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FREQUENCY TEMPERATURE CHARACTERISTICS OF THE x-LENGTH STRIP RESONATORS OF AT-CUT QUARTZ

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Abstract

The equations of motion of the coupled thickness-shear, thickness-length flexure, width-shear and width-length flexure deduced by neglecting of the piezoelectric coupling deduced from a system of one-dimensional equations of motion for AT-cut quartz strip resonators (by Lee and Wang, 1992) is employed for the study of the frequency temperature characteristics of the x-length AT-cut quartz strip resonators. The computed dispersion curves, frequency spectrum and the thicknes-shear resonance frequency-temperature curves (the last two as a function of dimensions ratios) are given.

Introduction

The analysis of the vibrations of AT-cut quartz strips of narrow width and finite length has been published by Lee and Wang [1] in 1992. In the mentioned paper, one-dimensional equations for the modes of vibration in strip width and for frequencies upto and including the fundamental thickness-shear have been deduced from the two-dimensional, first-order equations for piezoelectric crystal plates, given by Lee, Syngellakis, and Hou [2], by expanding the mechanical displacements and electric potentials in series of trigonometric functions of the width coordinate. The neglecting of the piezoelectric properties and elastic stiffness c_{56} of the plate made it possible to select four groups of the modes of vibrations. They were the thickness-shear and thickness length flexure vibration and their first twist-overtone, the length-extension, width-stretch, and symmetric width-shear vibrations and the width-shear, width-length flexure and antisymmetric width-stretch vibrations.

The thickness-shear resonance is the main resonance of the strip for applications. The thickness-shear resonance frequency temperature dependence can be predicted from the frequency equation of the coupled thickness-shear and thickness-length flexure vibration of the strip given by Lee and Wang in [1] when the temperature changes of the elastic stiffnesses and thermal expansion coefficients are included. But the more precise analysis of the effect of length-to-thickness (a/b) and width-to-thickness (c/b) ratios of the strip on the resonance frequency temperature dependence requires to consider also the influence of the coupling with the other modes of vibrations.

By neglecting the coupling of the anti-symmetric width-stretch mode with width-shear and width-length flexure modes of vibrations for c/b less then 3.78 Lee deduced a set of four coupled displacement equations of motion from the one-dimensional equations for strip resonator given in [1]. These four displacement equations of motion accommodate the coupling of thickness-shear, thickness-length flexure, width shear and width-length flexure vibrations.

In the present paper, the frequency equation of the four coupled displacement equations for AT-cut quartz crystal strip is obtainedby setting piezoelectric constants $e_{i6} = 0$. The temperature dependent material properties are included in the dispersion relation and frequency equation. The resonance frequency temperature dependences as the functions of a/b and c/b ratios are computed.

Temperature Dependent Material Properties

The AT-cut quartz strip resonators shown with its coordinates and dimensions in Fig. 1 is considered in subsequent discussion. Similarly as in references [4], [5] and [6] we express the influence of the thermally biased homogenius strain by means of terms β_{ki} and D_{ijkl}.

The term β_{ki} is given by the relation [4], [5]

$$\beta_{ki} = \delta_{ki} + \alpha_{ki}^{\Theta}, \qquad (1)$$

where δ_{ki} is a *Kronecker delta* and

$$\alpha_{ki}^{\Theta} = \alpha_{ki}^{(1)}\Theta + \alpha_{ki}^{(2)}\Theta^2 + \alpha_{ki}^{(3)}\Theta^3. \qquad (2)$$

In (2) $\alpha_{ki}^{(n)}$ are n-th order thermal expansion coefficients (measured by Bechmann, Ballato and Lukaszek [6] and corrected by Kosinski, Gualtieri and Ballato [7]) and Θ is the temperature change, $\Theta = T - T_0$.

The tensors α_{ki}^{Θ} and β_{ki} are predominantly diagonal tensors with the off-digonal terms in the order of magnitude of 10^{-6} as compared with the diagonal terms. Therefore, by neglecting the off-diagonal terms, we have ($\beta_{ki} = 0$ for $k = i$).

$$\beta_k = \beta_{kk} = 1 + \alpha_{kk}^{\Theta}, \quad \text{(nosum)}. \qquad (3)$$

The term D_{ijkl} is given by the relation

$$D_{ijkl} = C_{ijkl} + D_{ijkl}^{(1)}\Theta + D_{ijkl}^{(2)}\Theta^2 + D_{ijkl}^{(3)}\Theta^3, \qquad (4)$$

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where

$$D^{(q)}_{ijkl} = \frac{1}{q!}C^{(q)}_{ijkl} + C_{ijklmn}\alpha^{\Theta}_{mn}, \tag{5}$$

and C_{ijkl} and C_{ijklmn} are the second and third order elastic stiffness of quartz, while $C^{(1)}_{ijkl}$, $C^{(2)}_{ijkl}$ and $C^{(3)}_{ijkl}$ are respectively the first temperature derivatives, second effective temperature derivatives and third effective temperature derivatives. Values of the temperature derivatives were calculated and reported in reference [5] and [6]. The magnitudes of D_{ijkl} where reported in reference [5].

Coupled Thickness-Shear, Thickness-Length Flexure, Width-Shear and Width-Length Flexure Vibrations

The displacement equations of motion of the coupled TSh, tlF, WSh and wlF modes of vibrations including of the thermal expansion coefficients β_i are

$$\begin{gathered}
c_{56}(\beta_2\beta_3 u^{(00)}_{3,11} + \frac{\alpha_1}{c}\beta_1\beta_2 u^{(01)}_{1,1}) \\
+c_{66}(\beta_2^2 u^{(00)}_{2,11} + \frac{\alpha_1}{b}\beta_1\beta_2 u^{10)}_{1,1}) = \varrho(1+R)\ddot{u}^{(00)}_2, \\
c_{55}(\beta_3^2 u^{(00)}_{3,11} + \frac{\alpha_1}{c}\beta_1\beta_3 u^{(01)}_{1,1}) \\
+c_{56}(\beta_2\beta_3 u^{(00)}_{2,11} + \frac{\alpha_1}{b}\beta_1\beta_3 u^{(10)}_{1,1}) = \frac{\varrho}{\alpha_2^2}(1+R)\ddot{u}^{(00)}_3, \\
\bar{c}_{11}\beta_1^2 u^{(01)}_{1,11} - \frac{2\alpha_1}{c}c_{55}(\beta_1\beta_3 u^{(00)}_{3,1} + \frac{\alpha_1}{c}\beta_1^2 u^{(01)}_1) \\
-\frac{2\alpha_1}{c}c_{56}(\beta_1\beta_2 u^{(00)}_{2,1} + \frac{\alpha_1}{b}\beta_1^2 u^{(10)}_1) = \varrho(1+R)\ddot{u}^{(01)}_1, \\
\tilde{c}_{11}u^{(10)}_{1,11} - \frac{2\alpha_1}{b}c_{66}(\beta_1\beta_2 u^{(00)}_{2,1} + \frac{\alpha_1}{b}\beta_1^2 u^{(10)}_1) \\
-\frac{2\alpha_1}{b}c_{56}(\beta_1\beta_3 u^{(00)}_{3,1} + \frac{\alpha_1}{c}\beta_1^2 u^{(10)}_1) = \varrho(1+2R)\ddot{u}^{(10)}_1,
\end{gathered} \tag{6}$$

and the stress-displacement relations

$$\begin{aligned}
T^{(01)}_1 &= 2\bar{c}_{11}\beta_1 u^{(01)}_{1,1}, \\
T^{(10)}_1 &= 2\tilde{c}_{11}\beta_1 u^{(10)}_{1,1}, \\
T^{(00)}_5 &= 4(c_{55}(\beta_3 u^{(00)}_{3,1} + \frac{\alpha_1}{c}\beta_1 u^{(01)}_1) \\
&+ c_{56}(\beta_2 u^{(00)}_{2,1} + \frac{\alpha_1}{b}\beta_1 u^{(10)}_1)), \\
T^{(00)}_6 &= 4(c_{56}(\beta_3 u^{(00)}_{3,1} + \frac{\alpha_1}{c}\beta_1 u^{(01)}_1) \\
&+ c_{66}(\beta_2 u^{(00)}_{2,1} + \frac{\alpha_1}{b}\beta_1 u^{(10)}_1)).
\end{aligned} \tag{7}$$

We choose the modes of vibrations to have the form

$$\begin{aligned}
u^{(01)}_1 &= \sum_{r=1}^{4} A_{1r}\cos\xi_r x_1 e^{i\omega t}, \\
u^{(10)}_1 &= \sum_{r=1}^{4} A_{2r}\cos\xi_r x_1 e^{i\omega t}, \\
u^{(00)}_2 &= \sum_{r=1}^{4} A_{3r}\sin\xi_r x_1 e^{i\omega t}, \\
u^{(00)}_3 &= \sum_{r=1}^{4} A_{4r}\sin\xi_r x_1 e^{i\omega t}
\end{aligned} \tag{8}$$

which satisfy (6), provided

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} A_{1r} \\ A_{2r} \\ A_{3r} \\ A_{4r} \end{pmatrix} = 0, \tag{9}$$

where

$$\begin{aligned}
a_{11} &= \frac{\bar{c}_{11}}{c_{66}}\beta_1^2\bar{\xi}_r^2 + (\frac{b}{c})^2 c^*_{55}\beta_1^2 - \Omega^2(1+R), \\
a_{12} &= \frac{b}{c}c^*_{56}\beta_1^2, \\
a_{13} &= \sqrt{2}\frac{b}{c}c^*_{56}\beta_1\beta_2\bar{\xi}_r, \\
a_{14} &= \sqrt{2}\frac{b}{c}c^*_{55}\beta_1\beta_3\bar{\xi}_r, \\
a_{21} &= a_{12}, \\
a_{22} &= c^*_{11}\beta_1^2\bar{\xi}_r^2 + c^*_{66}\beta_1^2 - \Omega^2(1+2R), \\
a_{23} &= \sqrt{2}c^*_{66}\beta_1\beta_2\bar{\xi}_r, \\
a_{24} &= \sqrt{2}c^*_{56}\beta_1\beta_3\bar{\xi}_r, \\
a_{31} &= \frac{1}{2}a_{13}, \\
a_{32} &= \frac{1}{2}a_{23}, \\
a_{33} &= c^*_{66}\beta_2^2\bar{\xi}_r^2 - \Omega^2(1+R), \\
a_{34} &= c^*_{56}\beta_1\beta_3\bar{\xi}_r^2, \\
a_{41} &= \frac{1}{2}a_{14}, \\
a_{42} &= \frac{1}{2}a_{24}, \\
a_{43} &= a_{34}, \\
a_{44} &= c^*_{56}\beta_3^2\bar{\xi}_r^2 - \frac{\Omega^2}{\alpha_2^2}(1+R).
\end{aligned} \tag{10}$$

The normalized frequency and wave number are defined by

$$\Omega = \frac{\omega}{\frac{\pi}{2b}\sqrt{\frac{c_{66}}{\varrho}}}, \qquad \bar{\xi}_r = \frac{2b}{\pi}\xi_r. \tag{11}$$

The dispersion relation

$$[\ a_{ij}\ ] = 0 \tag{12}$$

yield four frequency branches as shown in Fig. 2.

For traction-free ends of the strip, we required, at $x_1 = \pm a$

$$T^{(01)}_1 = T^{(10)}_1 = T^{(00)}_5 = T^{(00)}_6 = 0. \tag{13}$$

Substitution of (8) into (7) and, in turn, into (13) results in

$$\sum_{r=1}^{4}\gamma_{1r}\bar{\xi}_r A_{4r}\sin\xi_r a = 0,$$

$$\sum_{r=1}^{4}\gamma_{2r}\bar{\xi}_r A_{4r}\sin\xi_r a = 0,$$

$$\sum_{r=1}^{4}\Big(\frac{1}{\sqrt{2}}\frac{b}{c}c^*_{55}\gamma_{1r}\beta_1+\frac{1}{\sqrt{2}}c^*_{56}\gamma_{2r}\beta_1+c^*_{56}\gamma_{3r}\beta_2 + c^*_{55}\beta_3\Big)A_{4r}\cos\xi_r a=0,$$

$$\sum_{r=1}^{4}\Big(\frac{1}{\sqrt{2}}\frac{b}{c}c^*_{56}\gamma_{1r}\beta_1+\frac{1}{\sqrt{2}}c^*_{66}\gamma_{2r}\beta_1+c^*_{66}\gamma_{3r}\beta_2 + c^*_{56}\beta_3\Big)A_{4r}\cos\xi_r a=0, \tag{14}$$

where

$$\gamma_i j=\frac{A_{ij}}{A_{4j}}, \qquad j=1,2,3,4 \tag{15}$$

are the amplitude ratios which can be computed from (14).

The vanishing of the determinant of the coefficients matrix of (14) gives the frequency equation which must be solved in conjunction with dispersion relation (12).

The elastic stiffnesses used in equations given above depend on the temperature and are defined by the relations

$$\begin{aligned}
\bar{c}_{pq} &= D_{ppqq}-\frac{D_{pp22}D_{22qq}}{D_{2222}} \quad p,q=1,3\\
\bar{c}_{p4} &= D_{pp23}-\frac{D_{pp22}D_{2223}}{D_{2222}} \quad p=1,3\\
\bar{c}_{4q} &= D_{23qq}-\frac{D_{2322}D_{22qq}}{D_{2222}} \quad q=1,3\\
\bar{c}_{44} &= D_{2323}-\frac{D_{2322}D_{2223}}{D_{2222}}\\
c_{55} &= D_{1313}\\
\hat{c}_{pq} &= \bar{c}_{pq}-\frac{\bar{c}_{p4}\bar{c}_{4q}}{\bar{c}_{44}} \quad p,q=1,3\\
\tilde{c}_{11} &= \hat{c}_{11}-\frac{\hat{c}_{13}\hat{c}_{31}}{\hat{c}_{33}},\\
c^*_{11} &= \frac{\tilde{c}_{11}}{c_{66}}, \qquad c^*_{55}=\frac{c_{55}}{c_{66}},\\
c^*_{56} &= \frac{c_{56}}{c_{66}}, \qquad c^*_{66}=\frac{D_{1212}}{c_{66}},
\end{aligned} \tag{16}$$

where

$$c_{66}=[D_{1212}]_{\Theta=0}.$$

We note that in (16) no summation over the repeated indices.

Frequency-Temperature Characteristics of Thickness-Shear Resonance

Computational result of resonance frequency as a function of the length-to-thickness ratio a/b for a fixed width-to-thickness ratio c/b = 3.78, R = 0 and AT-cut ($\theta = 35.167^0$) is shown in Fig. 3.

It can be seen from the frequency spectrum of the strip given in Fig. 3 that for predominant thickness-shear vibrations, the strip resonators must have the a/b ratios near the values 11.05, 12.91, 14.64, 16.36 and 18.05. The thickness shear resonance frequency-temperature characteristic for these a/b ratios are given in Fig. 4. The influence of the a/b ratio is greater for the small values of the a/b ratio.

The resonance-frequency temperature characteristics for a few ratios a/b near the value a/b = 11.05 are given in Fig. 5. The resonance-frequency temperature dependence changes very rapidly if the a/b ratio is far from the inflexion point of the frequency vs a/b ratio curve.

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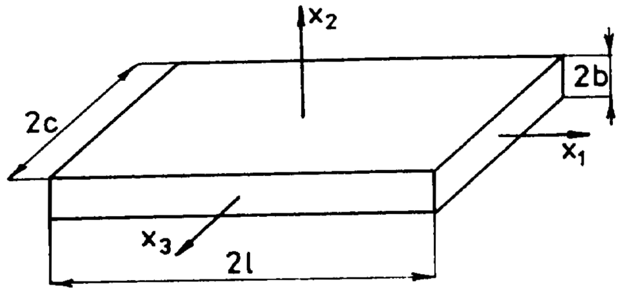

Fig. 1. An x-length AT-cut quartz strip resonator.

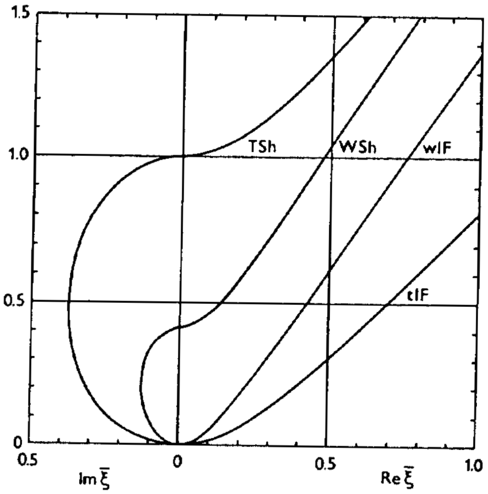

Fig. 2. Dispersion curves of coupled thicness-shear (TSh), thickness-length flexure (tlF), width-shear (WSh) and width-length flexure (wlF) vibrations of an AT-cut strip with $c/b = 3.78$.

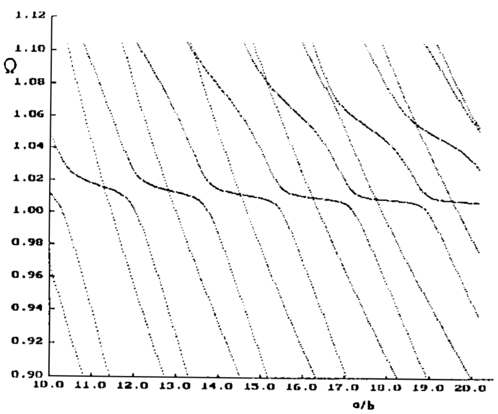

Fig. 3. Ω vs. a/b of coupled thicness-shear, thickness-length flexure, width-shear and width-length flexure vibrations in an AT-cut quartz strip with $c/b = 3.78$.

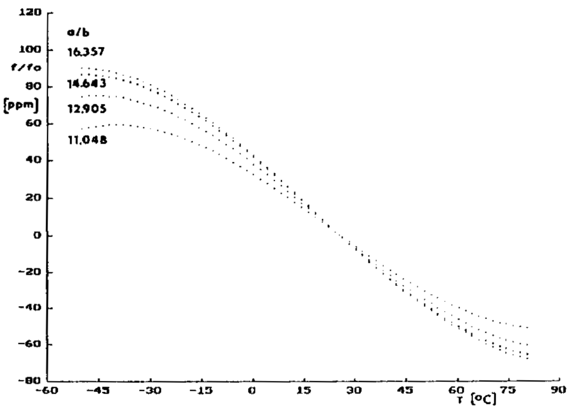

Fig. 4. Predict thickness-shear resonance frequency temperature curves for AT-cut ($\theta = 35.167^0$) quartz strip for a/b ratio variable and $c/b = 3.78$.

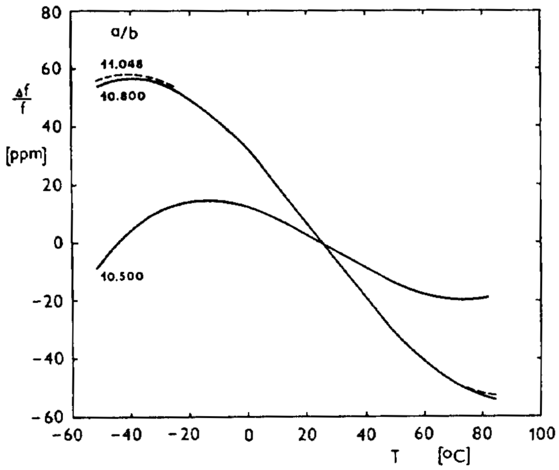

Fig. 5. Predict thickness-shear resonance frequency temperature curves for AT-cut ($\theta = 35.167^0$) quartz strip for three values of the ratio a/b near $a/b = 11.05$ and $c/b = 3.78$.