by the following transfer function:

$$G_c(s)=\frac{99(s+10)(s^2+s+10)}{(s+0.03)(s+4.4)(s^2+13.6s+112)}.$$

Integral action is then added by replacing the pole at 0.03 with a pole at the origin:

$$G_c(s)=\frac{99(s+10)(s^2+s+10)}{s(s+4.4)(s^2+13.6s+112)}.$$

The feedback system formed using this controller and the original plant is stable. This system is simulated with a unit step reference input. The measurement error is generated by putting discrete-time white noise into the highpass filter given above. This measurement noise has a maximum amplitude approximately equal to 1. The results of this simulation are plotted in Figure 6.8 and show that zero steady-state error is achieved. Note that this controller inverts the stable plant dynamics, which is typical when the measurement errors are small.

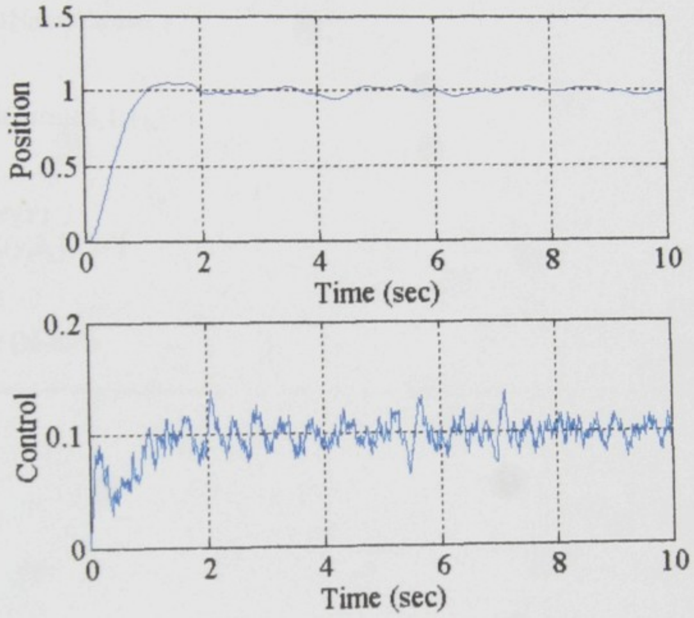


FIGURE 6.8 Integral control for example 6.3

Table 6.1 MATLAB commands to compute Hinf controller design and simulation for Example 6.1

```
clear,clc,
% Define the wind torque bound.
Wb=70;
% Define the output weighting.
Wq=13;
% Define the plant model (with normalized inputs).
A=[0  1
  0 -0.1];
Bu=[0 0.001]';
Bw=[  0   0
   0.001*Wb 0];
Cy=[Wq 0
   0 0];
Cm=[1 0];
Dyw=zeros(2);
Dyu=[0 1]';
Dmw=[0 1];
Dmu=0;
Plant=pck(A,[Bw Bu],[Cy;Cm],[Dyw Dyu;Dmw Dmu]);
% Define the performance bound.
gamma=90;
gm2=1/gamma/gamma;
```

```
% Generate the steady-state H-infinity suboptimal controller.
[Kss,Gcl]=hinfsyn(Plant,1,1,gamma,gamma,0.1);
% Define the time vector for simulation.
t=0:0.1:100;
% Define the inputs for simulating the closed loop system.
infun='[1; randn(1)]';
wv=siggen(infun,t);
% Simulate the closed loop system.
y=trsp(Gcl,wv);
% Generate the unweighted output.
yscaled=mmult([1/Wq 0;0 1],y);
% Plot the simulation results.
set(0,'DefaultAxesFontName','times')
set(0,'DefaultAxesFontSize',16)
set(0,'DefaultTextFontName','times')
figure(1)
clf
subplot(211),vplot(sel(yscaled,1,1),'r-')
grid
xlabel('Time (sec)')
ylabel('Angle error (deg)')
subplot(212),vplot(sel(y,2,1),'r-')
grid
xlabel('Time (sec)')
ylabel('Control torque (N-m)')
```

7 Conclusions

This is the final chapter of the thesis. It will summarize the contribution of the thesis both theory and practice.

7.1 What has been done....

Several important results of this thesis are knowledge about robust control, analysis of this control system and behaviour of some plant by this kind of control system. Robust stability is one of the most important subject which during this thesis was widely discussed. Furthermore this thesis has been focused on a H_∞ linear optimal control method. According to our knowledge, the solution, presented in the thesis, represents the first attempt to the basic concepts and fundamental issues of control using state model of system, mathematically well-defined methods to improve unstructured and structured uncertainty, robust stability and full information control and estimation.
The contribution of this thesis may be summarized into the following items:

- **Basic concepts and fundamental issues of control using state model of system**

The state equations in both continuous-time and discrete-time were presented. The state-space and transfer function models of general feedback control system were discussed. In this part of thesis we defined the concepts of controllability and observability and practical tests for determining if a system is controllable and/or observable. Principal gains, which define a range of frequency-dependent gains for the system with application of the singular value decomposition were mentioned. Tracking performance was defined as the ability of the control system to match the output to a desired value. Cost function provided a means of comparing the performance of competing control system design.

- **Robustness**

A number of uncertainty models are presented: single transfer function perturbations, termed unstructured uncertainty; and multiple transfer function perturbations, termed structured uncertainty. Parametric perturbations, which arise frequently in applications, are special case of transfer function perturbations where the transfer function is a scalar. A feedback system is termed robustly stable if the resulting system is internally stable for all admissible perturbations. The analysis of stability robustness of systems with structure uncertainty leads to the defined of structure singular value. The **D**-scaling method for computing bounds on the SSV was presented. A feedback system is said to possess robust performance if the resulting system is internally stable and meets certain performance objectives for all admissible perturbations.

- **Controller parameterization**

The H_∞ optimal full information control problem is to find a feedback controller that utilizes the plant state and the disturbance input, and minimizes the closed-loop system ∞-norm. This feedback controller is also required to internally stabilize the system in the infinite-time case. Differential game theory is used to generate a solution to the full information problem. A suboptimal controller is generated that satisfies a bound on the closed-loop system ∞-norm. This suboptimal controller is state feedback with a gain that is given in terms of the solution of Hamiltonian equation or, equivalently, in terms of solution of a riccati equation. The H_∞ estimation problem is to estimate a linear combination of the plant state, given measurements and know plants inputs .The resulting estimator should minimize the ∞-norm of the transfer function between the disturbance inputand the estimation error.

- **H_∞ Controller**

The H_∞ optimal control problem is to find a feedback controller that minimizes the closed-loop system ∞-norm. The solution of this control problem can be divided into the synthesis of full information controller and the synthesis of an output estimator. This structure for the H_∞ controller design is similar to LQG controller, but technically violates the separation principal. The estimator in this design depends on the full information control and also incorporates the worst-case disturbance. H_∞ control is often desirable when the specification are given in terms of bounds on the output.

The structured singular value of the closed-loop plants can be used to specify robust performance. The maximum of this SSV can be used as a cost function for an optimal control problem. Minimizing this cost function then maximizes the robust performance of he system.

- **Matlab**

Efficient design and development of control systems requires availability of a programming environment offering high-level support for matrix manipulations and simulations as well as easy-to-use user interface including graphing capabilities. Multiple tools satisfying these requirements were developed for PC applications, most of them taking inspiration in Matlab computing environment by MathWork, Inc.No similar support, however, is available for the parallel computing systems.

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Appendix

A1 L_2 Stability

An alternative definition of input-output stability can be given in terms of the system gain, where the gain is defined in terms of the signal 2-norm.

DEFINITION: A system is L_2 stable if for all inputs with finite signal 2-norm, the system gain is bounded:

$$\frac{\|y(t)\|_2}{\|u(t)\|_2} < M$$

THEOREM: A causal, linear time-invariant system is L_2 stable if and only if all of its poles have negative real parts.

PROOF: The proof begins by showing that the system is L_2 stable if all of the poles have negative real parts. The 2-norm of the output can be written in terms of the impulse response:

$$\|y(t)\|_2^2 = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\mathbf{g}(\tau_1)u(t-\tau_1)d\tau_1\right\}^T\left\{\int_{-\infty}^{\infty}\mathbf{g}(\tau_2)u(t-\tau_2)d\tau_2\right\}dt$$

$$= tr\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}^T(\tau_1)\mathbf{g}(\tau_2)u(t-\tau_2)u^T(t-\tau_1)d\tau_2 d\tau_1 dt\right\}.$$

The fact that the trace is invariant under cyclic permutations (A2.3) has been used in generating the expression containing the triple integral. An integral is bounded from above by the integral of the absolute value:

$$\|y(t)\|_2^2 \le tr\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{g}^T(\tau_1)||\mathbf{g}(\tau_2)||u(t-\tau_2)||u^T(t-\tau_1)|d\tau_2 d\tau_1 dt\right\}.$$

Interchanging the order of integration yields

$$\|y(t)\|_2^2 \le tr\left\{\int_{-\infty}^{\infty}|\mathbf{g}^T(\tau_1)|\int_{-\infty}^{\infty}|\mathbf{g}(\tau_2)|\left\{\int_{-\infty}^{\infty}|u(t-\tau_2)||u^T(t-\tau_1)|dt d\tau_2 d\tau_1\right\}d\tau_2 d\tau_1\right\}.$$

The final integral is a correlation that is maximized when $\tau_1 = \tau_2$, implying that

$$\|y(t)\|_2^2 \le tr\left\{\int_{-\infty}^{\infty}|\mathbf{g}^T(\tau_1)|d\tau_1\int_{-\infty}^{\infty}|\mathbf{g}(\tau_2)|d\tau_2\left\{\int_{-\infty}^{\infty}|u(t)||u^T(t)|dt\right\}\right\}.$$

Given that the system is BIBO stable (i.e., all poles have negative real parts), every element in the impulse response is absolutely integrable:

$$\|y(t)\|_2^2 \le tr\left\{M^T M\int_{-\infty}^{\infty}|u(t)||u^T(t)|dt\right\} = \int_{-\infty}^{\infty}|u^T(t)|M^T M|u(t)|dt$$

$$\le \int_{-\infty}^{\infty}|u^T(t)|\overline{\sigma}(M^T M)|u(t)|dt = \overline{\sigma}(M^T M)\int_{-\infty}^{\infty}u^T(t)u(t)dt \qquad \text{A(1.1)}$$

$$= \overline{\sigma}(M^T M)\|u(t)\|_2^2,$$

Where

$$\mathbf{M} = \int_{-\infty}^{\infty}|\mathbf{g}(\tau)|d\tau$$

Taking the square root of (A1.1), we hav

$$\|y(t)\|_2 \leq \sqrt{\bar{\sigma}\left(\mathbf{M}^T\mathbf{M}\right)}\|u(t)\|_2,$$

Which shows the system is L_2 stable when all poles have negative real parts.

It remains to show that L_2 stability implies that all poles have negative real parts. Let

$$u(t) = u_0\delta(t).$$

Then

$$\|y(t)\|_2^2 = \int_0^\infty u_0^T \mathbf{g}^T(t)\mathbf{g}(t)u_0 dt.$$

If system has an unstable pole, then the impulse response contains an unstable mode (it is assumed that there are no pole-zero cancellations), and some element of $\mathbf{g}(t)$ does not approach zero as time goes to infinity. Therefore, the above integral is infinite for some u_0.

A2 Controllability and Observability Grammians

Given the stable system

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t); y(t) = \mathbf{C}x(t) \tag{A2.1}$$

The controllability grammian is

$$\mathbf{L}_c = \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt. \tag{A2.2}$$

THEOREM: The controllability grammian can be computed by solving the following Lyapunov equation:

$$\mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T. \tag{A2.3}$$

PROOF: Differentiating the integral yields

$$\frac{de^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}}{dt} = \mathbf{A}e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} + e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t}\mathbf{A}^T.$$

Integrating both sides of this equation yields

$$\int_0^\infty \frac{de^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}}{d\tau} d\tau = \mathbf{A}\int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau + \int_0^\infty e^{\mathbf{A}t}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau\mathbf{A}^T$$

$$= \mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T;$$

$$e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau}\Big|_{\tau=0}^{\infty} = \mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T.$$

The expression on the left equals zero at the upper limit, since (A2.1) is stable. Evaluating at the lower limit then yields

$$-\mathbf{B}\mathbf{B}^T = \mathbf{A}\mathbf{L}_c + \mathbf{L}_c\mathbf{A}^T.$$

The observability grammian of the system (A2.1) is defined as

$$\mathbf{L}_0 = \int_0^\infty e^{\mathbf{A}^T t}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}t} dt. \tag{A2.4}$$

THEOREM: This observability grammian can be computed by solving the following Lyapunov equation:

$$\mathbf{A}^T\mathbf{L}_0 + \mathbf{L}_0\mathbf{A} = -\mathbf{C}^T\mathbf{C}. \tag{A2.5}$$

PROOF: The proof is similar to that presented for the controllability grammian and is therefore omitted.

THEOREM: If there exists a solution to (A2.3) such that $\mathbf{L}_c$ is positive semidefinite, then

$$\lim_{t\to\infty} e^{\mathbf{A}t}\mathbf{B} = 0. \tag{A2.6}$$

PROOF: The positive semidefinite solution of (A2.3) is given by (A2.2), which can be written

$$\mathbf{L}_c = \int_0^\infty \left\| e^{\mathbf{A}t}\mathbf{B} \right\|_E^2 dt.$$

The existence of this integral requires that the integrand approach zero as time goes to infinity, which implies (A2.6).

THEOREM: If there exists a solution to (A2.5) such that $\mathbf{L}_0$ is positive semidefinite, then

$$\lim_{t\to\infty} \mathbf{C}e^{\mathbf{A}t} = 0. \qquad \text{(A2.7)}$$

PROOF: The proof is similar to that presented for (A2.6) and is therefore omitted.

THEOREM: We are given a system with the state matrix **A** and the output matrix C. If the system is observable and

$$\lim_{t\to\infty} \mathbf{C}e^{\mathbf{A}t} = 0. \qquad \text{(A2.8)}$$

Then the system is stable.

PROOF: Assume that **A** has an eigenvalue λ with a non-negative real part, and an associated eigenvector ϕ:

$$\mathbf{A}\phi = \lambda\phi.$$

The spectral theory of matrices, states that

$$e^{\mathbf{A}t}\phi = e^{\lambda t}\phi.$$

Premultiplying by **C** and taking the limit as time approaches infinity yields

$$0 = \lim_{t\to\infty}\left(\mathbf{C}e^{\mathbf{A}t}\right)\phi = \lim_{t\to\infty} e^{\lambda t}(\mathbf{C}\phi).$$

This implies that $\mathbf{C}\phi = 0$, since

$$\lim_{t\to\infty} e^{\lambda t} \neq 0.$$

If $\mathbf{C}\phi = 0$, the homogeneous (no input) system response with the initial condition $x(0) = \phi$ is

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\phi = e^{\lambda t}(\mathbf{C}\phi) = 0.$$

Further, the homogeneous system response with the initial condition $x(0) = 2\phi$ is identical to the above response:

$$y(t) = \mathbf{C}e^{\mathbf{A}t}2\phi = 2e^{\lambda t}(\mathbf{C}\phi) = 0.$$

Therefore, the state cannot be found from a record of the output, and the system unobservable. This contradiction indicates that **A** has no eigenvalues with non-negative real parts; that is, the system is stable.

A3 The Push-Through Theorem

THEOREM:

$$(\mathbf{I}+\mathbf{GK})^{-1}\mathbf{G} = \mathbf{G}(\mathbf{I}+\mathbf{KG})^{-1}. \qquad \text{(A8.2)}$$

PROOF:

$$\mathbf{G}+\mathbf{GKG} = \mathbf{G}+\mathbf{GKG}.$$
$$(\mathbf{I}+\mathbf{GK})\mathbf{G} = \mathbf{G}(\mathbf{I}+\mathbf{KG});$$
$$(\mathbf{I}+\mathbf{GK})^{-1}(\mathbf{I}+\mathbf{GK})\mathbf{G}(\mathbf{I}+\mathbf{KG})^{-1}$$
$$= (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{G}(\mathbf{I}+\mathbf{GK})(\mathbf{I}+\mathbf{KG})^{-1};$$
$$\mathbf{G}(\mathbf{I}+\mathbf{GK})^{-1} = (\mathbf{I}+\mathbf{KG})\ \ -1\mathbf{G}.$$

A4 Properties of the System ∞-Norm

The system ∞-norm is defined in terms of the transfer function matrix of the system:

$$\|\mathbf{G}\|_\infty = \sup_\omega \bar{\sigma}(\mathbf{G}(j\omega)). \tag{A4.1}$$

The ∞-norm represents the maximum gain of the system over all possible sinusoidal inputs.
THEOREM: The 2-norm of the output can be bounded by the product of the system ∞-norm and the 2-norm of the input:

$$\|y\|_2 = \|\mathbf{G}u\|_2 \le \|\mathbf{G}\|_\infty \|u\|_2 . \tag{A4.2}$$

PROOF: The signal 2-norm of the output is given as

$$\|y\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} y^\dagger(j\omega)y(j\omega)d\omega} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|y(j\omega)\|_E^2 d\omega},.$$

where $\|\bullet\|_E$ is the Euclidean norm. The Euclidean norm of the output is bounded:

$$\|y(j\omega)\|_E^2 = \bar{\sigma}(j\omega)\|u(j\omega)\|_E^2 .$$

The signal 2-norm of the output can then be bounded:

$$\|y\|_2 \le \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\sigma}(j\omega)\|u(j\omega)\|_E^2 d\omega} .$$

The maximum singular value at a specific frequency is bounded by the supremum over all frequencies:

$$\|y\|_2 \le \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\sigma}(j\omega)\|u(j\omega)\|_E^2 d\omega} \le \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \{\sup \bar{\sigma}(j\omega)\}\|u(j\omega)\|_E^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathbf{G}\|_\infty^2 \|u(j\omega)\|_E^2 d\omega} = \|\mathbf{G}\|_\infty \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|u(j\omega)\|_E^2 d\omega} = \|\mathbf{G}\|_\infty \|u\|_2$$

The bound of the preceding proof is nearly achieved for some input. In fact, the ∞-norm is given by the supremum:

$$\|\mathbf{G}\|_\infty = \sup_{u\neq 0} \frac{\|\mathbf{G}u\|_2}{\|u\|_2} \tag{A4.3}$$

THEOREM:

$$\|\mathbf{G}\|_\infty = \sup_{u\neq 0} \frac{\|\mathbf{G}u\|_2}{\|u\|_2}$$

PROOF: Using (A9.2),

$$\|\mathbf{G}\|_\infty \ge \frac{\|\mathbf{G}u\|_2}{\|u\|_2} \forall u \neq 0$$

which implies

$$\|\mathbf{G}\|_\infty \ge \sup_{u\neq 0} \frac{\|\mathbf{G}u\|_2}{\|u\|_2}$$

The input direction and frequency that yield the maximums in the ∞-norm are defined as follows:

$$\|\mathbf{G}\|_\infty = \sup_\omega \bar{\sigma}(\mathbf{G}(j\omega)) = \bar{\sigma}(\mathbf{G}(j\omega_0)) = \max_{u(j\omega_0)} \frac{\mathbf{G}(j\omega_0)u(j\omega_0)}{u(j\omega_0)} = \frac{\mathbf{G}(j\omega_0)u_0}{u_0} .$$

where u_0 is a unit vector. Note that u_0 is the right singular vector associated with the largest singular value. For the input

$$u(j\omega) = u_0\delta(\omega - \omega_0),$$

$$\frac{\|\mathbf{G}u\|_2}{\|u\|_2} = \frac{\|\mathbf{G}(j\omega_0)u_0\delta(\omega-\omega_0)\|_2}{\|u_0\delta(\omega-\omega_0)\|_2} = \frac{\mathbf{G}(j\omega_0)u_0\|\delta(\omega-\omega_0)\|_2}{u_0\|\delta(\omega-\omega_0)\|_2} = \frac{\mathbf{G}(j\omega_0)u_0}{u_0} = \|\mathbf{G}u\|_\infty$$

The above proof is not rigorous, since the supremum gas been treated as a maximum. The supremum says only that the limit can be approached arbitrarily closely, not that the value can be achieved. The theorem remains unchanged when using the actual supremum, but the proof must include a limiting argument that has been omitted.

In addition to the ordinary properties of norms, the infinity norm gas the following property:

THEOREM:

$$\|\mathbf{GH}\|_\infty \leq \|\mathbf{G}\|_\infty \|\mathbf{H}\|_\infty \tag{A4.4}$$

PROOF: By equation (A9.3), for any $u \neq 0$,

$$\|\mathbf{GH}u\|_2 \leq \|\mathbf{G}\|_\infty \|\mathbf{H}u\|_2 \leq \|\mathbf{G}\|_\infty \|\mathbf{H}\|_\infty \|u\|_2$$

Rearranging and taking the supremum of both sides of this equation concludes the proof:

$$\|\mathbf{GH}\|_\infty = \sup_{u\neq 0} \frac{\|\mathbf{GH}u\|_2}{\|u\|_2} \leq \|\mathbf{G}\|_\infty \|\mathbf{H}\|_\infty$$

The infinity norm can be generalized to operate over finite time intervals:

$$\|\mathbf{GH}\|_{\infty,[t_0,t_f]} = \sup_{u\neq 0} \frac{\|\mathbf{G}u\|_{2,[t_0,t_f]}}{\|u\|_{2,[t_0,t_f]}}. \tag{A4.5}$$

THEOREM: The finite-time ∞-norm of the time-varying system described by the state model

$$\dot{x}(t) = \mathbf{A}(t)\,x(t) + \mathbf{B}(t)\,u(t);\, y(t) = \mathbf{C}(t)\,x(t) + \mathbf{D}(t)\,u(t)$$

is finite:

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \leq \infty$$

provided all matrices are continuous.

PROOF: The solution of this state model (assuming zero initial conditions) is

$$\mathbf{G}u = y(t) = \int_{t_0}^{t_f} \mathbf{g}(t,\tau)u(\tau)d\tau + \mathbf{D}(t)u(t)$$

where

$$\mathbf{g}(t,\tau) = \mathbf{C}(t)\mathbf{\Phi}(t,\tau)\mathbf{B}(t)$$

is the impulse response of the system without the input-to-output coupling term $\mathbf{D}(t)$. The signal 2-norm of the output is

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} = \left\| \int_{t_0}^{t_f} \mathbf{g}(t,\tau)u(\tau)d\tau + \mathbf{D}(t)u(t) \right\|_{2,[t_0,t_f]}$$

Using the triangle inequality, and noting that the integral is simply a sum, we have

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \leq \int_{t_0}^{t_f} \|\mathbf{g}(t,\tau)u(\tau)\|_{2,[t_0,t_f]} d\tau + \|\mathbf{D}(t)u(t)\|_{2,[t_0,t_f]}$$

$$= \int_{t_0}^{t_f} \sqrt{\|\mathbf{g}(t,\tau)u(\tau)\|_E^2 dt} d\tau + \sqrt{\int_{t_0}^{t_f} \|\mathbf{D}(t)u(t)\|_E^2 dt}$$

Where the norms in the final expression are Euclidean vector norms. The maximum singular value provides a bound on the gain of a matrix, which implies

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \leq \int_{t_0}^{t_f} \sqrt{\bar{\sigma}^2[\mathbf{g}(t,\tau)]\,\|u(\tau)\|_E^2 dt} d\tau + \sqrt{\int_{t_0}^{t_f} \bar{\sigma}^2[\mathbf{D}(t)]\,\|u(t)\|_E^2 dt}$$

Noting that $\mathbf{D}(t)$ and $\mathbf{g}(t,\tau)$ are continuous and therefore bounded over a finite time interval, a maximum value for the maximum singular values in this expression exists:

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \le \int_{t_0}^{t_f} \sqrt{\max_{t,\tau\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{g}(t,\tau)]\right\} \|u(\tau)\|_E^2 dt d\tau} + \sqrt{\int_{t_0}^{t_f} \max_{t\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{D}(t)]\right\} \|u(t)\|_E^2 dt}$$

Since these maximums are both independent of t and τ, they can be removed from the integrals:

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \le \sqrt{(t_f - t_0) \max_{t,\tau\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{g}(t,\tau)]\right\}} \int_{t_0}^{t_f} \|u(\tau)\|_2 dt + \sqrt{\max_{t\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{D}(t)]\right\}} \sqrt{\int_{t_0}^{t_f} \|u(t)\|_2^2 dt}.$$

Now, the integral of the norm can be bounded using the Cauchy-Schwarz inequality

$$\int_{t_0}^{t_f} \|u(\tau)\|_2 d\tau = \int_{t_0}^{t_f} \|u(\tau)\|_2 .1 d\tau \le \sqrt{t_f - t_0} \sqrt{\int_{t_0}^{t_f} \|u(\tau)\|_2^2 d\tau}.$$

Substituting this result into the above inequality yields

$$\|\mathbf{G}u\|_{2,[t_0,t_f]} \le \left((t_f - t_0) \sqrt{\max_{t,\tau\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{g}(t,\tau)]\right\}} + \sqrt{\max_{t\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{D}(t)]\right\}} \right) \|u\|_{2,[t_0,t_f]}^2.$$

Dividing by the signal 2-norm of the input yields

$$\frac{\|\mathbf{G}u\|_{2,[t_0,t_f]}}{\|u\|_{2,[t_0,t_f]}} \le (t_f - t_0) \sqrt{\max_{t,\tau\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{g}(t,\tau)]\right\}} + \sqrt{\max_{t\in[t_0,t_f]} \left\{\bar{\sigma}^2[\mathbf{D}(t)]\right\}},$$

Which shows that the finite time system ∞-norm is bounded.

A5 A Bound on the System ∞-Norm

A bound on the system ∞-norm can be given based on the eigenvalues of a Hamiltonian matrix. A related bound can be given in terms of the solution of an algebraic Riccati equation. These theorems are both versions of the bounded real lemma.

THEOREM: The infinity norm of the generic stable system,

$$\dot{x}(t) = \mathbf{A}(t)\, x(t) + \mathbf{B}(t)\, u(t) \tag{A5.1a}$$

$$y(t) = \mathbf{C}(t)\, x(t) + \mathbf{D}(t)\, u(t) \tag{A5.1b}$$

is bounded:

$$\|\mathbf{GH}\|_\infty = \sup_\omega \bar{\sigma}[\mathbf{G}(j\omega)] = \sup_\omega \bar{\sigma}\left[\mathbf{C}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right] < \gamma \tag{A5.2}$$

if and only if

$$\bar{\sigma}(\mathbf{D}) < \gamma, \tag{A5.3a}$$

and the Hamiltonian matrix

$$\mathcal{H}_\infty = \begin{bmatrix} \mathbf{A} + \mathbf{B}(\gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T\mathbf{C} & \vdots & \mathbf{B}(\gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D})^{-1}\mathbf{B}^T \\ \cdots & & \cdots\cdots\cdots \\ -\mathbf{C}^T(\mathbf{I} + \mathbf{D}(\gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T)\mathbf{C} & \vdots & -\mathbf{A}^T - \mathbf{C}^T\mathbf{D}(\gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D})^{-1}\mathbf{B}^T \end{bmatrix} \tag{A5.3b}$$

has no eigenvalues on the imaginary axis.

PROOF: Condition (A5.3a) is obtained by noting that $\mathbf{D}$ is the high-frequency limit of the transfer function:

$$\lim_{\omega\to\infty} \mathbf{G}(j\omega) = \lim_{\omega\to\infty} \left(\mathbf{C}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right) = \mathbf{D}$$

Taking the maximum singular value yields

$$\lim_{\omega\to\infty} \bar{\sigma}[\mathbf{G}(j\omega)] = \bar{\sigma}(\mathbf{D}) \tag{A5.4}$$

and the infinity norm is bounded from below:

$$\|\mathbf{G}\|_\infty \ge \bar{\sigma}(\mathbf{D})$$

Therefore, the infinity norm may be less than γ only if the maximum singular value of $\mathbf{D}$ is less than γ.

Condition (A5.3b) is then derived assuming that (A5.3a) is satisfied. The maximum singular value of the transfer function is bounded:

$$\bar{\sigma}[\mathbf{G}(j\omega)] < \gamma,$$

if and only if

$$eig\left[\mathbf{G}^{\dagger}(j\omega)\mathbf{G}(j\omega)\right] < \gamma^2$$

where eig[•] indicates the eigenvalues, and the inequality implies that all of the eigenvalues are less than γ^2. Note that the eigenvalues of this product are strictly real, and the singular values of $\mathbf{G}(j\omega)$ are the positive square roots of these eigenvalues. Further, the eigenvalues of this product are less than γ^2 if and only if :

$$eig\left[\gamma^2\mathbf{I} - \mathbf{G}^{\dagger}(j\omega)\mathbf{G}(j\omega)\right] > 0. \tag{A5.5}$$

This bound is satisfied in the limit as ω approaches infinity (A5.4) provided (A5.3a) is assumed. Further, the eigenvalues in (A5.5) are continuous function of ω. Therefore, the bound (A5.5) is satisfied for all frequencies if and only if the eigenvalues in (A5.5) do not cross the origin; that is

$$\det\left[\gamma^2\mathbf{I} - \mathbf{G}^{\dagger}(j\omega)\mathbf{G}(j\omega)\right] \neq 0.$$

for all ω. This statement is equivalent to the condition that $\left[\gamma^2\mathbf{I} - \mathbf{G}^{\dagger}(j\omega)\mathbf{G}(j\omega)\right]$ has no zeros on the imaginary axis.

The system zeros can be found from the state model. The state model of $\mathbf{G}^{\dagger}(j\omega)$ is

$$\begin{aligned}\dot{\tilde{x}}(t) &= -\mathbf{A}^T\tilde{x}(t) + \mathbf{C}^T u(t)\\ y(t) &= \mathbf{B}^T\tilde{x}(t) + \mathbf{D}^T u(t),\end{aligned}$$

which can be verified by computing the transfer function of this system and comparing it to $\mathbf{G}(j\omega)$. The state model of $\left[\gamma^2\mathbf{I} - \mathbf{G}^{\dagger}(j\omega)\mathbf{G}(j\omega)\right]$ is then

$$\begin{bmatrix}\dot{x}(t)\\ \cdots\cdots \\ \dot{\tilde{x}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{C}^T\mathbf{C} & \vdots & -\mathbf{A}^T\end{bmatrix}\begin{bmatrix}x(t)\\ \cdots\cdots \\ \tilde{x}(t)\end{bmatrix} - \begin{bmatrix}-\mathbf{B}\\ \cdots\cdots \\ \mathbf{C}^T\mathbf{D}\end{bmatrix}u(t);$$

$$y(t) = \begin{bmatrix}\mathbf{D}^T\mathbf{C} & \vdots & \mathbf{B}^T\end{bmatrix}\begin{bmatrix}x(t)\\ \cdots\cdots \\ z(t)\end{bmatrix} + (\gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D})u(t)$$

The zeros of this system are the solutions of

$$\det\begin{bmatrix} s\mathbf{I}\begin{bmatrix}\mathbf{A} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{C}^T\mathbf{C} & \vdots & -\mathbf{A}^T\end{bmatrix} & \vdots & \begin{bmatrix}-\mathbf{B}\\ \cdots\cdots \\ \mathbf{C}^T\mathbf{D}\end{bmatrix}\\ \cdots\cdots\cdots\cdots\cdots\cdots & & \cdots\cdots\\ \begin{bmatrix}\mathbf{D}^T\mathbf{C} & \vdots & \mathbf{B}^T\end{bmatrix} & \vdots & \gamma^2\mathbf{I} - \mathbf{D}^T\mathbf{D}\end{bmatrix} = 0$$

The determinant of this block matrix can be expanded , which yields

$$\det\left(\gamma^2\mathbf{I}-\mathbf{D}^T\mathbf{D}\right)\det\left(s\mathbf{I}-\begin{bmatrix}\mathbf{A} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{C}^T\mathbf{C} & \vdots & -\mathbf{A}^T\end{bmatrix}-\begin{bmatrix}-\mathbf{B}\\ \cdots\cdots\\ \mathbf{C}^T\mathbf{D}\end{bmatrix}(\gamma^2\mathbf{I}-\mathbf{D}^T\mathbf{D})^{-1}\left[\mathbf{D}^T\mathbf{C} \;\vdots\; \mathbf{B}^T\right]\right)=0.$$

The first determinant is not equal to zero due to the bound on $\mathbf{D}$, so the second determinant must equal zero:

$$\det\left(s\mathbf{I}-\begin{bmatrix}\mathbf{A} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{C}^T\mathbf{C} & \vdots & -\mathbf{A}^T\end{bmatrix}-\begin{bmatrix}-\mathbf{B}\\ \cdots\cdots\\ \mathbf{C}^T\mathbf{D}\end{bmatrix}(\gamma^2\mathbf{I}-\mathbf{D}^T\mathbf{D})^{-1}\left[\mathbf{D}^T\mathbf{C} \;\vdots\; \mathbf{B}^T\right]\right)=0.$$

Note that this is an eigenequation, and the system $\left[\gamma^2\mathbf{I}-\mathbf{G}^\dagger(j\omega)\mathbf{G}(j\omega)\right]$ has no zeros on the imaginary axis if and only if the matrix

$$\mathcal{H}=\begin{bmatrix}\mathbf{A} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{C}^T\mathbf{C} & \vdots & -\mathbf{A}^T\end{bmatrix}-\begin{bmatrix}-\mathbf{B}\\ \cdots\cdots\\ \mathbf{C}^T\mathbf{D}\end{bmatrix}(\gamma^2\mathbf{I}-\mathbf{D}^T\mathbf{D})^{-1}\left[\mathbf{D}^T\mathbf{C} \;\vdots\; \mathbf{B}^T\right]$$

has no eigenvalues on the imaginary axis.

The eigensolution of a Hamiltonian matrix is related to the solution of an algebraic Riccati equation in the previous chapters. This is also the case for the Hamiltonian matrix appearing in the bound on the infinity norm. The second condition for this bound can be given in terms of this Riccati equation.

THEOREM: The infinity norm of the generic stable system (A5.1) is bounded:

$$\|\mathbf{G}(s)\|_\infty<\gamma, \tag{A5.6}$$

if and only if

$$\bar{\sigma}(\mathbf{D})<\gamma \tag{A5.7a}$$

and there exists a symmetric matrix $\mathbf{P}$ that satisfies the following algebraic Riccati equation:

$$\mathbf{P}(\mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C})+(\mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C})^T\mathbf{P}+\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{C}^T(\mathbf{I}+\mathbf{D}\mathbf{R}^{-1}\mathbf{D}^T)\mathbf{C}=0 \tag{A5.7b}$$

such that

$$\mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} \tag{A5.7c}$$

is stable (i.e., has all eigenvalues with negative real parts). Note that $\mathbf{R}=\gamma^2\mathbf{I}+\mathbf{D}^T\mathbf{D}$.

PROOF OF IF: Suppose a matrix $\mathbf{P}$ exists that satisfies (A5.7b). Performing a similarity transformation on the Hamiltonian matrix (A5.3b) yields

$$\begin{bmatrix}\mathbf{I} & \vdots & 0\\ \cdots & & \cdots\\ -\mathbf{P} & \vdots & \mathbf{I}\end{bmatrix}\mathcal{H}\begin{bmatrix}\mathbf{I} & \vdots & 0\\ \cdots & & \cdots\\ \mathbf{P} & \vdots & \mathbf{I}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} & \vdots & \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\\ \cdots & & \cdots\\ \mathbf{X} & \vdots & -(\mathbf{A}+\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}+\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P})^T\end{bmatrix},$$

where

$$\mathbf{X}=\mathbf{P}\mathbf{A}+\mathbf{A}^T\mathbf{P}+\mathbf{C}^T\mathbf{C}+\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}+\mathbf{C}^T\mathbf{D}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\mathbf{C}^T\mathbf{D}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{C}.$$

Note that $\mathbf{X}$ equals the left side of (A5.7b) and is therefore equal to zero, making the above matrix block triangular. The eigenvalues of this block triangular matrix equal the sum of the eigenvalues of the block on the diagonal, none of which lie on the imaginary axis. Further, since the eigenvalues of a matrix are invariant under similarity transformation, none of the eigenvalues of $\mathcal{H}$ lie on the imaginary axis, which implies $\|\mathbf{G}(s)\|_\infty<\gamma$.

A6 Hilbert Space

A Hilbert space is vector space **H** with an inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ such that the norm defined by

$$|\mathbf{f}| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$$

turns **H** into a complete metric space. If the inner product does not so define a norm, it is instead known as an inner product space.

Examples of Finite-dimensional Hilbert spaces include

1. The real numbers R^n with $\langle \mathbf{v}, \mathbf{u} \rangle$ the vector dot product of **v** and **u**.

2. The complex numbers C^n with $\langle \mathbf{v}, \mathbf{u} \rangle$ the vector dot product of **v** and the complex conjugate of **u**.

An example of an Infinite-dimensional Hilbert space is $\mathbf{L}^n$, the set of all functions $f : R \to R$ such that the integral of f^2 over the whole real Line is finite. In this case, the Inner Product is

$$\langle f, g \rangle = \int f(x) g(x) dx.$$

A Hilbert space is always a Banach space, but the converse need not hold.

A7 L$_2$ - Space

A Hilbert space in which a Bracket Product is defined by

$$\langle \phi, \varphi \rangle \equiv \int \varphi^{\dagger} \phi dx$$

and which satisfies the following conditions

$$\langle \phi, \varphi \rangle^{\dagger} = \langle \phi, \varphi \rangle e$$

$$\langle \phi \mid \lambda_1 \varphi_1 + \lambda_2 \varphi_2 \rangle = \lambda_1 \langle \phi \mid \varphi_1 \rangle + \lambda_2 \langle \phi \mid \varphi_2 \rangle$$

$$\langle \lambda_1 \varphi_1 + \lambda_2 \varphi_2 \mid \phi \rangle = \lambda_1^{\dagger} \langle \phi_1 \mid \varphi \rangle + \lambda_2^{\dagger} \langle \phi_2 \mid \varphi \rangle$$

$$\langle \varphi \mid \varphi \rangle \in R \geq 0$$

$$\left| \langle \varphi_1 \mid \varphi_2 \rangle^2 \right| = \langle \varphi_1 \mid \varphi_1 \rangle \langle \varphi_2 \mid \varphi_2 \rangle.$$

The last of these is Schwarz's Inequality.

To get more information about L$_2$ and Hilbert spaces please