On Consecutive Labeling of Plane Graphs

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ABSTRACT: *This paper concerns a labeling problem of the plane graphs* P_{mn}. *The present paper describes a magic vertex labeling and a consecutive labeling of type (0, 1, 1). These labelings combine to a consecutive labeling of type (1, 1, 1).*

I. Introduction

The notions of consecutive and magic labelings have their origin in classical Chinese mathematics of the 13th century, see (**1**). Only recently have these labelings been investigated using contemporary notions of graph theory. The notions of consecutive and magic labelings of type (i, j, k) for plane graphs were defined by Lih in (**1**) where magic labelings of type (1, 1, 0) for wheels, friendship graphs and prisms are given. Magic labelings of type (1, 1, 1) for fans, planar bipyramids and ladders are described in (**2**) and for Möbius ladders can be found in (**3**).

II. Terminology and Notation

The graphs considered here will be finite. For $m \geqslant 1$, $n = 2k \geqslant 4$ and $n \neq 6$ we denote by P_{mn} a graph consisting of both n-sided faces and $mn/2$ interior 6-sided faces, embedded in the plane and labeled, as in Fig. 1.

Let p, q and t be, respectively, the number of vertices, edges and 6-sided faces. A labeling of type (1, 1, 1) is a bijection from the set $\{1, 2, \ldots, p+q+t+2\}$ onto the vertices, edges and faces of plane graph P_{mn}. The weight of a face under labeling is the sum of the label of the face and the labels of vertices and edges surrounding that face. A labeling is said to be magic if for every integer s all s-sided faces have the same weight. We allow different weights for different s. We say that a labeling is consecutive if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. We allow different sets for different s.

We make the convention that $x_{j,n+1} = x_{j,1}$, $a = |\cos{(i\pi)}/2|$, $b = |\sin{(i\pi)}/2|$, $c = |\cos{(j\pi)}/2|$ and $d = |\sin{(j\pi)}/2|$ (for $i \in I$ and $j \in J$) to simplify later notations. Set $I = \{1, 2, \ldots, n\}$ and set $J = \{0, 1, 2, \ldots, m\}$.

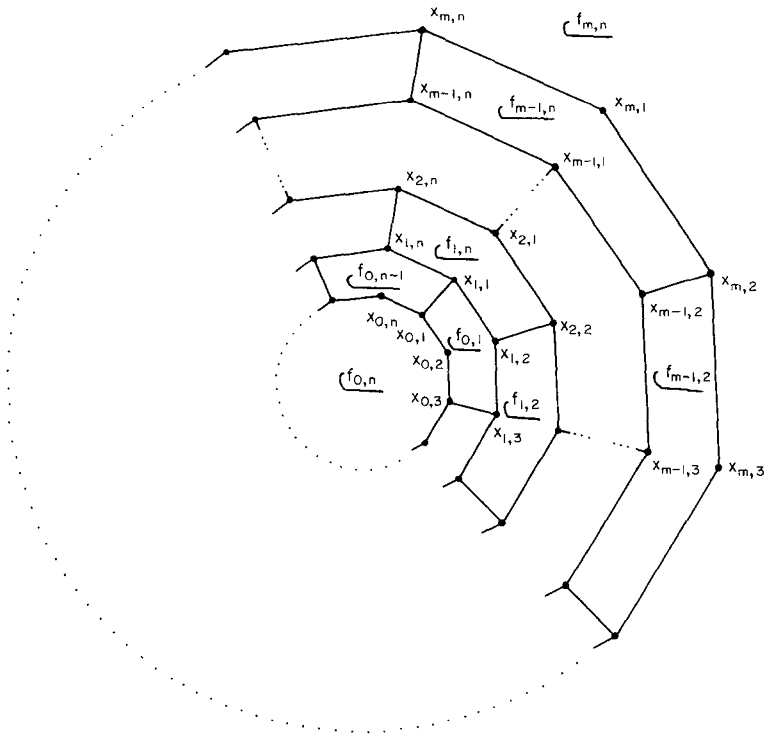

FIG. 1.

III. The Vertex Labeling

Define the vertex labeling g_1 of a plane graph P_{mn} as follows:

$$g_1(x_{j,i}) = ac\left[(m+1)\left(n-\frac{i}{2}\right)-j\right]+ad\left[(m+1)\frac{i}{2}+j+1\right]$$

for $i \in I-\{n\}$ and $j \in J$,

$$= c[n(m+1)-j]+d(j+1) \quad \text{for} \quad i=n \quad \text{and} \quad j \in J,$$

$$= bc[\tfrac{1}{2}(m+1)(i-1)+j+1]+bd\left[(m+1)\left(n-\frac{i-1}{2}\right)-j\right]$$

for $i \in I$ and $j \in J$.

Theorem I

The vertex labeling g_1 of P_{mn} is magic if $m \geqslant 1$ and n is even, $n \geqslant 4$, $n \neq 6$.

Proof. First we shall show that the vertex labeling g_1 uses each integer $1, 2, \ldots, p$. We will consider only the case where m is even. The other case (where m is odd) is dealt with similarly.

If i is odd and j is even then $g_1(x_{j,i})$ is equal successively to

$$[1, 3, 5, \ldots, m-1, m+1], \quad [m+2, m+4, m+6, \ldots, 2m, 2m+2], \ldots,$$

$$\left[\frac{n}{2}(m+1)-m, \frac{n}{2}(m+1)-m+2, \ldots, \frac{n}{2}(m+1)-2, \frac{n}{2}(m+1)\right];$$

if i is even and j is odd, then $g_1(x_{j,i})$ successively attain values of

$$[2, 4, 6, \ldots, m], \quad [m+3, m+5, m+7, \ldots, 2m+1], \ldots,$$

$$\left[\frac{n}{2}(m+1)-m+1, \frac{n}{2}(m+1)-m+3, \ldots, \frac{n}{2}(m+1)-1\right];$$

if i and j are even, then $g_1(x_{j,i})$ successively assume values of

$$\left[\frac{n}{2}(m+1)+1, \frac{n}{2}(m+1)+3, \frac{n}{2}(m+1)+5, \ldots, \frac{n}{2}(m+1)+m-1, \frac{n}{2}(m+1)+m+1\right],$$

$$\left[\frac{n}{2}(m+1)+m+2, \frac{n}{2}(m+1)+m+4, \frac{n}{2}(m+1)+m+6, \ldots, \frac{n}{2}(m+1)+2m,\right.$$

$$\left.\frac{n}{2}(m+1)+2m+2\right], \ldots, \quad [n(m+1)-m, n(m+1)-m+2,$$

$$n(m+1)-m+4, \ldots, n(m+1)-2, n(m+1)]$$

and finally if i and j are odd, then $g_1(x_{j,i})$ is equal successively to

$$\left[\frac{n}{2}(m+1)+2, \frac{n}{2}(m+1)+4, \ldots, \frac{n}{2}(m+1)+m\right],$$

$$\left[\frac{n}{2}(m+1)+m+3, \frac{n}{2}(m+1)+m+5, \ldots, \frac{n}{2}(m+1)+2m+1\right], \ldots,$$

$$[n(m+1)-m+1, n(m+1)-m+3, \ldots, n(m+1)-1].$$

It is not difficult to check that (if $i+j$ is odd, $i \in I$ and $j \in J-\{m\}$) the weight of each 6-sided face

$$w(f_{j,i}) = g_1(x_{j,i}) + g_1(x_{j,i+1}) + g_1(x_{j,i+2}) + g_1(x_{j+1,i}) + g_1(x_{j+1,i+1}) + g_1(x_{j+1,i+2})$$

is $3p+2$ and the weights of both n-sided faces are

$$w(f_{0,n}) = \sum_{i=1}^{n} g_1(x_{0,i}) = w(f_{m,n}) = \sum_{i=1}^{n} g_1(x_{m,i}) = \frac{n}{2}(p+1).$$

This proves that the vertex labeling g_1 is magic.

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IV. The Edge and Face Labelings

Define the edge labeling and the face labeling of a plane graph P_{mn} as follows:

$$g_2(x_{j,i}x_{j+1,i}) = ad\left[\frac{n}{2}(2m+2+j)+\frac{i}{2}\right]+bc\left[n\left(m+1+\frac{j}{2}\right)+\frac{i+1}{2}\right]$$

for $i \in I$ and $j \in J-\{m\}$,

$g_2(x_{j,i}x_{j,i+1})$ is equal to the vertex labeling $g_1(x_{j,i})$,

$$g_2(f_{j,i}) = ad[n(2m+1)-\tfrac{1}{2}(nj+i)+1]+bc\left[n\left(2m+1-\frac{j}{2}\right)+\frac{1-i}{2}\right]$$

for $i \in I$ and $j \in J-\{m\}$,

$$= (2m+1)n+1 \quad \text{for } i=n \quad \text{and} \quad j=0,$$

$$= (2m+1)n+2 \quad \text{for } i=n \quad \text{and } j=m.$$

Theorem II

The labeling g_2 of P_{mn} is consecutive if $m \geqslant 1$ and n is even, $n \geqslant 4$, $n \neq 6$.

Proof. It is simple to verify analogously to the proof of Theorem I, that the labeling g_2 uses each integer $1, 2, \ldots, p, p+1, \ldots, q, q+1, \ldots, q+t+2$. It remains to be shown that the weights of all 6-sided (n-sided) faces constitute sets of consecutive integers. The weights of all 6-sided faces of P_{mn} constitute the set

$$\{W(f_{j,i}) : W(f_{j,i}) = g_2(x_{j,i}x_{j,i+1})+g_2(x_{j,i+1}x_{j,i+2})$$

$$+g_2(x_{j+1,i}x_{j+1,i+1})+g_2(x_{j+1,i+1}x_{j+1,i+2})$$

$$+g_2(x_{j,i}x_{j+1,i})+g_2(x_{j,i+2}x_{j+1,i+2})+g_2(f_{j,i})\,;$$

$i+j$ is odd, $i \in I$ and

$$j \in J-\{m\}\} = \{6mn+5n+4, 6mn+5n+5, 6mn+5n+6, \ldots, \tfrac{13}{2}mn+5n+3\}$$

and the weights of both n-sided faces constitute the set

$$\{W(f_{0,n}), W(f_{m,n})\} = \left\{\frac{n}{2}(p+4m+3)+1, \frac{n}{2}(p+4m+3)+2\right\},$$

where

$$W(f_{0,n}) = \sum_{i=1}^{n} g_2(x_{0,i}x_{0,i+1})+g_2(f_{0,n})$$

and

$$W(f_{m,n}) = \sum_{i=1}^{n} g_2(x_{m,i}x_{m,i+1})+g_2(f_{m,n}).$$

We can see that g_2 is consecutive labeling and the proof is completed.

V. Main Result

Theorem III

If n is even, $n \geqslant 4$, $n \neq 6$ and $m \geqslant 1$, then the plane graph P_{mn} has a consecutive labeling of type $(1, 1, 1)$.

Proof. Label the edges, faces and vertices of P_{mn} by $p+g_2$ and g_1, respectively. From the previous theorems it follows that the labeling obtained by combining these labelings is of type $(1, 1, 1)$ with labels from the set $\{1, 2, \ldots, p+q+t+2\}$, where the weights of 6-sided faces successively attain consecutive values $16p-n+6$, $16p-n+7, \ldots, 16p-n+t+5$ and the weights of both n-sided faces are $p(2n+3)+1$ and $p(2n+3)+2$. Thus the resulting labeling is consecutive.

VI. Conclusion

A labeling problem on the plane graphs P_{mn} has been discussed. The paper described a magic vertex labeling and a consecutive labeling of type $(0, 1, 1)$. These labelings combine to a consecutive labeling of type $(1, 1, 1)$.

References

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