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Note

Shortness coefficients of simple 3-polytopal graphs with edges of only two types

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Abstract

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We consider two classes of simple 3-polytopal graphs whose edges are incident with either two 5-gons or a 5-gon and q-gon ($q = 26$ or 27). We show that the shortness coefficient is less than 1 for both of these classes.

1. Introduction and results

In this note, a graph has no loops and no multiple edges and a multigraph has no loops but may have multiple edges.

Let $P(x_1, x_2, \dots, x_n)$ (or $P(x_1, x_n)$) denote a path (in a given graph) connecting x_i to x_{i+1} for $i = 1, 2, \dots, n-1$ with end vertices x_1 and x_n. For a graph G, let $v(G)$ and $h(G)$ denote the number of vertices and the length of a maximum cycle, respectively. The shortness coefficient $\rho(\mathcal{G})$ of an infinite class of graphs $\mathcal{G}$ is defined by

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{h(G)}{v(G)}, \quad \text{see [2].}$$

Let $G(p, q)$ denote the class of 3-connected trivalent planar graphs, i.e., simple 3-polytopal graphs, all of whose faces are p-gons and q-gons, $p < q$, $p \geq 3$. Let $S(p, q)$ denote the class of simple 3-polytopal graphs in which all the edges are incident with two p-gons or a p-gon and a q-gon, $p \neq q$, $p, q \geq 3$. Thus $S(p, q)$ is a subclass of $G(p, q)$ if $p < q$ and $G(q, p)$ if $p > q$.

In several papers, including [2, 3, 6, 7, 8, 9] it has been shown that the shortness coefficient is less than 1 for many classes of graphs in $G(p, q)$, as well as for some subclasses defined by imposing various special conditions.

Owens proved that each class $S(5, q)$, for $q \geq 28$ has a shortness coefficient which is less than 1 and asked whether there are some non-Hamiltonian members in the classes $S(5, q)$ for $12 \leq q \leq 23$ or $q = 27$ and whether $\rho(S(5, q)) < 1$, for $q = 24$, 25 or 26 [7, Problem 1 and 2]. Recently Jendrol' and Mihók [4] proved that all graphs of the class $S(5, 12)$ are Hamiltonian.

Our next theorem partially supplements these results

Theorem. (1) $\rho(S(5, 26)) \leq 209/210 < 1$.
(2) $\rho(S(5, 27)) \leq 439/440 < 1$.

2. Constructions and proof of the theorem

We begin by defining certain $S(5, q)$-subgraphs, that is, graphs which can occur as induced subgraphs in graphs of the class $S(5, q)$. For $m = 2, 3$ we shall use the graph denoted by V_m in [7]. Evidently, as shown in Fig. 1, the graphs V_m are planar and trivalent apart from three pairs of adjacent 2-valent vertices x_i, y_i ($i = 1, 2, 3$). The 2-valent vertices and the edges $x_i y_i$ which join them in pairs will be called half vertices and half edges since, when a copy of V_m occurs in a graphs of the class $S(5, q)$, these elements of V_m are identified with elements of other $S(5, q)$-subgraphs. All interior faces of V_m, that is, all faces except the exterior face are 5-gons. The exterior face is the face whose boundary contains all the half edges (see Fig. 1). The property of V_3 which makes it useful to us is stated in the following lemma.

Lemma 1. *Let G be a trivalent graph which contains a copy of V_3. Let C be a cycle in G which contains edges of both V_3 and G-V_3. Let C spans V_3 and let C*

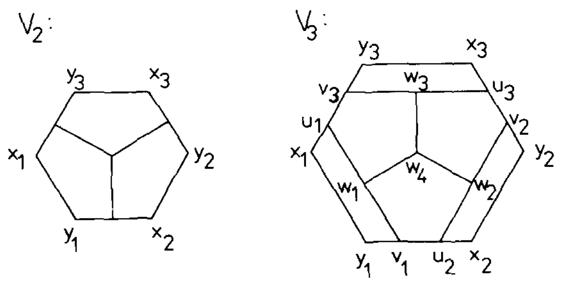

Fig. 1. The graphs V_m.

contains only one of the edges x_iu_i or y_iv_i, for $i = 1, 2$. After allowing for symmetry, $C \cap V_3$ must be of one of the following types:

Type (1) $P(x_1, x_2) \cup P(x_3, y_3)$,

Type (2) $P(y_1, y_3) \cup P(x_2, x_3)$.

Proof. We consider three cases.

Case 1: *Both edges y_1v_1 and x_2u_2 are in C.*

Then x_1u_1 and y_2v_2 are in V_3-C. Thus v_3u_1, u_1w_1, w_2v_2, v_2u_3 are in C. If v_1u_2 is in C, then

$$P(v_3, u_1, w_1, w_4, w_2, v_2, u_3) \quad \text{and} \quad P(v_3, w_3, u_3)$$

are in C, which is impossible. Thus v_1u_2 is in V_3-C, then

$$P(y_1, v_1, w_1, u_1, v_3) \quad \text{and} \quad P(x_2, u_2, w_2, v_2, u_3)$$

are in C, which is impossible, too.

Case 2: *Just one of the edges y_1v_1 or x_2u_2 (say y_1v_1) is in C.*

Then $P(x_1, y_1, v_1, u_2, w_2)$, $P(v_3, u_1, w_1, w_4)$ and $P(x_2, y_2, v_2)$ are in C. If w_2w_4 is in C, then $P(v_3, w_3, u_3, v_2)$ is in C, if not, then

$$w_2v_2 \quad \text{and} \quad P(y_3, v_3, u_1, w_1, w_4, w_3, u_3, x_3) \text{ are in } C.$$

In both these possibilities $C \cap V_3$ is of Type (1).

Case 3: *Neither y_1v_1 nor x_2u_2 is in C.*

Then x_1u_1 and y_2v_2 are both in C, also $P(y_1, x_1, u_1)$, $P(w_1, v_1, u_2, w_2)$ and $P(x_2, y_2, v_2)$ are in C. If w_2v_2 is in C, then

$$P(w_1, w_4, w_3, u_3) \quad \text{and} \quad P(u_1, v_3, y_3)$$

are in C, if w_2v_2 is not in C, then

$$u_1w_1, \qquad P(v_3, w_3, w_4, w_2) \quad \text{and} \quad P(x_3, u_3, v_2)$$

are in C. In both these possibilities $C \cap V_3$ is of Type (2).

This completes the lemma. □

The following lemma is easily verified. We omit its proof.

Lemma 2. *Let G be a trivalent graph which contains a copy of V_2. Let C by a cycle in G which contains edges of both V_2 and G-V_2. For $i = 1, 2$ let us suppose that $P(x_i, y_i)$ is in C and the intersection of $P(x_i, y_i)$ with V_2 consists of the vertices x_i and y_i only. (The edge x_iy_i is not in C.) Then $v(V_2) - v(C \cap V_2) \geq 1$.*

Let D_1 be as shown in Fig. 2. For all $n \geq 1$ we shall use the graph denoted by D_n in [7]. It is an $S(5, q)$-subgraph with two half edges x_1y_1, x_2y_2 and may be defined inductively for all n as follows: Let D_n be the graph obtained from D_{n-1} and D_1 by identifying the half edge x_2y_2 of D_{n-1} with the half edge x_1y_1 of D_1 and then deleting these labels. The interior faces of D_n are all 5-gons. The property of

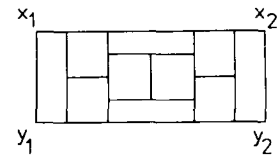

Fig. 2. D_1.

D_n ($n = 1, 2$) which makes them useful to us is stated in the following lemma, equivalent to [1, Lemmas 2.1, 2.2 and 2.3]. We omit the straightforward, but tedious, proof.

Lemma 3. *Let G be a trivalent graph which contains a copy of D_n ($n = 1, 2$). Let C be a cycle in G which spans D_n and let C contains edges of both D_n and G-D_n. After allowing for symmetry, $C \cap D_1$ must be of one of the following types:*

Type (i) $P(x_1, y_2)$,

Type (ii) $P(x_1, y_1) \cup P(x_2, y_2)$,

and $C \cap D_2$ must be only of Type (i).

Let W be as shown in Fig. 3. It is an $S(5, 27)$-subgraph obtained from D_1, V_3, two copies of D_2 and two copies of V_2 by identifying suitable half edges. We present the following lemma.

Lemma 4. *Let G be a trivalent graph which contains a copy of W. Let C be a cycle in G which spans W and contains edges of both W and G-W. Then $C \cap W$ must be of type*

$$P(x_1, y_1) \cup P(x_2, y_2) \cup P(x_3, y_3)$$

and the edges x_1y_1, x_2y_2 are not in C.

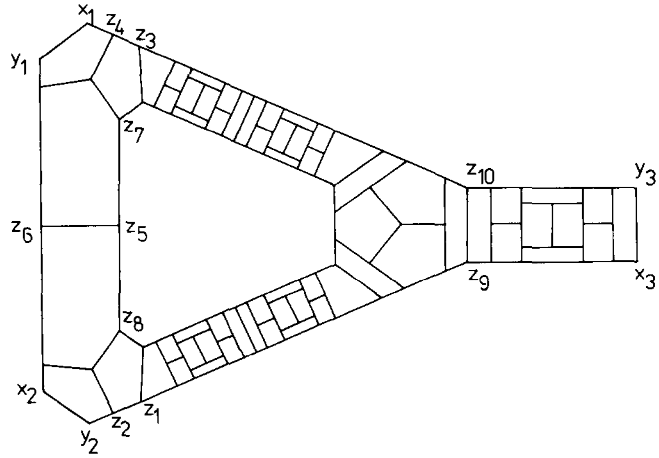

Fig. 3. The graph W.

Proof. First note that by using Lemma 3 to the copies of D_2, we can apply Lemma 1 to a copy of V_3 in W.

If $C \cap V_3$ is of Type (1), then there are $P(z_{10}, y_3)$ and $P(z_9, x_3)$ in C. This contradicts Lemma 3 used for a copy of D_1. It follows that $C \cap V_3$ is of Type (2) in W. Then the edges z_1z_2, z_3z_4 are not in C (Lemma 3 used for two copies of D_2). Let us suppose that z_5z_6 is not in C, then

$$P(z_7, z_8) \cup P(z_8, z_9) \cup P(z_9z_{10}) \cup P(z_{10}z_7)$$

is a cycle in C, which is impossible. Thus after allowing for symmetry,

$$P(y_1, z_6, z_5, z_8, z_9, z_{10}, z_7, z_4, x_1) \quad \text{and} \quad P(x_2, z_2, y_2)$$

are in C. This implies the lemma. □

To construct graph G of the class $S(5, q)$, where q is given, similarly as in [7], we take copies of V_m and D_n (with suitable values of m and n) and join them together by identifying half edges. To specify the pattern of joins and the values of m and n, we use a 2-connected trivalent planar multigraph (basic multigraph of G in the sequel) with suitable labels.

A vertex with label m denotes V_m, an edge with label $5n$ denotes D_n and incidence between the vertex and edge indicates that V_m and D_n have a half edge identified. An unlabelled edge joining vertices with labels m and m' indicates that the corresponding copies of V_m and $V_{m'}$ have a half edge identified.

The success of the construction depends on the possibility of choosing a basic multigraph with the suitable parameters m and n so that all faces of the final graph, other than interior faces of copies of V_m or D_n are q-gons. In every case the final graph is 3-connected.

In the following we describe our constructions of basic multigraphs of graphs in the classes of $S(5, 26+t)$, for $t = 0$ or 1. Certain multigraphs which occur repeatedly as submultigraphs will be denoted by capital letters and represented in diagrams by labelled rectangles. For $t = 0$ or 1 and for all $n \geq 0$ we shall use the submultigraph denoted by L_t^n. It may be defined inductively for all n as follows: For $t = 0$, 1 let L_t^0 be as shown in Fig. 4. For $n \geq 1$ let L_t^n be the submultigraph obtained from 4-t copies of L_t^{n-1} as shown in Fig. 5. The 'dangling' edges are not part of the submultigraphs but show how it is to be joined into a multigraph. The numbers in the surroundings of the rectangles or edges are the numbers of vertices by which the corresponding subgraphs (in final graph) contribute to the adjoining faces of the final graph, in whose basic multigraph they occur. Now we take the trivalent multigraph on two vertices, with suitable labels of vertices and replace each edge by a copy of L_t^n as shown in Fig. 6. Evidently the resulting multigraph, which we denote by B_t^n, is a 2-connected trivalent planar multigraph. It is easily verified that all faces of the corresponding final graph (which we denote by G_t^n), other than interior faces of copies of V_m or D_n, are $(26+t)$-gons. This implies that G_t^n is in $S(5,\ 26+t)$. The next lemma shows that G_t^n is non-Hamiltonian.

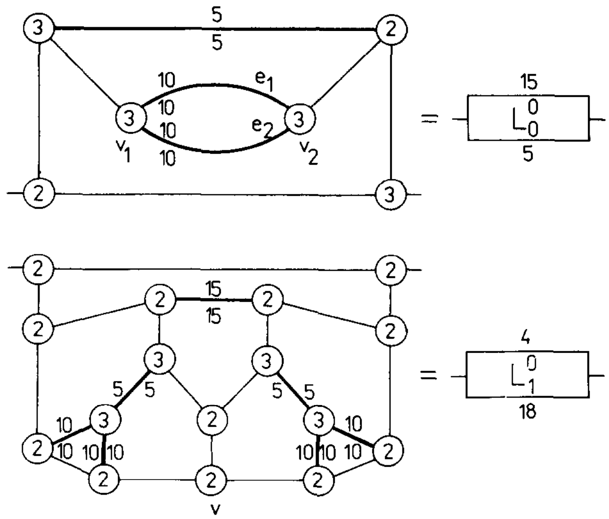

Fig. 4. The multigraphs L_t^0.

Lemma 5. *For $t = 0$ or 1 and for $n \geq 0$, $v(G_t^n) - h(G_t^n) \geq 3(4-t)^n$.*

Proof. Let C be any maximum cycle in G_t^n. Let M_t be a subgraph in G_t^n which is the corresponding subgraph of submultigraph L_t^0 in B_t^n. Suppose that C spans M_t. In the case $t=0$, L_0^0 contains two vertices v_1 and v_2 which are incident with two

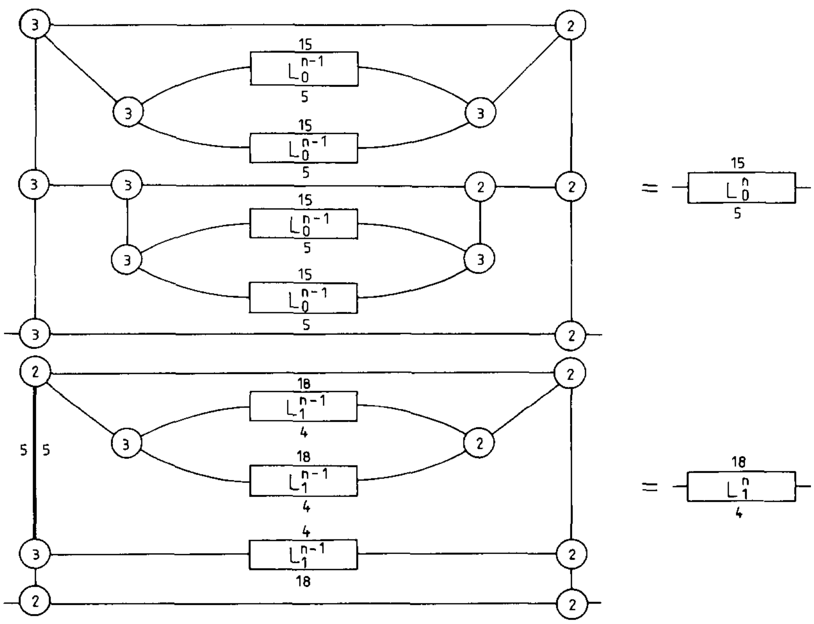

Fig. 5. The multigraphs L_t^n.

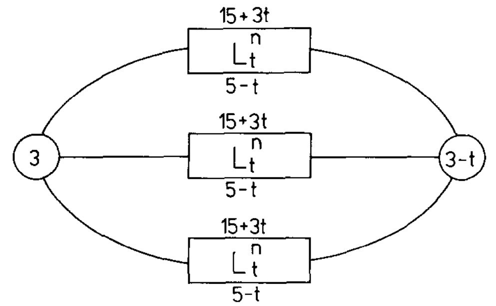

Fig. 6. The multigraph B_t^n.

edges e_1 and e_2 (see Fig. 4). For v_i and e_i $(i = 1, 2)$, let V_3^i and D_2^i denote the corresponding copies of V_3 and D_2 in M_0, respectively. If C spans M_0, then $C \cap D_2^i$ must be of Type (i) (Lemma 3). Thus we can apply Lemma 1 to V_3^i. If both of $C \cap V_3^1$ and $C \cap V_3^2$ are of Type (1), then there exists a cycle in

$$C \cap (V_3^1 \cup D_2^1 \cup V_3^2 \cup D_2^2),$$

which is impossible. Lemmas 1 and 3 imply, that all remaining possibilities for $C \cap (V_3^1 \cup V_3^2)$ lead also to a contradiction. In the case $t = 1$, L_1^0 contains a vertex v (see Fig. 4). Let V_2' denote its corresponding copy of V_2 in M_1. It is seen that the subgraph M_1 contains two copies of W joined with V_2' each by one of its half edges. From Lemmas 4 and 2 it follows that $v(V_2') - v(C \cap V_2') \geq 1$, which is impossible. Since in both cases ($t = 0$ or 1) we get a contradiction, no such spanning cycle C exist. It is easily verified that if G_t^n contains $3(4-t)^n$ copies of M_t, then C misses at least $3(4-t)^n$ vertices of G_t^n. The lemma follows immediately. □

Now we prove the theorem. If in counting the vertices of an $S(5, q)$-subgraph we reckon a half vertex as $\frac{1}{2}$, then $v(V_2) = 7$, $v(V_3) = 13$ and $v(D_n) = 20n$. It is easy to verify that $v(M_0) = 166$, $v(M_1) = 396$ and $v(G_t^n) = 26 + 3((4-t)^n v(M_t) + 44((4-t)^n - 1))$, for $t = 0$ or 1.

For $t = 0$ or 1 we denote $\langle G_t^n \rangle$ an infinite sequence of graphs G_t^n, for all $n \geq 0$. Evidently $\langle G_t^n \rangle$ is a subclass of $S(5, 26 + t)$. Since

$$\rho(S(5, 26+t)) \leq \lim_{n\to\infty} h(G_t^n)/v(G_t^n),$$

it follows that

$$\rho(S(5, 26+t)) \leq \lim_{n\to\infty} 1 - \frac{v(G_t^n) - h(G_t^n)}{v(G_t^n)}$$
$$= 1 - \frac{1}{v(M_t) + 44} < 1.$$

This completes the proof of the theorem.

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