Multiplication by Wavelet Matrix – Efficient Implementation

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Abstract

Stiffness matrix of the Dirichlet problem $-(au')' = f$ with a homogeneous boundary value condition in a spline wavelet basis has $O(n \log n)$ non-zero elements [4]. We show that for a constant function a it is just $O(n)$ and moreover we show that it can be stored in $O(1)$ elements. This leads to a linear-time algorithm for multiplication by the wavelet matrix.

Keywords: Dirichlet problem; efficient implementation; Galerkin method; spline wavelets.

Introduction

Our aim is to solve the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u = f \quad &\text{on} \quad \Omega = (0,1)^d \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

in the higher dimension d by the Galerkin method in a wavelet basis which is a tensor product of wavelet bases in one dimension. The Galerkin method leads to a system of linear algebraic equations with a matrix constructed from matrices (d_{ij}), (g_{ij}), where

$$\begin{aligned} d_{ij} &= \int_0^1 \varphi_i'(x)\varphi_j'(x)\,\mathrm{d}x, \\ g_{ij} &= \int_0^1 \varphi_i(x)\varphi_j(x)\,\mathrm{d}x \end{aligned} \tag{1}$$

and φ_i are basis functions. We need an efficient storage of matrices (d_{ij}), (g_{ij}) and as we solve the system by an iterative method, we also need an efficient implementation of multiplication of matrices by a vector.

In this paper we describe what a wavelet basis is, its construction and then we show that the stiffness matrix (d_{ij}) for the one-dimensional Poisson equation has $O(n)$ non-zero elements, and moreover it can be stored in $O(1)$ space. In the last section we show a numerical experiment on one-dimensional Poisson equation.

1 Spline Wavelet Basis

In this section we first describe the spaces V_n of scaling functions based on quadratic splines. Then we deal with the wavelet basis of V_n. First we formulate desired properties for wavelets and explain their consequences, then we show some constructions of wavelet bases.

1.1 Scaling Functions

Spline wavelet basis consists of scaling and wavelet functions. We use two following functions to construct them:

$$\varphi_{bd}(x) = \begin{cases} 2x - \frac{3}{2}x^2 & \text{for } x \in [0,1] \\ \frac{1}{2} - x + \frac{1}{2}x^2 & \text{for } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_{in}(x) = \begin{cases} \frac{1}{2}x^2 & \text{for } x \in [0,1] \\ \frac{1}{2} + x - x^2 & \text{for } x \in [1,2] \\ \frac{1}{2} - x + \frac{1}{2}x^2 & \text{for } x \in [2,3] \\ 0 & \text{otherwise} \end{cases}$$

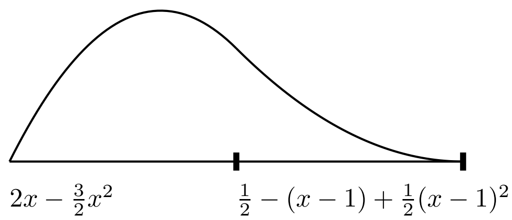

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Fig. 1. *Boundary scaling function*

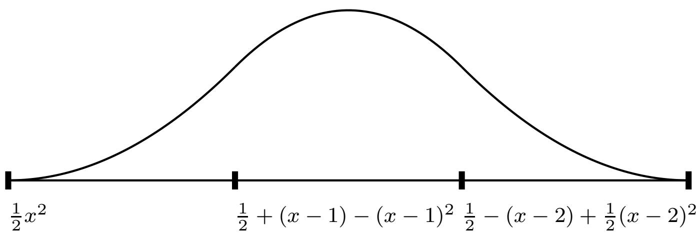

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Fig. 2. *Inner scaling function*

Both are $\mathscr{C}^1$ piecewise quadratic functions.

Definition 1 *Let $\ell \in \mathbb{N}$. We call functions*

$$\varphi_{\ell,0}(x) = \varphi_{bd}(2^\ell x) \tag{2}$$

$$\varphi_{\ell,i}(x) = \varphi_{in}(2^\ell x - i + 1) \quad \text{for } i = 1, \ldots, (2^\ell - 2) \tag{3}$$

$$\varphi_{\ell,2^\ell-1}(x) = \varphi_{bd}(2^\ell(1-x)) \tag{4}$$

scaling functions at the level ℓ.

Note that from (3) it follows that the values of integrals

$$\left.\begin{array}{rcl} d_{\ell,i,j} &=& \int_0^1 \varphi'_{\ell,i}\varphi'_{\ell,j}\,\mathrm{d}x \\ g_{\ell,i,j} &=& \int_0^1 \varphi_{\ell,i}\varphi_{\ell,j}\,\mathrm{d}x \end{array}\right\} \quad \text{for } i,j=1,\dots,(2^\ell-2) \tag{5}$$

depend just on the difference $(i-j)$ and the value ℓ. Furthermore, for two different levels ℓ_1, ℓ_2 it holds

$$\begin{array}{rcl} d_{\ell_1,i,j} &=& 2^{\ell_1-\ell_2} d_{\ell_2,i,j} \\ g_{\ell_1,i,j} &=& 2^{\ell_2-\ell_1} g_{\ell_2,i,j} \end{array} \tag{6}$$

Similarly from (2), (4) it follows that

$$\left.\begin{array}{rcl} \int_0^1 \varphi'_{\ell_1,0}\varphi'_{\ell_1,i}\,\mathrm{d}x &=& 2^{\ell_1-\ell_2}\int_0^1 \varphi'_{\ell_2,0}\varphi'_{\ell_2,i}\,\mathrm{d}x \\ \int_0^1 \varphi'_{\ell_1,2^\ell-1}\varphi'_{\ell_1,i}\,\mathrm{d}x &=& 2^{\ell_1-\ell_2}\int_0^1 \varphi'_{\ell_2,2^\ell-1}\varphi'_{\ell_2,i}\,\mathrm{d}x \\ \int_0^1 \varphi_{\ell_1,0}\varphi_{\ell_1,i}\,\mathrm{d}x &=& 2^{\ell_2-\ell_1}\int_0^1 \varphi_{\ell_2,0}\varphi_{\ell_2,i}\,\mathrm{d}x \\ \int_0^1 \varphi_{\ell_1,2^\ell-1}\varphi_{\ell_1,i}\,\mathrm{d}x &=& 2^{\ell_2-\ell_1}\int_0^1 \varphi_{\ell_2,2^\ell-1}\varphi_{\ell_2,i}\,\mathrm{d}x \end{array}\right\} \quad \text{for } i=1,\dots,(2^\ell-2) \tag{7}$$

1.2 Wavelets

Let us denote

$$\mathscr{S}_\ell = \{\varphi_{\ell,i} : i = 0\dots 2^\ell-1\}$$

and

$$V_\ell = \operatorname{span}\mathscr{S}_\ell \qquad \text{for } \ell\in\mathbb{N}.$$

It holds that

$$V_\ell \subset V_{\ell+1} \qquad \text{for } \ell\in\mathbb{N}.$$

In the next section we describe some constructions of sets $\mathscr{W}_\ell \subset V_{\ell+1}$ which consist of functions $\psi_{\ell,0},\dots\psi_{\ell,2^\ell-1}$. We require the following properties:

1. The set $\mathscr{S}_\ell \cup \mathscr{W}_\ell$ forms a basis of $V_{\ell+1}$.
2. Functions $\psi\in\mathscr{W}_\ell$ have vanishing moments of order 0, 1 and 2. It means that

$$\int_{\mathrm{supp}\,(\psi)} x^m\psi(x)\,\mathrm{d}x = 0 \qquad \text{for } m=0,1,2.$$

We use property 1 iteratively to construct the basis of V_ℓ

$$\mathscr{S}_{l_0} \cup \bigcup_{i=l_0}^{l} \mathscr{W}_i \tag{8}$$

for an appropriate ℓ_0.

The property of vanishing moments has two very important consequences. The first one is more sparse approximation of the solution. The second one concerns matrices (1) – they are less sparse in the basis (8) than in the basis $\mathscr{S}_\ell$, but vanishing moments imply a lot of zero entries of matrices (1) even if supports of functions in the base overlap – more precisely

$$\int_{[0,1]} \psi_i\psi_j = \int_{[0,1]} \psi'_i\psi'_j = 0$$

whenever ψ_i is a quadratic function on the support of ψ_j. In the general case we can write ψ_j as a sum

$$\psi_j(x) = ax^2 + bx + c + \sum_k d_k(\max\{(x - x_k), 0\})^2.$$

We can then express integrals as sums

$$\begin{aligned} \int_{[0,1]} \psi_i \psi_j &= \sum_k d_k \int_{x_k}^1 \psi_i(x)(x - x_k)^2 \, \mathrm{d}x, \\ \int_{[0,1]} \psi_i' \psi_j' &= \sum_k 2d_k \int_{x_k}^1 \psi_i'(x)(x - x_k) \, \mathrm{d}x. \end{aligned} \tag{9}$$

We use these formulas in cases when levels ℓ_i, ℓ_j of functions ψ_i, ψ_j respectively differ by at least two – then the sum consists of at most two terms.

1.3 Construction of Wavelets

We need wavelets satisfying properties from the previous section. Several such constructions are known: inner bi-orthogonal wavelets [3], boundary bi-orthogonal wavelets [1] and three types of short wavelets [2].

We show construction of one type of short wavelets:

$$\psi_{\ell,0} = -\tfrac{5}{2}\varphi_{\ell+1,0} + \tfrac{47}{12}\varphi_{\ell+1,1} - \tfrac{13}{4}\varphi_{\ell+1,2} + \varphi_{\ell+1,3}, \tag{10}$$

$$\psi_{\ell,i} = -\tfrac{1}{4}\varphi_{\ell+1,2i-1} + \tfrac{3}{4}\varphi_{\ell+1,2i} - \tfrac{3}{4}\varphi_{\ell+1,2i+1} + \tfrac{1}{4}\varphi_{\ell+1,2i+2} \quad \text{for } i = 1, \ldots, 2^\ell - 2, \tag{11}$$

$$\psi_{\ell,2^\ell-1} = -\varphi_{\ell+1,2^{\ell+1}-4} + \tfrac{13}{4}\varphi_{\ell+1,2^{\ell+1}-3} - \tfrac{47}{12}\varphi_{\ell+1,2^{\ell+1}-2} + \tfrac{5}{2}\varphi_{\ell+1,2^{\ell+1}-1}. \tag{12}$$

In general case wavelets of the ℓ-th level are linear combinations of scaling functions of the $(\ell+1)$-th level. There are a few boundary wavelets on both edges – as in (10), (12). We denote their number at each edge by n_{bw}.

Left-edge boundary wavelets are

$$\psi_{\ell,i} = \sum_{j=0}^{n_1} a_{i,j}\varphi_{\ell+1,j}, \quad i = 0, \ldots, (n_{bw} - 1). \tag{13}$$

The upper bound n_1 in the sum corresponds to the support of boundary wavelets

$$\operatorname{supp} \psi_{\ell,i} = [0, (n_1 + 2)2^{-\ell-1}] \quad \text{for } i < n_{bw}.$$

We will measure the support in the units of $2^{-\ell-1}$ and denote $l_{bw} = n_1 + 2$.

Right-edge boundary wavelets are

$$\psi_{\ell,2^\ell-1-i} = \sum_{j=0}^{n_1} a_{i,j}\varphi_{\ell+1,2^{\ell+1}-1-j}, \quad i = 0, \ldots, (n_{bw} - 1)$$

with the same coefficients a_{ij} and the same upper bound l_{bw} of the length.

Inner wavelets $\psi_{\ell,i}$ for

$$i = n_{bw}, \ldots, 2^\ell - n_{bw} - 1$$

are constructed by

$$\psi_{\ell,i} = \sum_{j=0}^{n_2} a_j\varphi_{\ell+1,2(i-n_{bw})+1+j}. \tag{14}$$

It holds

$$\operatorname{supp} \psi_{\ell,i} = [2(i-n_{bw})2^{-\ell-1}, (2(i-n_{bw})+n_2+2)2^{-\ell-1}]. \tag{15}$$

We denote $l_{iw} = n_2 + 2$ the length of the support of inner wavelets in the unit $2^{-\ell-1}$.

All constructions are such that the support of the last inner wavelet contains the point $x = 1$. From here it follows

$$(2(2^\ell - 2n_{bw} - 1) + n_2 + 2)2^{-\ell-1} = 1$$

and as $l_{iw} = n_2 + 2$ it follows that

$$(2(2^\ell - 2n_{bw} - 1) + l_{iw})2^{-\ell-1} = 1$$

which gives by a straightforward calculation

$$l_{iw} = 2(2n_{bw} + 1). \tag{16}$$

Another straightforward calcualation gives that the center of the support of $\psi_{\ell,i}$ is at the point

$$x = \left(i + \tfrac{1}{2}\right) 2^{-\ell}. \tag{17}$$

In the next section we will use the value $l_w = \max\{l_{bw}, l_{iw}\}$ and the maximal number of discontinuities n_d which bases function can have. As two different discontinuities are distant at least $2^{-\ell-1}$ it holds $n_d \leq l_w - 1$.

Due to the construction of wavelets similar property to (5), (6), (7) it holds:

$$\begin{aligned} \int_0^1 \varphi'_{\ell_1,i}\varphi'_{\ell_1,j}\,\mathrm{d}x &= 2^{\ell_1-\ell_2}\int_0^1 \varphi'_{\ell_2,i}\varphi'_{\ell_2,j}\,\mathrm{d}x \\ \int_0^1 \varphi_{\ell_1,i}\varphi_{\ell_1,j}\,\mathrm{d}x &= 2^{\ell_2-\ell_1}\int_0^1 \varphi_{\ell_2,i}\varphi_{\ell_2,j}\,\mathrm{d}x \end{aligned} \tag{18}$$

and also

$$\begin{aligned} \int_0^1 \varphi'_{\ell,i}\varphi'_{\ell,j}\,\mathrm{d}x &= \int_0^1 \varphi'_{\ell,i+k}\varphi'_{\ell,j+k}\,\mathrm{d}x \\ \int_0^1 \varphi_{\ell,i}\varphi_{\ell,j}\,\mathrm{d}x &= \int_0^1 \varphi_{\ell,i+k}\varphi_{\ell,j+k}\,\mathrm{d}x \end{aligned} \tag{19}$$

whenever all involved wavelets are inner ones.

2 Wavelet Matrix

In this section, we first show that matrices (1) in a wavelet basis can be stored in a constant space. Next we show in the Theorem that they have at most linear non-zero elements with respect to the order N of the matrices.

2.1 Storage of Matrices

We store matrices (1) in blocks as in Figure 3. Let us denote $B(\ell)$ the block corresponding to scaling functions and to wavelets at the ℓ-th level. Blocks $B(4)$ and $B(5)$ as well as the top-left block corresponding to scaling functions are stored as a whole. Elements of blocks $B(\ell)$, $\ell = 6, \ldots$ are computed by the formulas (9). We store in an array both the discontinuities of scaling functions and integrals

$$\int_{x_k} \psi(x)(x-x_k)^m\,\mathrm{d}x \qquad m = \begin{cases} 1 \text{ for the matrix } g \\ 2 \text{ for the matrix } d \end{cases}$$

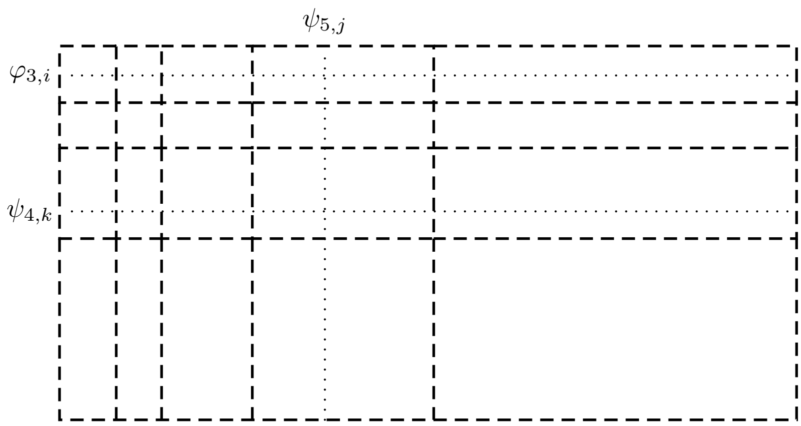

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Fig. 3. *Structure of a wavelet matrix*

Now we describe how to pair discontinuities and integrals for inner wavelets (rows corresponding to boundary wavelets are zero due to choice of ℓ_0 in (8) – it is chosen in such a way that the support of the left-edge boundary wavelet does not overlap the support of the right-edge boundary wavelet). We assign the central point of its support (17)

$$x = \left(i + \tfrac{1}{2}\right) 2^{-\ell}$$

to the wavelet $\psi_{\ell,i}$ and from it we measure points x_k of discontinuities of scaling functions. They are positioned at

$$1 \cdot 2^{-3}, 2 \cdot 2^{-3}, \ldots, 6 \cdot 2^{-3}, 7 \cdot 2^{-3}, \tag{20}$$

so they can be inside the support of $\psi_{\ell,i}$ at the points

$$\ldots, (i-1)2^{-\ell}, i2^{-\ell}, (i+1)2^{-\ell}, (i+2)2^{-\ell}, \ldots \tag{21}$$

Inner wavelets are $l_w 2^{-\ell-1}$ long, so we have $n := l_w/2 - 1 = 2n_{nb}$ points (16)

$$(i-n+1)2^{-\ell}, \ldots i2^{-\ell} \tag{22}$$

on the left of x and n points

$$(i+1)2^{-\ell}, \ldots (i+n)2^{-\ell} \tag{23}$$

on the right of x. In total we have $2n = l_w - 1$ points and we will index them by an index $j = 0, \ldots, l_w - 2$. Given discontinuity at $k2^{-3}$ we have an equation

$$(i - n + 1 + j)2^{-\ell} = k2^{-3},$$

and so

$$i = k2^{\ell-3} + n - 1 - j.$$

As the value of the integral depends just on j and ℓ and the dependence on ℓ is given by (9), we have 8 scaling functions $\times$ 7 discontinuities $\times$ $(l_w - 2)$ integrals and the cycle of the length $56(l_w - 2)$ to construct any block $B(6), \ldots$.

Let us denote by $B(\ell_1, \ell_2)$ blocks of wavelet functions of the levels ℓ_1 and ℓ_2. In case that ℓ_1 and ℓ_2 differ by at least two we use similar idea as described above for the blocks of scaling functions. Other blocks are stored in a compressed way, separately boundary wavelets and inner wavelets and to reconstruct the whole block we use (18), (19).

2.2 Number of Non-Zero Elements

Matrices (1) in a wavelet basis have a block structure. The first block in a row and in a column corresponds to scaling functions of the third level, the second one to wavelets of the third level, then the other levels follow (Figure 3).

In the following theorem, we show that matrices (1) in a wavelet basis have at most linear number of non-zero elements with respect to the order N of the matrices.

Theorem. Let $n \in \mathbb{N}$, $n \geq 3$,

$$\mathscr{B}_n = \{\varphi_{3,i} : i = 0, \ldots, 7\} \cup \bigcup_{\ell=4}^{n} \{\psi_{\ell,i} : i = 0, \ldots, < 2^{\ell-1} - 1\},$$

let N be $|\mathscr{B}_n| = 2^n$, l_w be the maximum of length of basis functions in $2^{-\ell}$ units, n_d be a maximal number of discontinuities of basis functions. Then the matrices (1) have at most $(2n_d l_w - 2n_d + 2l_w - 1)N$ non-zero elements.

PROOF.

- *Square diagonal blocks:* the integral is zero for basis functions with disjoint supports. Supports of neighbour functions are shifted by 2^{-l+1} – see (15) – so every function meets in its support at most $(l_w - 1)$ functions and every row will contain at most $(l_w - 1)$ non-zero elements. In total the N rows in diagonal blocks contain at most $(l_w - 1)N$ non-zero elements.

- *Blocks above the diagonal:* every basis function has at most n_d points of discontinuity and every discontinuity is met by at most $(l_w/2 - 1)$ wavelets of a given finer level. So every row in every block contains at most $n_d(l_w/2 - 1)$ non-zero elements. We will calculate them by blocks in the same column from right to left and get at most $n_d(l_w/2 - 1)(N/2 + N/4 + N/8 + \cdots + 8) < n_d(l_w/2 - 1)N$ non-zero elements.

- *In blocks under diagonal* there is the same amount of non-zero elements as over diagonal as matrices are symmetric.

We get

$$(l_w - 1)N + 2n_d(l_w/2 - 1)N = (n_d l_w - 2n_d + l_w - 1)N$$

in total. □

3 Numerical Experiment

We tested our implementation on a one-dimensional Poisson equation on the interval $[0, 1]$ with the solution

$$u(x) = (1 - x)(1 - e^{-50x}). \tag{24}$$

The columns of the table contain:

- Level of wavelet basis L.
- Matrix order N.
- Number of iterations (we use conjugate gradient method) #CG.
- Total time of conjugate gradient method in seconds.
- Time per cycle in seconds.
- L_2 norm of error of a solution.

We run it on a processor with 2.4 GHz frequency.

Note that the very slowly increasing number of iterations shows that wavelet basis is well-conditioned.

Tab. 1. *Results of numerical experiment*

L	N	#CG	time of CG(s)	time per cycle	L_2 norm
12	32 768	47	0.46	0.0098	2.1×10^{-12}
13	65 536	47	0.92	0.020	2.6×10^{-13}
14	131 072	49	1.9	0.040	3.3×10^{-14}
15	262 144	49	4.0	0.082	4.1×10^{-15}
16	524 288	50	8.3	0.17	5.2×10^{-16}
17	1 048 576	51	17	0.34	6.5×10^{-17}
18	2 097 152	51	35	0.69	8.1×10^{-18}

Source: Own

Conclusion

We presented an efficient implementation of multiplication by a wavelet matrix. Our next aim is to use it to solve a multi-dimensional Poisson equation on hypercube with a basis which is a tensor product of presented basis.

Acknowledgments

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Literature

[1] ČERNÁ, D.; FINĚK, V.: Construction of optimally conditioned cubic spline wavelets on the interval. In: *Advances in Computational Mathematics* 34(2): pp. 219-252, 2011. ISSN 1019-7168.

[2] ČERNÁ, D.; FINĚK, V.: The construction of well-conditioned wavelet basis based on quadratic B-splines. To appear In: *ICNAAM (T. E. Simos, G. Psihoyios, Ch. Tsitouras eds.)*, American Institute of Physics, New York, 2012.

[3] COHEN, A.; DAUBECHIES, I.; FEAVEAU, J.-C.: Biorthogonal bases of compactly supported wavelets. In: *Commun. Pure Apply. Math.* 45: pp. 485-560, 1992. ISSN 1097-0312.

[4] URBAN, K.: *Wavelet Methods for Elliptic Partial Differential Equations*. Oxford University Press, 2009. ISBN 978-0-19-852605-6.

RNDr. Martina Šimůnková, Ph.D.

Násobení waveletovou maticí – efektivní implementace

Matice tuhosti Dirichletovy okrajové úlohy $-(au')' = f$ v bázi splajnových waveletů má dle K. Urbana $O(n\log n)$ nenulových prvků. Ukážeme, že pro konstantní funkci a jich je ve skutečnosti $O(n)$ a popíšeme algoritmus, který počítá násobení vektoru touto maticí v $O(n)$ operacích. V implementaci používáme wavelety na bázi kvadratických splajnů.

Die Multiplikation der Wavelet-Matrix – eine effektive Implementierung

Die Zähigkeitsmatrix der Dirichlet-Randaufgabe $-(au')' = f$ auf der Basis der Spline-Wavelets hat nach K. Urban $O(n\log n)$ Nicht-Null-Elemente. Wir zeigen, dass es für die konstante Funktion a in Wirklichkeit $O(n)$ gibt, und wir beschreiben einen Algorithmus, der die Multiplikation des Vektors durch diese Matrix in $O(n)$-Operationen berechnet. In der Implementierung benutzen wir Wavelets auf der Basis quadratischer Splines.

Mnożenie macierzą falkową – efektywne wdrażanie

Macierz sztywności Dirichleta krańcowego zadania $-(au')' = f$ w bazie falek splajnowych ma wg K. Urbana $O(n\log n)$ elementów niezerowych. Pokazano, że dla funkcji stałej a jest ich w rzeczywistości $O(n)$ oraz opisano algorytm, który oblicza mnożenie wektora poprzez tę macierz w $O(n)$ operacjach. Wdrażane są falki na bazie splajnów drugiego stopnia.