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On the exact values of coefficients of coiflets

Research Article

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Abstract: In 1989, R. Coifman suggested the design of orthonormal wavelet systems with vanishing moments for both scaling and wavelet functions. They were first constructed by I. Daubechies [15, 16], and she named them coiflets. In this paper, we propose a system of necessary conditions which is redundant free and simpler than the known system due to the elimination of some quadratic conditions, thus the construction of coiflets is simplified and enables us to find the exact values of the scaling coefficients of coiflets up to length 8 and two further with length 12. Furthermore for scaling coefficients of coiflets up to length 14 we obtain two quadratic equations, which can be transformed into a polynomial of degree 4 for which there is an algebraic formula to solve them.

MSC: 65T60

Keywords: orthonormal wavelet • coiflet • exact value of filter coefficients

1. Introduction

Approximation properties of multiresolution analysis and the smoothness of wavelet and the scaling functions depend on the number of vanishing wavelet moments. In [14] Daubechies constructed orthonormal wavelets with arbitrary number N of vanishing wavelet moments and minimal length of support $2N-1$. The filter coefficients were computed there by an analytical method and exact values could be found only for filters up to length 6. In [26] Shann and Yen calculated the exact values of the filter coefficients of Daubechies wavelets of length 8 and 10. Other approaches for constructing Daubechies wavelets which enable us to find exact values of some coefficients can be found in [9, 10, 23, 24].
In addition to the orthogonality, compact support and vanishing wavelet moments, Coifman has suggested that also requiring vanishing scaling moments has some advantages. In practical applications these wavelets are useful due to their "nearly linear phase" and "almost interpolating property", see [22]. Daubechies created coiflets by prescribing an

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equal number N of vanishing wavelet moments and vanishing scaling moments for even N and the length of support $3N$, see [15, 16]. It was noticed in [4] that these coiflets have one more additional vanishing scaling moment than is imposed. Tian constructed coiflets with N vanishing moments for odd N and the length of support $3N-1$ in [27, 29]. Burrus and Odegard constructed coiflets with N vanishing moments for odd N and length of support $3N+1$ which have two additional vanishing scaling moments, see [7]. In this paper the computation of exact values of filter coefficients of coiflets up to filter length 14 is presented.

There exist a number of coiflet filter design methods, such as Newton's method [16, 25] or iterative numerical optimization [7]. These methods enable us to derive one particular solution for each system. The convergence and the obtained solution depend on the initial starting point though, and thus it is difficult to find all possible solutions. Moreover, the coefficients for length greater than 16 are given with less precision due to the roundoff error [15]. As an alternative one can use the Gröbner basis method [1, 6, 21]. This method is geared toward solving a polynomial system of equations with finite solutions. The idea consists of finding a new set of equations equivalent to the original set, which can be solved more easily. The advantage of such an approach is that solutions can be computed to arbitrary precision and that in some cases it gives all possible solutions for a given system of polynomial equations. In this paper we derive a redundant free and simplified system of equations and then aplly the Gröbner basis method. Using this approach we are able to find some exact values of filter coefficients and all possible solutions for filters up to length 20.

2. Preliminaries

The scaling function ϕ, which generates a coiflet, is constructed as the solution to the scaling equation

$$\phi = 2\sum_{k\in\mathbb{Z}} h_k \phi(2\cdot - k), \tag{1}$$

where the scaling coefficients $\{h_k\}$ are determined so that the corresponding scaling functions and wavelets have required properties.

Definition 2.1.

An orthonormal wavelet ψ with compact support is called a *coiflet of order* N, if the following conditions are satisfied:

i) $\displaystyle\int_{-\infty}^{\infty} x^n \psi(x)\,dx = 0 \quad \text{for} \quad n = 0, \ldots N-1,$

ii) $\displaystyle\int_{-\infty}^{\infty} x^n \phi(x)\,dx = \delta_n \quad \text{for} \quad n = 0, \ldots N-1,$

where ϕ is the scaling function corresponding to ψ and δ_n is Kronecker delta, i.e. $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$.

Since the length of support also plays a role, it is common to consider a wavelet satisfying *i*) and *ii*) which has the minimal length of support. The existence of a coiflet for an arbitrary order N is still an open question. We rewrite this definition in terms of filter coefficients $\{h_k\}$. It is known that for an orthonormal wavelet with compact support the number of filter coefficients is an even number, which we denote by $2M$.

Lemma 2.1.

Let $\{h_k\}_{k=N_1}^{N_2}$ *be real coefficients with* $N_2 = N_1 + 2M - 1$. *If the orthonormal wavelet corresponding to the scaling function* $\phi(\cdot) = 2\sum_{k=N_1}^{N_2} h_k\, \phi(2\cdot - k)$ *is a coiflet of order N, then the following three conditions are satisfied:*

i) $\displaystyle\delta_m = 2\sum_{j=0}^{N_2-N_1-2m} h_{N_1+j}h_{N_1+2m+j} \quad \text{for} \quad 0 \le m \le M-1,$

ii) $\sum_{k=N_1}^{N_2} h_k k^n = \delta_n \quad \text{for} \quad 0 \le n \le N-1,$

iii) $\sum_{k=N_1}^{N_2} (-1)^k h_k k^n = 0 \quad \text{for} \quad 0 \le n \le N-1.$

Condition i) is necessary but not sufficient for a wavelet to be orthonormal. Conditions ii) and iii) are equivalent to vanishing wavelet and vanishing scaling function moments, respectively. In summary, the conditions in Lemma 2.1 are only necessary. It is known that they are not sufficient to generate a coiflet system. There exist functions given by (1) with filter coefficients satisfying $i) - iii$) from Lemma 2.1 which are very rough. Hence after finding coefficients satisfying $i) - iii$), orthonormality should be verified, for example using the Cohen [11] or Lawton [20] conditions. Typically there are multiple wavelets satisfying these conditions and some of them, despite zero wavelet moments, are very rough. Likewise, in spite of zero scaling function moments, some are not at all symmetric. In practical applications the most regular wavelet or the wavelet with the most symetrical scaling function is typically chosen.

3. Construction

It is well known that coiflets have more vanishing scaling moments than required in the above definition. This was first noted by G. Beylkin et al. in [4]. In this paper, we derive a redundant free and simpler definition of coiflets. The key component of our approach is formed by the following Theorem.

Theorem 3.1.
Let $N_2 = N_1 + 2M - 1$. Then

$$\delta_m = 2 \sum_{j=N_1}^{N_2-2m} h_j h_{j+2m} \quad \text{for} \quad 0 \le m \le M-1 \tag{2}$$

is equivalent to

$$\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i \left(a_i a_{2n-i} + b_i b_{2n-i}\right) \quad \text{for} \quad 0 \le n \le M-1, \tag{3}$$

where

$$a_i = \sum_{k=0}^{M-1} (N_1+2k)^i h_{N_1+2k} \qquad \text{and} \qquad b_i = \sum_{k=0}^{M-1} (N_1+2k+1)^i h_{N_1+2k+1}. \tag{4}$$

Proof. Since condition (2) is equivalent to the condition

$$|m(\omega)|^2 + |m(\omega+\pi)|^2 = 1, \tag{5}$$

where

$$m(\omega) = \sum_{k=N_1}^{N_2} h_k e^{-ik\omega},$$

we can repeat the proof of Theorem 3.1 in [19] with some obvious changes. □

Due to Theorem 3.1 we are now able to impose necessary conditions on filter coefficients to generate a coiflet which are equivalent to conditions from Lemma 2.1 and the arising system is without redundant conditions.

Corollary 3.1.

Let $\{h_k\}_{k=N_1}^{N_2}$ be real coefficients, $N_2 = N_1 + 2M - 1$ and let a_i and b_i be defined by (4). Then conditions $i) - iii)$ from Lemma 2.1 are equivalent to the following conditions:

$$i^*)\ 0 = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}) \quad \text{for} \quad N \leq n \leq M-1,$$

$$ii^*)\ \ a_0 = b_0 = \frac{1}{2},$$

$$iii^*)\ \ a_n = b_n = 0 \quad \text{for} \quad 1 \leq n \leq N-1,$$

$$iv^*)\ \ a_{2n} + b_{2n} = 0 \quad \text{for} \quad N \leq 2n \leq 2N-2.$$

Proof. It is clear that $ii)$ and $iii)$ are equivalent to $ii*)$ and $iii*)$. The rest follows from Theorem 3.1. □

The consequence of this Corollary is that the minimal length of support of coiflet of order N is $3N$ for even N and $3N-1$ for odd N and that some coiflets have more vanishing moments than are imposed. Thus, we have three classes of coiflets, see Table 1.

Table 1. The length of filter $2M$, the number of vanishing scaling and wavelet moments for coiflet of order N

N	2M	number of vanishing scaling moments		number of vanishing wavelet moments	
		set	actual	set	actual
even	3N	N	N+1	N	N
odd	3N-1	N	N	N	N
odd	3N+1	N+1	N+2	N	N

Now we further simplify the system by replacing some quadratic conditions by linear ones.

Lemma 3.1.

Let a_i, b_i be defined by (4). Then a_i is a linear combination of $a_0, \dots a_{M-1}$ for $i \geq M$, and b_i is a linear combination of $b_0, \dots b_{M-1}$ for $i \geq M$.

Proof. Coefficients $h_{N_1}, h_{N_1+2}, \dots h_{N_1+2M-2}$ are a solution of the system of linear algebraic equations (4). Since the matrix of this system is regular, the solution exists and is unique. a_i is a linear combination of $h_{N_1}, h_{N_1+2}, \dots h_{N_1+2M-2}$ and thus for $i \geq M$, a_i is a linear combination of a_i for $0 \leq i \leq M-1$:

$$a_i = c_{i0}a_0 + c_{i1}a_1 + \dots c_{iM-1}a_{M-1},$$

where the coefficients of this linear combinations are given by

$$\begin{pmatrix} 1 & N_1 & N_1^2 & \dots & N_1^{M-1} \\ 1 & N_1+2 & (N_1+2)^2 & \dots & (N_1+2)^{M-1} \\ \vdots & & & & \vdots \\ 1 & N_1+2M-2 & (N_1+2M-2)^2 & \dots & (N_1+2M-2)^{M-1} \end{pmatrix} \begin{pmatrix} c_{i0} \\ c_{i1} \\ \vdots \\ c_{iM-1} \end{pmatrix} = \begin{pmatrix} N_1^i \\ (N_1+2)^i \\ \vdots \\ (N_1+2M-2)^i \end{pmatrix}.$$

The situation for b_i is similar. □

Now we summarize the construction procedure which enables us to find exact values of coefficients of coiflets up to length of support 14:

1. For a given N, take the system of algebraic equations given by Corollary 3.1.
2. Replace $a_M, \dots a_{2M-2}$ by linear combinations of $a_0, \dots a_{M-1}$ and $b_M, \dots b_{2M-2}$ by linear combinations of $b_0, \dots b_{M-1}$.
3. Solve the arising system for $a_0, \dots a_{M-1}, b_0, \dots b_{M-1}$. For larger N use the Gröbner basis method to simplify the system.
4. Compute filter coefficients $h_{N_1}, \dots, h_{N_2}$ by solving the system of linear algebraic equations (4).

4. Examples

At last we provide two examples to illustrate our approach based on Corollary 3.1.

Example 4.1.

For $N = 4$ and $N_1 = -5$, the following system will be obtained:

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0,$$

$$a_4 + b_4 = 0 \quad \text{and} \quad a_6 + b_6 = 0, \tag{6}$$

$$a_8 + b_8 + 140\, b_4^2 = 0 \quad \text{and} \quad a_{10} + b_{10} + 840\, b_4 b_6 - 252(a_5^2 + b_5^2) = 0. \tag{7}$$

Now a_6, a_8, a_{10}, b_6, b_8, b_{10} are linear combinations of $a_0 \dots a_5$, $b_0 \dots b_5$. We find these linear combinations and substitute them into (6) and (7). Then after simplification we obtain the system

$$-135 + 12b_4 + 8b_4^2 = 0, \quad a_4 + b_4 = 0,$$

$$75 - 10b_4 + 4b_5 = 0, \quad 32a_5^2 + 12300b_4 - 28575 = 0.$$

In this case we can easily find both real solutions in closed form. See Table 2 and Table 3.

Example 4.2.

For $N = 5$ and $N_1 = -5$, the following system will be obtained:

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0,$$

$$a_6 + b_6 = 0 \quad \text{and} \quad a_8 + b_8 = 0, \tag{8}$$

$$a_{10} + b_{10} - 252(a_5^2 + b_5^2) = 0 \quad \text{and} \quad a_{12} + b_{12} - 1584\, b_5 b_7 - 1584\, a_5 a_7 + 924(a_5^2 + b_5^2) = 0. \tag{9}$$

Now a_7, a_8, a_{10}, a_{12}, b_6, b_8, b_{10}, b_{12} are linear combinations of $a_0 \dots a_6$, $b_0 \dots b_6$. We find these linear combinations and substitute them into (8) and (9). Consequently we simplify the arising system and finally compute its Gröbner bases. The following system is obtained:

$$11419648\, b_5^4 + 246374400\, b_5^3 - 13765248000\, b_5^2 - 497539800000\, b_5 - 4303042734375 = 0,$$

Table 2. Error of approximation of the function f by 50 coefficients for coiflets of order N, length of support $2M$ and Sobolev exponent of smoothness α

N	2M	α	L^∞ of error $\times 10^{-6}$	L^2 norm of error $\times 10^{-7}$	H^1 seminorm of error $\times 10^{-4}$
1	4	0.604	743	1986	1358
1	4	0.050	2800	7332	5642
2	6	0.041	402	978	706
2	6	1.232	44	116	46
2	6	0.590	184	469	234
2	6	1.022	83	200	87
3	8	0.147	103	225	137
3	8	1.775	2	6	1
3	8	1.422	20	31	13
3	8	0.936	44	97	33
3	8	1.464	15	33	10
3	8	1.773	3	5	1

$$298890000\,a_5 - 5709824\,b_5^3 + 3945600\,b_5^2 + 6931764000\,b_5 + 94943559375 = 0,$$

$$8\,a_6 + 64\,b_5 + 525 = 0, \qquad -525 - 64\,b_5 + 8\,b_6 = 0.$$

Then by using an algebraic formula for the solution of polynomials of degree 4 we obtain two different real roots:

$$b_5 = \frac{15\left(\sqrt{15}u^{3/4} - 4010u^{1/6}v^{1/4} \pm \sqrt{15}\sqrt{w}\right)}{11152u^{1/6}v^{1/4}},$$

where

$$u = 4854802096 + 369\sqrt{15}\sqrt{66685436848043}, \quad v = 8475076\,u^{1/3} + 697\,u^{2/3} - 3366028373,$$

$$w = 16950152\,u^{1/3}\sqrt{v} - 697\sqrt{v}\,u^{2/3} + 3366028373\sqrt{v} + 13383342756\sqrt{15}\sqrt{u}.$$

Once we have the values of b_5, we simply find a_5, a_6, and b_6. And finally we transform coefficients a_i and b_i to scaling coefficients h_i.

5. Properties of coiflets

Let us now mention the properties of such constructed wavelets. It is well-known that the approximation properties depend on the number of vanishing wavelet moments. More precisely, let $P_j f$ be an approximation of $f \in L^2(\mathbb{R})$ on level j, i.e.

$$P_j f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{j,k} \rangle\, \phi_{j,k}, \tag{10}$$

and for $J < j$,

$$P_j f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{J,k} \rangle\, \phi_{J,k} + \sum_{l=J}^{j-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_{l,k} \rangle\, \psi_{l,k}, \tag{11}$$

where $\phi_{l,k} = 2^{l/2}\phi\left(2^l \cdot - k\right)$ and $\psi_{l,k} = 2^{l/2}\psi\left(2^l \cdot - k\right)$ for $l, k \in \mathbb{Z}$. Let us further denote $I_{l,k} = \operatorname{supp}\, \phi_{l,k}$, $J_{l,k} = \operatorname{supp}\, \psi_{l,k}$. The wavelet coefficients satisfy

$$\langle f, \psi_{l,k} \rangle = \int_{-\infty}^{\infty} f(x)\, 2^{l/2}\psi\left(2^l x - k\right) dx, \tag{12}$$

Table 3. Scaling coefficients of coiflets of order N, length of filter $2M$ and Sobolev exponent α

	n	h_n		n	h_n
$N=1, 2M=2$	0	$\frac{1}{2}$		1	$\frac{7+\sqrt{7}}{16}$
Haar wavelet	1	$\frac{1}{2}$		2	$\frac{1+\sqrt{7}}{16}$
$N=1, 2M=4$	-1	$\frac{3}{8}-\frac{\sqrt{3}}{8}$		3	$\frac{-3-\sqrt{7}}{32}$
$\alpha=0.604$	0	$\frac{3}{8}+\frac{\sqrt{3}}{8}$	$N=2, 2M=6$	-2	$\frac{1-\sqrt{7}}{32}$
	1	$\frac{1}{8}+\frac{\sqrt{3}}{8}$	$\alpha=1.022$	-1	$\frac{5+\sqrt{7}}{32}$
	2	$\frac{1}{8}-\frac{\sqrt{3}}{8}$	most symmetrical	0	$\frac{7+\sqrt{7}}{16}$
$N=1, 2M=4$	-1	$\frac{3}{8}+\frac{\sqrt{3}}{8}$		1	$\frac{7-\sqrt{7}}{16}$
$\alpha=0.050$	0	$\frac{3}{8}-\frac{\sqrt{3}}{8}$		2	$\frac{1-\sqrt{7}}{16}$
	1	$\frac{1}{8}-\frac{\sqrt{3}}{8}$		3	$\frac{-3+\sqrt{7}}{32}$
	2	$\frac{1}{8}+\frac{\sqrt{3}}{8}$	$N=3, 2M=8$	-1	$\frac{15}{64}+\frac{3\sqrt{1495}}{1664}$
$N=2, 2M=6$	-1	$\frac{9+\sqrt{15}}{32}$	$\alpha=0.147$	0	$\frac{59}{128}-\frac{\sqrt{1495}}{832}$
$\alpha=0.041$	0	$\frac{13-\sqrt{15}}{32}$		1	$\frac{15}{64}-\frac{9\sqrt{1495}}{1664}$
	1	$\frac{3-\sqrt{15}}{16}$		2	$\frac{15}{128}+\frac{3\sqrt{1495}}{832}$
	2	$\frac{3+\sqrt{15}}{16}$		3	$\frac{5}{64}+\frac{9\sqrt{1495}}{1664}$
	3	$\frac{1+\sqrt{15}}{32}$		4	$-\frac{15}{128}-\frac{3\sqrt{1495}}{832}$
	4	$\frac{-3-\sqrt{15}}{32}$		5	$-\frac{3}{64}-\frac{3\sqrt{1495}}{1664}$
$N=2, 2M=6$	-1	$\frac{9-\sqrt{15}}{32}$		6	$\frac{5}{128}+\frac{\sqrt{1495}}{832}$
$\alpha=1.232$	0	$\frac{13+\sqrt{15}}{32}$	$N=3, 2M=8$	-1	$\frac{15}{64}-\frac{3\sqrt{1495}}{1664}$
	1	$\frac{3+\sqrt{15}}{16}$	$\alpha=1.775$	0	$\frac{59}{128}+\frac{\sqrt{1495}}{832}$
	2	$\frac{3-\sqrt{15}}{16}$		1	$\frac{15}{64}+\frac{9\sqrt{1495}}{1664}$
	3	$\frac{1-\sqrt{15}}{32}$		2	$\frac{15}{128}-\frac{3\sqrt{1495}}{832}$
	4	$\frac{-3+\sqrt{15}}{32}$		3	$\frac{5}{64}-\frac{9\sqrt{1495}}{1664}$
$N=2, 2M=6$	-2	$\frac{1+\sqrt{7}}{32}$		4	$-\frac{15}{128}+\frac{3\sqrt{1495}}{832}$
$\alpha=0.590$	-1	$\frac{5-\sqrt{7}}{32}$		5	$-\frac{3}{64}+\frac{3\sqrt{1495}}{1664}$
	0	$\frac{7-\sqrt{7}}{16}$		6	$\frac{5}{128}-\frac{\sqrt{1495}}{832}$

and if $f \in C^N(J_{l,k})$, then expanding f about $\frac{k}{2^l}$, it follows by Taylor's formula that for all $x \in J_{l,k}$,

$$f(x) = f\left(\frac{k}{2^l}\right) + f'\left(\frac{k}{2^l}\right)\left(x - \frac{k}{2^l}\right) + \ldots + \frac{f^{(N-1)}\left(\frac{k}{2^l}\right)}{(N-1)!}\left(x - \frac{k}{2^l}\right)^{N-1} + \frac{f^{(N)}(\xi)}{N!}\left(x - \frac{k}{2^l}\right)^N, \tag{13}$$

where ξ depends on x and belongs to the interval $J_{l,k}$. If ψ has N vanishing moments, i.e. if condition $i)$ in Definition 2.1 is satisfied, then the first N terms don't contribute and

$$|\langle f, \psi_{l,k}\rangle| = \left|\int_{-\infty}^{\infty} \frac{f^{(N)}(\xi(x))}{N!}\left(x - \frac{k}{2^l}\right)^N 2^{l/2}\psi\left(2^l x - k\right) dx\right| \le C 2^{-l(N+1/2)}, \tag{14}$$

where

$$C = \frac{\max_{\xi \in J_{l,k}} \left|f^{(N)}(\xi)\right|}{N!} \int_{J_{l,k}} |y|^N \psi(y)\, dy. \tag{15}$$

Thus for l large, the wavelet coefficients are small except for those which are near singularities of the function f or its derivatives. Small coefficients can be set to zero and the function f can be represented by a small number of coefficients.

This compression property of wavelets has many applications. Most important are data compression, signal analysis and efficient adaptive schemes for PDE's. Note that more vanishing wavelet moments imply a faster decay of wavelet coefficients and that only local smoothness of the function f is involved in the above estimate. It was observed in [2] that also regularity of the scaling function plays a role. We confirmed in our experiments that this is true for coiflets as well. As an example, let us consider

$$\begin{aligned} f(x) &= x^5 && \text{if} \quad 0 \le x \le 0.5, \\ &= (1-x)^5 && \text{if} \quad 0.5 < x \le 1, \\ &= 0 && \text{otherwise,} \end{aligned}$$

and its n-term approximation

$$f_n(x) = \sum_{\lambda=(l,k)\in\Lambda_\phi^n} \langle f, \phi_\lambda \rangle \phi_\lambda + \sum_{\lambda=(l,k)\in\Lambda_\psi^n} \langle f, \psi_\lambda \rangle \psi_\lambda, \tag{16}$$

where $\Lambda_\phi^n \subset \{\lambda = (J,k), k \in \mathbb{Z}\}$, $\Lambda_\psi^n \subset \{\lambda = (l,k), J \le l < j, k \in \mathbb{Z}\}$ and $\Lambda_\phi^n \cup \Lambda_\psi^n$ is the set of indexes of the n largest coefficients. In our case, the coarsest level is $J = 3$, the finest level is $j = 9$ and the number of preserved coefficients is $n = 50$. The function f has a sharp derivative near the point $x = 0.5$ and the approximation is automatically refined near this point. Errors of approximation for some of the constructed coiflets are shown in Table 2. We can see that the most regular coiflet of prescribed order gives the best result.

The significance of vanishing scaling moments highly depends on the type of application. In [16], it is proved that all real orthonormal wavelets with compact support are asymmetric. However, vanishing scaling moments result in "almost symmetry" of the scaling function and filter. In image coding, more symmetry would result in greater compressibility for the same perceptual error and it makes it easier to deal with the boundaries of the image. Vanishing scaling moments also cause a "nearly linear phase", which is a desired quality in many applications, e.g. transmission of audio and video signals, because it does not cause phase distortion. In numerical analysis, vanishing scaling moments are important due to their "almost interpolating property". It means that any $f \in C_0^N(\mathbb{R})$ can be approximated by

$$f_j = 2^{-j/2} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k} \tag{17}$$

and if the number of vanishing scaling and wavelet moments is N then this approximation satisfies the following estimate

$$\left\| f - f_j \right\| \le C 2^{-jN}, \tag{18}$$

where C depends only on f and the scaling function ϕ, see [28]. Due to this property, some types of operators can be treated efficiently. Thus coiflets have some interesting properties and for some applications are more suitable than orthonormal wavelets with vanishing wavelet moments only. The price to pay is of course the length of support, which can make the computation more expensive. We should also mention that we can obtain symmetric wavelets by giving up orthonormality. Symmetric biorthogonal wavelets were constructed in [12], and construction of biorthogonal coiflets can be found in [28, 29]. However, there are applications where orthogonality plays a role and the disadvantage of biorthogonal wavelets is their bad stability when adapted to the interval, see [5, 13].

In literature, one can find coiflets which are the most symmetrical among all coiflets of a given order and length of support, see [7, 15, 16, 27, 29]. As we could see above, these coiflets need not be the best, and other solutions of equations given in Corollarry 3.1 may be better suited for some types of applications. Typically the most regular coiflet for a given order N has the best compression property and due to the almost interpolating property and the ability to generate a stable wavelet basis on a bounded domain it seems to be very well suited for some applications, e.g. numerical solutions of PDE's.

Table 4. Scaling coefficients of coiflets of order N, length of filter $2M$ and Sobolev exponent α

	n	h_n		n	h_n
$N=3, 2M=8$	-2	$\frac{21}{640}-\frac{3\sqrt{31}}{320}$	$N=3, 2M=8$	-3	$-\frac{1}{32}-\frac{\sqrt{7}}{128}$
$\alpha=1.422$	-1	$\frac{51}{320}+\frac{3\sqrt{31}}{640}$	$\alpha=1.773$	-2	$-\frac{3}{128}$
	0	$\frac{257}{640}+\frac{9\sqrt{31}}{320}$	most symmetrical	-1	$\frac{9}{32}+\frac{3\sqrt{7}}{128}$
	1	$\frac{147}{320}-\frac{9\sqrt{31}}{640}$		0	$\frac{73}{128}$
	2	$\frac{63}{640}-\frac{9\sqrt{31}}{320}$		1	$\frac{9}{32}-\frac{3\sqrt{7}}{128}$
	3	$\frac{-47}{320}+\frac{9\sqrt{31}}{640}$		2	$-\frac{9}{128}$
	4	$\frac{-21}{640}+\frac{3\sqrt{31}}{320}$		3	$-\frac{1}{32}+\frac{\sqrt{7}}{128}$
	5	$\frac{9}{320}-\frac{3\sqrt{31}}{640}$		4	$\frac{3}{128}$
$N=3, 2M=8$	-2	$\frac{21}{640}+\frac{3\sqrt{31}}{320}$	$N=4, 2M=12$	-5	$\frac{7}{1024}+\frac{\sqrt{31}}{1024}-\frac{\sqrt{336+82\sqrt{31}}}{2048}$
$\alpha=0.936$	-1	$\frac{51}{320}-\frac{3\sqrt{31}}{640}$	$\alpha=1.707$	-4	$\frac{7}{2048}-\frac{3\sqrt{31}}{2048}$
	0	$\frac{257}{640}-\frac{9\sqrt{31}}{320}$		-3	$-\frac{53}{1024}-\frac{3\sqrt{31}}{1024}+\frac{5\sqrt{336+82\sqrt{31}}}{2048}$
	1	$\frac{147}{320}+\frac{9\sqrt{31}}{640}$		-2	$-\frac{39}{2048}+\frac{11\sqrt{31}}{2048}$
	2	$\frac{63}{640}+\frac{9\sqrt{31}}{320}$		-1	$\frac{151}{512}+\frac{\sqrt{31}}{512}-\frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	3	$\frac{-47}{320}-\frac{9\sqrt{31}}{640}$		0	$\frac{555}{1024}-\frac{7\sqrt{31}}{1024}$
	4	$\frac{-21}{640}-\frac{3\sqrt{31}}{320}$		1	$\frac{151}{512}+\frac{\sqrt{31}}{512}+\frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	5	$\frac{9}{320}+\frac{3\sqrt{31}}{640}$		2	$-\frac{47}{1024}+\frac{3\sqrt{31}}{1024}$
$N=3, 2M=8$	-3	$-\frac{1}{32}+\frac{\sqrt{7}}{128}$		3	$-\frac{53}{1024}-\frac{3\sqrt{31}}{1024}-\frac{5\sqrt{336+82\sqrt{31}}}{1024}$
$\alpha=1.464$	-2	$-\frac{3}{128}$		4	$\frac{51}{2048}+\frac{\sqrt{31}}{2048}$
	-1	$\frac{9}{32}-\frac{3\sqrt{7}}{128}$		5	$\frac{7}{1024}+\frac{\sqrt{31}}{1024}+\frac{\sqrt{336+82\sqrt{31}}}{1024}$
	0	$\frac{73}{128}$		6	$-\frac{11}{2048}-\frac{\sqrt{31}}{2048}$
	1	$\frac{9}{32}+\frac{3\sqrt{7}}{128}$			
	2	$-\frac{9}{128}$			
	3	$-\frac{1}{32}-\frac{\sqrt{7}}{128}$			
	4	$\frac{3}{128}$			

6. Conclusion

The arising system from the Corollary 3.1 is redundant-free, more simple (due to the elimination of some quadratic conditions) and enables us to find directly the exact values of the scaling coefficients of coiflets up to length 8 and two further with length 12 in closed form. The results are given in Table 3, Table 4 and Table 5. We verified orthonormality using the Lawton criterion, and all the results correspond to orthonormal scaling functions. As mentioned earlier, the solutions are not of the same quality, since smoothness and symmetry also play a role. For this reason the most symmetrical scaling function among all scaling functions of order N is denoted in the Tables, and the Sobolev exponents of smoothness are computed using the method from [17, 31]. Furthermore, for the remaining coiflets up to length 14 we obtain two quadratic equations of two variables. These can be transformed into polynomials of degree 4, for which there is an algebraic formula to find solutions in closed form. We do not provide these solutions because of their length and complicated structure. Moreover, one can use our approach to find all possible solutions to a given system up to the length of filter 20. For longer filters the computation failed since the coefficients of the polynomials in the Gröbner basis were too large.

Table 5. Scaling coefficients of coiflets of order N, length of filter $2M$ and Sobolev exponent α

	n	h_n
$N = 4, 2M = 12$	-5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} + \frac{\sqrt{336+82\sqrt{31}}}{2048}$
$\alpha = 2.174$	-4	$\frac{7}{2048} - \frac{3\sqrt{31}}{2048}$
	-3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} - \frac{5\sqrt{336+82\sqrt{31}}}{2048}$
	-2	$-\frac{39}{2048} + \frac{11\sqrt{31}}{2048}$
	-1	$\frac{151}{512} + \frac{\sqrt{31}}{512} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	0	$\frac{555}{1024} - \frac{7\sqrt{31}}{1024}$
	1	$\frac{151}{512} + \frac{\sqrt{31}}{512} - \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	2	$-\frac{47}{1024} + \frac{3\sqrt{31}}{1024}$
	3	$-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$
	4	$\frac{51}{2048} + \frac{\sqrt{31}}{2048}$
	5	$\frac{7}{1024} + \frac{\sqrt{31}}{1024} - \frac{\sqrt{336+82\sqrt{31}}}{1024}$
	6	$-\frac{11}{2048} - \frac{\sqrt{31}}{2048}$

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References

[1] Adams W.W., Loustaunau P., An Introduction to Gröbner Bases, American Mathematical Society, 1994

[2] Antonini M., Barlaud M., Mathieu P., Daubechies I., Image coding using wavelet transforms, IEEE Trans. Image Process., 1992, 1, 205–220

[3] Beylkin G., On the representation of operators in bases of compactly supported wavelets, SIAM J. Numer. Anal., 1992, 29, 1716–1740

[4] Beylkin G., Coifman R.R., Rokhlin V., Fast wavelet transforms and numerical algorithms I, Comm. Pure Appl. Math., 1991, 44, 141–183

[5] Bittner K., Urban K., Adaptive wavelet methods using semiorthogonal spline wavelets: Sparse evaluation of nonlinear functions, preprint

[6] Buchberger B., An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, PhD thesis, University of Inssbruck, Austria, 1965 (in German)

[7] Burrus C.S., Odegard J.E., Coiflet Systems and Zero Moments, IEEE Trans. Signal Process., 1998, 46, 761–766

[8] Burrus C.S., Gopinath R.A., On the moments of the scaling function ψ_0, Proceedings of the ISCAS-92, 1992, 963–966

[9] Černá D., Finěk V., On the computation of scaling coefficients of Daubechies wavelets, Cent. Eur. J. Math., 2004, 2, 399–419

[10] Chyzak F., Paule P., Scherzer O., Schoisswohl A., Zimmermann B., The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval, Experiment. Math., 2001, 10, 67–86

[11] Cohen A., Ondelettes analyses multirésolutions et filtres miroir en quadrature, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1990, 7, 439–459

[12] Cohen A., Daubechies I., Feauveau J.C., Biorthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math., 1992, 45, 485–500
[13] Dahmen W., Kunoth A., Urban K., Biorthogonal spline wavelets on the interval - stability and moment conditions, Appl. Comput. Harmon. Anal., 1999, 6, 132–196
[14] Daubechies I., Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math., 1988, 41, 909–996
[15] Daubechies I., Orthonormal bases of compactly supported wavelets II Variations on a theme, SIAM J. Math. Anal., 1993, 24, 499–519
[16] Daubechies I., Ten lectures on wavelets, SIAM, Philadelphia, 1992
[17] Eirola T., Sobolev characterization of compactly supported wavelets, SIAM J. Math. Anal., 1992, 23, 1015–1030
[18] Finěk V., Approximation properties of wavelets and relations among scaling moments, Numer. Funct. Anal. Optim., 2004, 25, 503–513
[19] Finěk V., Approximation properties of wavelets and relations among scaling moments II, Cent. Eur. J. Math., 2004, 2, 605–613
[20] Lawton W.M., Necessary and sufficient conditions for constructing orthonormal wavelet bases, J. Math. Phys., 1991, 32, 57–61
[21] Lebrun J., Selesnick I., Grobner bases and wavelet design, J. Symb. Comp., 2004, 37, 227–259
[22] Monzón L., Beylkin G., Hereman W., Compactly supported wavelets based on almost interpolating and nearly linear phase filters (coiflets), Appl. Comput. Harmon. Anal., 1999, 7, 184–210
[23] Regensburger G., Scherzer O., Symbolic computation for moments and filter coefficients of scaling functions, Ann. Comb., 2005, 9, 223–243
[24] Regensburger G., Parametrizing compactly supported orthonormal wavelets by discrete moments, Applicable Algebra in Engineering, Communication and Computing, 2007, 18, 583–601
[25] Resnikoff H.L., Wells R.O., Wavelet analysis. The scalable structure of information, Springer-Verlag, New York, 1998
[26] Shann W.C., Yen C.C., On the exact values of orthonormal scaling coefficients of length 8 and 10, Appl. Comput. Harmon. Anal., 1999, 6, 109–112
[27] Tian J., The mathematical theory and applications of biorthogonal Coifman wavelet systems, Ph.D. thesis, Rice University, Houston, TX, 1996
[28] Tian J., Wells R.O. Jr., Vanishing moments and biorthogonal Coifman wavelet systems, Proceedings of 4th International Conference on Mathematics in Signal Processing, University of Warwick, England, 1997
[29] Tian J., Wells R.O. Jr., Vanishing moments and wavelet approximation, Technical Report, CML TR95-01, Rice University, January 1995
[30] Unser M., Approximation power of biorthogonal wavelet expansions, IEEE Transactions on Signal Processing, 1996, 44, 519–527
[31] Villemoes L.F., Energy moments in time and frequency for two-scale difference equation solutions and wavelets, SIAM J. Math. Anal., 1992, 23, 1519–1543