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Some remarks on domination in cubic graphs

Bohdan Zelinka

Department of Discrete Mathematics and Statistics, Technical University, Liberec, Czech Republic

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Abstract

We study three recently introduced numerical invariants of graphs, namely, the signed domination number γ_s, the minus domination number γ^- and the majority domination number γ_{maj}. An upper bound for γ_s and lower bounds for γ^- and γ_{maj} are found, in terms of the order of the graph.

1. Introduction

In this paper we study three numerical invariants of graphs concerning domination. All graphs will be finite, undirected, without loops and multiple edges.

The vertex set of a graph G will be denoted by $V(G)$. If x is a vertex of a graph, then $N[x]$ denotes the *closed neighbourhood* of x, i.e. the set consisting of x and of all vertices adjacent to x. If f is a function which assigns real numbers to vertices of a graph G and $S \subseteq V(G)$, then $f(S)$ is defined as $\sum_{x \in S} f(x)$.

A *signed dominating function* f of a graph G is defined in [3] as a function $f: V(G) \to \{-1, 1\}$ such that $f(N[x]) \geq 1$ for each $x \in V(G)$. A *minus dominating function* f is defined in [2] as a function $f: V(G) \to \{-1, 0, 1\}$ such that $f(N[x]) \geq 1$ for each $x \in V(G)$. Both these concepts are studied in [5]. A *majority dominating function* f is defined in [1] as a function $f: V(G) \to \{-1, 1\}$ such that $f(N[x]) \geq 1$ for at least $\frac{1}{2}|V(G)|$ vertices of G.

The minimum of $f(V(G)$ over all signed (minus, majority) dominating functions f of a graph G is called the *signed* (*minus*, *majority*) *domination number* of G and is denoted by $\gamma_s(G)$ ($\gamma^-(G)$, $\gamma_{\mathrm{maj}}(G)$).

In Section 2 we prove a best possible upper bound for $\gamma_s(G)$, where G is a cubic graph. This solves a problem from [5]. Next, in Section 3, we prove a sharp lower bound for $\gamma^-(G)$, where G is a cubic graph. We conclude the paper, in Section 4, by finding a best possible lower bound for $\gamma_{\mathrm{maj}}(G)$, where G is a cubic graph.

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2. An upper bound for $\gamma_s(G)$

In this section we prove an upper bound for $\gamma_s(G)$, where G is a cubic graph, and also show that this bound is best possible. This solves a problem from [5]. We start by proving the following lemma.

Lemma 1. *Let G be a cubic graph and let $A \subseteq V(G)$. The following assertions are equivalent*:

(i) *There exists a signed dominating function f of G such that $f(x) = -1$ for all $x \in A$, while $f(x) = 1$ for all $x \in V(G) - A$.*

(ii) *The distance between any two distinct vertices of A in G is at least* 3.

Proof. (i) $\Rightarrow$ (ii): Let $x \in A$. Since $f(x) = -1$ and $f(N[x]) \geqslant 1$, it follows that x is adjacent only to vertices which are assigned the value $+1$ by f, i.e. to vertices of $V(G) - A$. Hence, the distance from x to any other vertex of A is at least 2. Suppose there exists a vertex $y \in A$ such that $d(x, y) = 2$. Then there exists a vertex z such that xzy forms a path. But $f(N[z]) \leqslant f(z) + f(x) + f(y) + 1 = 1 + (-1) + (-1) + 1 = 0$, which is a contradiction. As x was chosen arbitrarily, the assertion is proved.

(ii) $\Rightarrow$ (i). Let $f(x) = -1$ for all $x \in A$ and $f(x) = 1$ for all $x \in V(G) - A$. If $x \in A$, then $N[x]$ contains three vertices of $V(G) - A$ and thus $f(N[x]) = 2$. If $x \in V(G) - A$, then $N[x]$ contains at most one vertex of A; for otherwise two distinct vertices of A would be joined by a path of length 2, which is a contradiction. Hence $f(N[x]) \geqslant 2$, which proves that f is a signed dominating function of G. □

We are now ready to prove the main result of this section.

Theorem 1. *If G is a cubic graph of order n, then*

$$\gamma_s(G) \leqslant \tfrac{4}{5} n$$

This bound is best possible.

Proof. Let f be a signed dominating function of G such that $f(V(G)) = \gamma_s(G)$ and let $A = \{x \in V(G) \mid f(x) = -1\}$. Lemma 1 implies that the set A has the property that any two of its distinct vertices are at distance at least 3 apart. Suppose there exists $z \in V(G) - A$ such that $d(z, A) \geqslant 3$. Then $g : V(G) \to \{-1, 1\}$ defined by $g(z) = -1$ and $g(x) = f(x)$ for all $x \in V(G) - \{z\}$ is a signed dominating function such that $g(V(G)) = f(V(G)) - 2$, which is contradiction. Hence, if $z \in V(G) - A$, there exists an $x \in A$ such that $d(z, x) \leqslant 2$. Let $a = |A|$. Since G is cubic, there are at most $3a$ vertices which are at distance 1 from vertices of A and at most $6a$ vertices which are at distance 2 from vertices of A. Therefore $n \leqslant 10a$, which implies $a \geqslant (1/10)n$. Hence $f(V(G) = (n - a) - a = n - 2a \leqslant \frac{4}{5} n$.

We now show that this bound is best possible by constructing a cubic graph G of order 10 such that $\gamma_s(G) = 8 = \frac{4}{5} 10$. Let $V(G) = \{u, v_1, v_2, v_3, w_{11}, w_{12}, w_{21}, w_{22},$

w_{31}, $w_{32}\}$ and let $E(G) = \{uv_1, uv_2, uv_3, v_1w_{11}, v_1w_{12}, v_2w_{21}, v_2w_{22}, v_3w_{31}, v_3w_{32}, w_{11}w_{21}, w_{21}w_{31}, w_{31}w_{12}, w_{12}w_{22}, w_{22}w_{32}, w_{32}w_{11}\}$. Define $f: V(G) \to \{-1, 1\}$ by $f(u) = -1$ and $f(x) = 1$ for all $x \in V(G) - \{u\}$. Then f is a signed dominating function of G, so that $\gamma_s(G) \leqslant f(V(G)) = 8$. Since no two vertices of G are assigned the value -1 by a signed dominating function, equality holds. □

This result was generalized by Henning [4] for r-regular graphs with arbitrary r. Namely he proved that

$$\gamma_s(G) \leqslant \frac{(r+1)^2}{r^2+4r-1} n \quad \text{for } r \text{ odd}$$

and

$$\gamma_s(G) \leqslant \frac{r+1}{r+3} n \quad \text{for } r \text{ even.}$$

3. A lower bound for $\gamma^-(G)$

In this section we determine a lower bound for $\gamma^-(G)$, where G is a cubic graph.

Suppose that f is a minus dominating function of a cubic graph G such that $f(V(G)) = \gamma^-(G)$. We denote $V^+ = \{x \in V(G) \mid f(x) = 1\}$, $V^- = \{x \in V(G) \mid f(x) = -1\}$, $V^0 = \{x \in V(G) \mid f(x) = 0\}$, $v^+ = |V^+|$, $v^- = |V^-|$, $v^0 = |V^0|$.

Before proceeding further, we prove four lemmas.

Lemma 2. $v^+ \geqslant 2v^-$.

Proof. Each vertex $x \in V^-$ is adjacent to at least two vertices of V^+; otherwise $f(N[x]) \leqslant 0$ for some $x \in V^-$. On the other hand, each vertex of V^+ is adjacent to at most one vertex of V^-. The number of edges joining V^+ with V^- is then at least $2v^-$ and at most v^+, which proves the assertion. □

Lemma 3. $v^+ \geqslant \frac{1}{4}n$.

Proof. Each vertex of $V^0 \cup V^-$ is adjacent to at least one vertex of V^+. Therefore $v^0 + v^- \leqslant 3v^+$. This implies $n = v^0 + v^- + v^+ \leqslant 4v^+$, which proves the assertion. □

Lemma 4. $v^- \leqslant \frac{1}{4}n$.

Proof. The set V^- is independent, therefore there are $3v^-$ edges joining vertices of V^- with vertices of $V^+ \cup V^0$. It follows that $v^+ + v^0 \geqslant 3v^-$, so that $n = v^+ + v^- + v^0 \geqslant 4v^-$. Hence $v^- \leqslant \frac{1}{4}n$. □

Lemma 5. $3v^+ \geqslant 5v^- + v^0$.

Proof. The sum of the degrees of the vertices of V^+ is $3v^+$. We shall now speak about degree units rather than about edges. We have $3v^+$ degree units; to each edge with one end vertex (or two end vertices) in V^+ one degree unit (or two degree units) corresponds. We now assign degree units to vertices of V^0 V^- as follows.

Each vertex of V^0 is adjacent to at least one vertex of V^+; thus for each $x \in V^0$ we choose one edge joining x with a vertex of V^+ and assign the degree unit corresponding to this edge to x. In such a way we assign one degree unit to each vertex of V^0. We now show that we can assign five degree units to each vertex of V^-. Let $x \in V^-$. The vertex x is adjacent either to three vertices of V^+, or to two vertices of V^+ and to one vertex of V^0. We assign the degree units corresponding to edges joining x with vertices of V^+ to x. In the second case the vertex $y \in v^0$ adjacent to x is adjacent to two vertices of V^+. One of the degree units corresponding to edges joining y with vertices in V^+ was already assigned to y; we assign the other to x. Note that each vertex of V^+ adjacent to a vertex of V^- is adjacent to at least one other vertex of V^+. In both cases we take two vertices of V^+ adjacent to x, at each of them we take one edge joining it with another vertex of V^+ (these edges may coincide) and we assign the corresponding degree units to x. Thus to each $x \in V^-$ five degree units are assigned. As each vertex of $V^+ \cup V^0$ is adjacent to at most one vertex of V^-, no degree unit is assigned to different vertices. This implies the assertion. $\square$

Theorem 2. *If G is a cubic graph of order n, then*

$$\gamma^-(G) \geqslant \tfrac{1}{4}n.$$

Proof. Lemma 3 implies that $v^+ \geqslant \frac{1}{4}n$. Let $p = v^+ - \frac{1}{4}n$. Then

$$v^- + v^0 = \tfrac{3}{4}n - p. \tag{1}$$

By Lemma 5 we have

$$5v^- + v^0 \leqslant 3v^+ = \tfrac{3}{4}n + 3p. \tag{2}$$

From the inequality (2) we subtract the inequality (1) and divide the result by four. We obtain $v^- \leqslant p$. Now

$$\gamma^-(G) = v^+ - v^- \geqslant \tfrac{1}{4}n + p - p = \tfrac{1}{4}n. \qquad \square$$

Theorem 3. *Let n be a positive integer divisible by four. Then there exists a cubic graph G of order n with the property that for any integer p such that $0 \leqslant p \leqslant \frac{1}{4}n$ there exists a minus dominating function f of G such that $f(V(G)) = \frac{1}{4}n$ and f assigns the value -1 to exactly p vertices.*

Proof. The simplest example of such a graph G is the disjoint union of $\frac{1}{4}n$ complete graphs with four vertices. In p of them we assign the value 1 to two vertices, the value

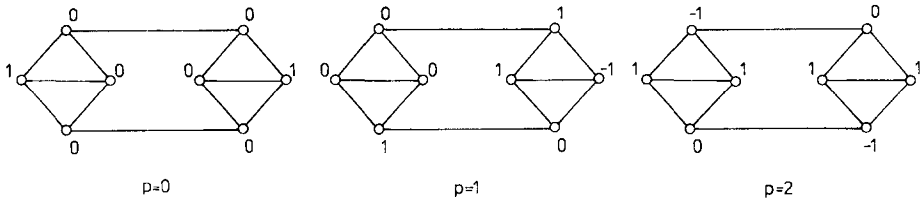

Fig. 1.

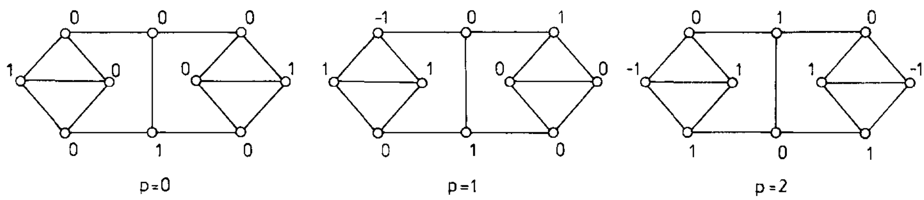

Fig. 2.

0 to one vertex and the value -1 to one vertex. In the remaining $\frac{1}{4}n - p$ ones we assign the value 1 to one vertex and the value 0 to three vertices. □

There are other examples, among them connected graphs. For $n = 8$ Fig. 1 gives an example.

Theorem 4. *Let $n \geqslant 10$ be an even integer nondivisible by 4. Then there exists a cubic graph G of order n with the property that for any integer p such that $0 \leqslant p \leqslant \frac{1}{4}(n-2)$ there exists a minus dominating function f of G such that $f(V(G)) = \frac{1}{4}(n+2)$ and f assigns the value -1 to exactly p vertices.*

Proof. An example for $n = 10$ is given in Fig. 2. In general, we take the graph which is the disjoint union of the depicted graph and of $\frac{1}{4}(n-10)$ complete graphs of order 4. The required minus dominating functions are constructed analogously to the proof of Theorem 3. □

Proposition. *For $n = 6$ there exist only two nonisomorphic cubic graphs with n vertices. They both have the minus domination number equal to $[\frac{1}{4}n] = 2$, but in both cases the corresponding minus domination function has no value -1.*

Proof. These two graphs are the complements of the circuit C_6 of length 6 and of the disjoint union $K_3 \cup K_3$ of two complete graphs of order 3. In both these graphs, whenever a minus dominating function has one value -1, it must have at least four values 1; the reader may verify it himself. Then the sum of values of that function is at least 3. No minus dominating function can have two or more values -1, because both these graphs have diameter 2 and no vertex can be adjacent to two vertices of value -1. On the other hand, in each of these graphs it is possible to assign the value 1 to two vertices and the value 0 to four vertices. In the complement of C_6 we assign the value 1 to two opposite vertices of C_6, in the complement of $K_3 \cup K_3$ to two vertices from different connected components of $K_3 \cup K_3$. $\square$

Henning (private communication) has generalized this result for r-regular graphs with arbitrary r. He proved that $\gamma^-(G) \geqslant n/(r+1)$ and this bound is sharp.

4. A lower bound for $\gamma'_{\mathrm{maj}}(G)$

In this section we prove a lower bound for $\gamma_{\mathrm{maj}}(G)$, where G is a cubic graph and also show that this bound is best possible.

Theorem 5. *If G is a cubic graph of order n, then*

$$\gamma_{\mathrm{maj}}(G) \geqslant -\tfrac{1}{4}n.$$

This bound is best possible.

Proof. Let f be the majority dominating function of G such that $f(V(G) = \gamma_{\mathrm{maj}}(G)$. Let $V^+ = \{x \in V(G) \mid f(x) = 1\}$, $V^- = [x \in V(G) \mid f(x) = -1\}$, $W^+ = \{x \in V(G) \mid f(N[x]) \geqslant 1\}$, $W^- = \{x \in V(G) \mid f(N[x]) \leqslant 0\}$. Furthermore, let $a = |V^- \cap W^+|$, $b = |V^+ \cap W^+|$, $c = |V^+ \cap W^-|$. We have $a + b = |W^+| \geqslant \frac{1}{2}n$. If $a < \frac{1}{8}n$, then $|V^+| = b + c \geqslant b \geqslant \frac{1}{2}n - a > \frac{1}{2}n - \frac{1}{8}n = \frac{3}{8}n$. Further $|V^-| = n - |V^+| < n - \frac{3}{8}n = \frac{5}{8}n$ and $\gamma_{\mathrm{maj}}(G) = |V^+| - |V^-| > \frac{3}{8}n - \frac{5}{8}n = -\frac{1}{4}n$. Thus in this case the assertion is true. We may therefore assume that $a \geqslant \frac{1}{8}n$. Each vertex of $V^- \cap W^+$ must be adjacent to three vertices of V^+; therefore there are $3a$ edges joining a vertex of $V^- \cap W^+$ with a vertex of V^+. There are at most b vertices of $V^- \cap W^+$ adjacent to vertices of $V^+ \cap W^+$, because each vertex of $V^+ \cap W^+$ may be adjacent to at most one vertex of V^-. A vertex of $V^+ \cap W^-$ may be adjacent to three vertices of $V^- \cap W^+$ and therefore $3a \leqslant b + 3c$. This implies $c \geqslant a - \frac{1}{3}b$. Further $b \geqslant \frac{1}{2}n - a$ and thus

$$|V^+| = b + c \geqslant b + a - \tfrac{1}{3}b = \tfrac{2}{3}b + a \geqslant \tfrac{2}{3}(\tfrac{1}{2}n - a) + a = \tfrac{1}{3}n + \tfrac{1}{3}a$$

Then

$$|V^-| = n - |V^+| \leqslant n - (\tfrac{1}{3}n + \tfrac{1}{3}a) = \tfrac{2}{3}n - \tfrac{1}{3}a$$

Hence

$$\gamma_{\mathrm{maj}}(G) = |V^+| - |V^-| \geqslant \tfrac{1}{3}n + \tfrac{1}{3}a - (\tfrac{2}{3}n - \tfrac{1}{3}a) = -\tfrac{1}{3}n + \tfrac{2}{3}a.$$

Since $a \geqslant \frac{1}{8}n$, we have

$$\gamma_{\mathrm{maj}}(G) \geqslant -\tfrac{1}{3}n + \tfrac{2}{3}\cdot\tfrac{1}{8}n = -\tfrac{1}{4}n.$$

Now let n be a positive integer divisible by eight; we shall construct a cubic graph of order n such that $\gamma_{\mathrm{maj}}(G) = -\frac{1}{4}n$. Take two disjoint vertex sets A, B such that $|A| = \frac{1}{8}n$, $|B| = \frac{3}{8}n$. Construct a circuit with the vertex set B. Join each vertex of A with three vertices of B in such a way that each vertex of B is adjacent to exactly one vertex of A. The result is a cubic graph G' of order $\frac{1}{2}n$. Let $G = G' \cup G''$, where G'' is a copy of G'. Define $f\colon V(G) \to \{-1, 1\}$ such that $f(x) = 1$ for all $x \in B$ and $f(x) = -1$ for all other vertices x of G. Then $f(V(G)) = -\frac{1}{4}n$ and thus $\gamma_{\mathrm{maj}}(G) = -\frac{1}{4}n$. □

Henning [4] has generalized this result for r-regular graphs with arbitrary r. He proved that

$$\gamma_{\mathrm{maj}}(G) \geqslant (1-r)/2(r+1)n \quad \text{for } r \text{ odd} \qquad \text{and} \qquad \gamma_{\mathrm{maj}}(G) \geqslant \frac{-r}{2(r+1)}n$$

for r even.

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