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Option Pricing with Simulation of Fuzzy Stochastic Variables

Abstract

During last decades the stochastic simulation approach, both via MC and QMC has been vastly applied and subsequently analyzed in almost all branches of science. Very nice applications can be found in areas that rely on modeling via stochastic processes, such as finance. However, since financial quantities as opposed to natural processes depend on human activity, their modeling is often very challenging. Many scholars therefor suggest to specify some parts of financial models by means of fuzzy set theory. Many financial problems, such as pricing and hedging of specific financial derivatives, are too complex to be solved analytically even in a crisp case, it can be efficient to apply (Quasi) Monte Carlo simulation. In this contribution a recent knowledge of fuzzy numbers and their approximation is utilized in order to suggest fuzzy-MC simulation to option price modeling in terms of fuzzy-random variables. In particular, we suggest three distinct fuzzy-random processes as an alternative to a standard crisp model. Application possibilities are shown on illustrative examples assuming a plain vanilla European put option under Brownian motion with fuzzy parameter (volatility), Brownian motion with fuzzy subordinator and Brownian motion with fuzzyfied subordinator. In each case the model result into a whole set of prices – thus, since we assume one of the input data as LU fuzzy number, we get the price in terms of the LU fuzzy number as well. The payoff function of analyzed put option can be obviously replaced by more complex payoff structure.

Key Words

random variable, fuzzy variable, option, simulation

JEL Classification: C46, E37, G17, G24

Introduction

Options, a specific nonlinear type of a financial derivative, play an important role in the economy. In particular, the usage of options allows one to reach a higher level of efficiency in terms of risk-return trade-off, whether through speculation or hedging strategy. The option's holder can exercise his right, eg. buy or sell an underlying asset, when he finds it useful. Obviously, it is the case of positive cash flowing from the option exercising. Otherwise the option matures unexploited. By contrast, the seller of the option has to act according to the instructions of the holder. This asymmetry of buyer/seller rights implies the needs of advanced technique for option pricing and hedging.

The standard ways to option pricing, as well as replication and hedging, dates back to 70's to the seminal papers of Black and Scholes (1973) and Merton (1973), Cox et al. (1979) or Boyle (1977). While Black and Scholes (1973) and Merton (1973) derived their respective models within continuous time by solving partial differential equations (and thereafter called Black-Scholes-Merton partial differential equations) for risk free portfolio consisting of option itself and its underlying asset, Cox et al. (1979) provided an approximative solution in two-stage discrete time setting via recursive backward procedure. Alternatively, Boyle (1979) suggested that in order to obtain the (discounted) expectation of the option payoff function the Monte Carlo simulation technique can be useful, ie. instead of riskless portfolio construction and utilization of no-arbitrage principle the risk-neutral world is assumed. It is a well-known result of quantitative finance, see eg. Duffie (2001), that these approaches are equivalent under the assumption of complete markets, or, at least, when an equivalent martingale measure exists.

Although the aforementioned approaches slightly differ in details, eg. only the model of Cox et al. (1979) can be used for pricing of American options, the general frame is the same – the underlying process is derived from Gaussian distribution and all parameters are either deterministic or probabilistic, ie. particular probabilities are assigned to the set of real numbers. However, in the real world, it is often difficult to obtain reliable estimates to input parameters. The reason can be that sufficiently long time series of data is lacking or the data are too heterogenous. Several research papers collected by Ribeiro et al. (1999) suggested that the fuzzy set theory proposed by Zadeh (1965) can be useful for financial engineering problems of such kind.

One of the first attempts to utilize the fuzzy set theory in option pricing dates back to Cherubini (1997). Later, for example Zmeškal (2001) applied a simplifying fuzzy-stochastic approach based on T-numbers to value a firm as a European call option (ie. a real option). The Zmeškal's assumption of real options in (2001) implied that besides the underlying process also the exercise price and maturity time were defined as fuzzy. By contrast, Yoshida (2003) assumed European financial options so that only standard parameters (mainly riskless rate/drift and volatility) were specified as fuzzy. The author also provided more rigorous analytical analysis, including a particular buyer/seller view and hedging possibilities, comparing to quite complicated global optimization approach of Zmeškal (2001). Similarly to Yoshida (2003), Wu eg. in (2004) and (2007) suggested another interesting approach to the treatment of the Black and Scholes model in the fuzzy-stochastic setting. The papers by these three authors were followed by several others, extending the analysis of Black-Scholes type models to eg. more general Lévy processes. Moreover, there exist another direction of research dealing with discrete binomial-type models or utility functions. However, except recent, but brief contribution of Nowak and Romaniuk (2010) there was no attempt to value an option with fuzzy parameters via Monte Carlo simulation approach.

In this paper, we try to fill the gap by suggesting three distinct types of potential underlying processes defined on the basis of fuzzy-random variables. More particularly, we assume (geometric) Brownian motion with fuzzy volatility and (geometric) Brownian motion with time t replaced either by fuzzy process or by fuzzified gamma

subordinator, which allows us to redefine in finance well known and commonly used Lévy type variance gamma model (Cont and Tankov, 2004) in terms of fuzzy-random subordinator. In the next section, we provide brief details about option pricing via plain Monte Carlo simulation within the risk neutral setting. Next, LU-fuzzy numbers and relevant operations with them are defined. After that, three potential candidates to option underlying asset price model are suggested. Finally, a European option price is evaluated assuming all three processes.

1. Standard approach

A financial derivative f that provides at maturity time T its owner, ie. party in the long position, a right to buy an underlying asset S , ie. a right to exercise the option paying the exercise price K , is called a *European call option*. If there is a right to sell the underlying asset, we refer to such derivative security as a *European put option*. There also exist *American options* – the right can be exercised at any time during the option life, and *Bermudan options* – the right can be exercised at prespecified finite set of points during option life. Furthermore, the right to exercise the option can be conditional to additional circumstances, eg. the path followed by the underlying asset price in the past. However, we restrict ourselves only to simple European call and put options, also called *plain vanilla options*. Denoting the underlying asset price at maturity time as S_T we can write the payoff function for European call and put options as follows:

$$\Psi_T^{\text{vanilla call}} = (S_T - K)^+ \quad (1)$$

and

$$\Psi_T^{\text{vanilla put}} = (K - S_T)^+ \quad (2)$$

respectively. For call option price at time $t < T$ it generally holds that:

$$f_t = e^{d_\tau} E \left[\Psi_T^{\text{vanilla call}} \right] = e^{d_\tau} E \left[(S_T - K)^+ \right], \quad (3)$$

where a discount factor d_τ relates to the probability measure under which the expectation operator E is evaluated and $\tau = T - t$ is the remaining time to maturity. Commonly, E^P denotes the real world expectation (under physical probability measure), while E^Q is used within the risk-neutral world, ie. , where r is a riskless rate valid over time interval τ . Since financial asset prices are often restricted to positive values only, geometric processes are commonly preferred. If, for example, $Z(t)$ denotes a stochastic process for log-returns of financial asset S , eg. a non-dividend paying stock, to model its price in time we have to evaluate the exponential function of $Z(t)$. It follows that under Q the key formula above can be rewritten into:

$$f_t = e^{-r\tau} E^Q \left[\left(S_T e^{r\tau + Z_\tau^Q} - K \right)^+ \right], \quad (4)$$

where Z_τ^Q is a (potentially compensated) realization of a suitable stochastic process over τ such that it is ensured that S is a martingale.

The optimal choice of Z_τ^Q depends on the assumptions (observations) about the returns of the underlying asset. If the process is sufficiently tractable, it can be solved

analytically leading to closed form formula, see eg. risk-neutral derivation of Black-Scholes model in Shereve (2004). Alternatively, we can utilize the law of large numbers and evaluate the expectation via Monte Carlo simulation, ie. sufficiently large number of independent scenarios is taken from the relevant probability distribution of (see eg. Glasserman (2004) or Broadie et al (1997) for comprehensive review of this technique):

$$f_t = e^{-rt} E^Q \left[\left(S_T e^{rt+Z_T^Q} - K \right)^+ \right] \approx \frac{e^{-rt}}{N} \sum_{i=1}^N \left(S_T e^{rt+Z_T^{Q(i)}} - K \right)^+, \quad (5)$$

where superscript (i) refers to i -th scenario from a given probability space. Obviously, if the information available about the source of uncertainty is not sufficient to select a reliable candidate for its stochastic evolution, one can prefer to replace Wiener process Z by a fuzzy-stochastic variable.

2. Fuzzy sets theory

Let R denotes the set of real numbers. A fuzzy number is usually called a mapping, which is normal (ie. there exists an element such that $A(x) = 1$), convex (ie. $A(\lambda x + (1-\lambda)y) \geq \min(A(x), A(y))$ for any $x, y \in R$ and $\lambda \in [0, 1]$), upper semicontinuous and $supp(A)$ is bounded, where $supp(A) = cl\{x \in R | A(x) > 0\}$ and cl is the closure operator. The most popular models of fuzzy numbers are the triangular and trapezoidal models investigated by Dubois and Prade (1980). Their popularity follows from the simple (fuzzy) calculus as addition or multiplication of fuzzy numbers which can be established for them. This is also a reason why we can find many recent papers on the approximation of fuzzy numbers by the mentioned models (see eg. Ban (2009a,b) and the references therein).

2.1 LU-fuzzy numbers

In order to model fuzzy numbers we will use a more advanced model of fuzzy numbers based on the interpolation of given knots using rational splines that was proposed by Guerra and Stefanini in (2005) and developed in (2006). This model generalizes the triangular fuzzy numbers and gives a broad variety of shapes enabling more precise representation of fuzzy real data, nevertheless, the calculus is still very simple.

It is well known that each fuzzy number has a representation using α -cuts. Recall that α -cut of a fuzzy number A is the common set and:

$$A(x) = \bigvee_{\substack{\alpha \in]0,1] \\ x \in A_\alpha}} \alpha. \quad (6)$$

Since the fuzzy numbers are upper semicontinuous real functions, then each α -cut may be replaced by its endpoints, say u for the left endpoint and v for the right endpoint. Hence, each fuzzy number can be completely represented by two functions u, v such that:

1. u is a bounded monotonic non-decreasing function which is left-continuous on $]0, 1]$ and the right-continuous for $\alpha=0$,

2. v is a bounded monotonic non-increasing function which is left-continuous on $]0,1]$ and the right-continuous for $\alpha=0$,
3. $u(\alpha) < v(\alpha)$ for any $\alpha \in [0,1]$.

The arithmetic operations between two fuzzy numbers A and B represented by pairs of functions can be introduced using a suitable manipulation of the functions u and v . For example, $A+B$ can be simply obtained by adding respective bounds. For further definitions of arithmetic operations, we refer to Stefanini et al (2006). Since the modeling of fuzzy numbers and the manipulation with them is not so simple in general, we use the parametric representation of fuzzy numbers proposed by the same authors.

3. Potential candidates for price modelling

As we have already argued, it can be very difficult to obtain reliable estimates for the parameters (eg. volatility or intensity of jumps) of the stochastic process $Z(t)$. It is the reason why many researchers suggest to define the underlying process in terms of fuzzy or fuzzy-random variables. In this section, three distinct fuzzy-random models are suggested as potential candidates to describe the option underlying asset price process; in particular, we assume (i) standard market model (Brownian motion) with fuzzy parameter, (ii) Brownian motion with fuzzy subordinator, and (iii) Brownian motion with fuzzyfied gamma subordinator.

Model 1 (standard market model with fuzzy parameter)

Let σ_{LU} be an LU-fuzzy number defined around crisp estimation of σ . Then we can model price returns by the following fuzzy-stochastic model:

$$Z(t) = \mu t + \sigma_{LU} \sqrt{t} \varepsilon. \quad (7)$$

Model 2 (Brownian motion with fuzzy subordinator)

Let x_{LU} is a non-negative LU-fuzzy number centred around t so that it can be a subordinator. Then we get the following alternative to the common assumption of Brownian motion:

$$Z(t) = \theta g(t) + \mathcal{G} \sqrt{g(t)} \varepsilon. \quad (8)$$

Model 3 (Brownian motion with fuzzyfied gamma subordinator)

Let g_{LU} is an LU-fuzzy number centred around random gamma variable. Then we can get another alternative model by using g_{LU} as a subordinator to the Brownian motion:

$$Z(t) = \theta x_{LU}(t) + \sigma \sqrt{x_{LU}(t)} \varepsilon. \quad (9)$$

4. Results

In order to evaluate the risk-neutral expectation via Monte Carlo simulation, we need to get models of the preceding section into the exponential and choose a proper ω_{LU} such that the complex process will be martingale when discounted by the riskless rate.

Comparative results of particular models for various input data are provided in Table 1. Let us assume put options written on stock price index in the form of a mutual fund price. For the illustrative example we derive the input data from the price observations of Pioneer stock fund over 5 years. The idea is that such option can serve as

Tab. 1 Output table of pricing algorithm for put options when various models are considered

| | Model 1 (BS model with fuzzy volatility σ_{LU} $S_0 = 100, K = 100, r = 0, T = 1$) | | | | | |
|--|--|------|---------|---------|-------|----------|
| $\sigma_m = 0.15$ $\sigma_s = 0.10$ | (1.22 | 3.10 | 6.76) | (22.56 | 14.66 | 6.76) |
| | (227 | 209 | 7023) | (-138 | -161 | -7023) |
| $\sigma_m = 0.25$ $\sigma_s = 0.10$ | (1.15 | 3.78 | 8.85) | (26.75 | 17.22 | 8.85) |
| | (235 | 202 | 8773) | (-128 | -148 | -8773) |
| $\sigma_m = 0.25$ $\sigma_s = 0.05$ | (1.04 | 2.90 | 5.73) | (14.33 | 9.28 | 5.73) |
| | (208 | 207 | 35 241) | (-155 | -180 | -35 241) |
| | Model 2 (Brownian motion with fuzzy subordinator x_{LU} $S_0 = 100, K = 100, r = 0, T = 1, \sigma = 0.15$) | | | | | |
| $\theta = 0.00$ $x_{LU} = [0.75, 0.25]$ | (3.73 | 4.83 | 5.93) | (8.20 | 7.14 | 5.76) |
| | (12.5 | 13.0 | 693) | (-138 | -161 | -7023) |
| | Model 3 (Brownian motion with fuzzyfied subordinator $g_{LU}(g(t))$ $S_0 = 100, K = 100, r = 0, T = 1, \sigma = 0.15$) | | | | | |
| $\theta = 0.00$ $\nu = 0.85$ | (0.44 | 1.75 | 6.15) | (19.63 | 13.77 | 6.15) |
| | (725 | 530 | 367) | (-1 028 | -661 | -367) |

Source: own calculations

Generally, we assume crisp values of initial price of the underlying asset ($S_0 = 100$), exercise price ($K = 100$), riskless rate ($r = 0$) and maturity ($T = 1$). Model 1 (first panel) is similar to BS model, except that the volatility of underlying asset price returns is defined as fuzzy random LU number over normal distribution $N(0.15; 0.1)$, with $s = 0.15$ being the most commonly observed value. For sensitivity reasons, we also consider $N(0.15; 0.05)$ and $N(0.20; 0.1)$.

By contrast, Model 2 (second panel) provides us the results of Brownian motion with subordinator defined as fuzzy random LU number over uniform distribution $U(0.5; 1.5)$. Within the model, a symmetry of log-returns can be assumed or it can be relaxed by setting suitable θ to obtain either positive and negative skew. Similarly, in the last panel, the fuzzy-option price assuming Model 3 is depicted for fuzzy random gamma process with variance parameter 0.85 allowing again both, symmetry and asymmetry of log returns by setting suitable θ .

Conclusion

Many issues of financial modeling and decision making require some knowledge about the future states. However, sometimes it is very difficult to get reliable parameterization of stochastic models. In this contribution we suggested an alternative approach to option valuation problem via Monte Carlo simulation by specifying three distinct types of fuzzy-random processes. Suggested models of financial returns can have very interesting impact on option pricing and hedging. First we should note that the BS option price would be really close to the mid-price of each resulting LU-fuzzy price.

Suggested approach opens a wide scope of application possibilities of fuzzy stochastic LU fuzzy numbers in option pricing, especially in cases when the knowledge about the input data do not justify application of more standard approaches based on stochastic variable.

In subsequent research, it can be interesting to study the effect of particular parameters on fuzzy-option price, compare it to real market data as well analyze the convergence of fuzzy-Monte Carlo simulation.

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