Optimization Problems under Two-Sided $(\max,\min)-$Linear Inequalities Constraints

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Abstract

Systems of so called two-sided $(\max,\min)-$linear inequalities with variables on both sides will be studied. Optimization problems, the objective function of which is equal to the maximum of a finite number of continuous functions of one variable are considered. The set of feasible solutions in described by a system of two-sided $(\max,\min)-$linear inequalities with variables on both sides. A finite algorithm for finding the optimal solution of the problem is proposed.

Keywords: Two-sided $(\max,\min)-$linear inequalities system; lower and upper bounds; max-min optimization problems.

Introduction

The algebraic structures in which $(\max,+)$ or $(\max,\min)$ replace addition and multiplication of the classical linear algebra have been appeared in the literature approximately since the sixties of the last century (see e.g. [1], [3], and [8]). In recently published book [2] readers can find the latest results concerning theory and algorithms for $(\max,+)-$linear systems of equations. A polynomial method for finding the maximum solution of the $(\max,\min)$-linear system has been proposed in [5]. A finite algorithm for finding the optimal solution of the optimization problems under $(\max,+)-$linear constraints has been introduced in [10]. A survey of some of the recent results concerning the $(\max,\min)$-linear systems of equations and inequalities and optimization problems under the constraints described by such systems of equations and inequalities is presented in [6]. Algorithm for optimization problems under one-sided $(\max,\min)-$linear equality constraints is introduced in [4]. Maximal solutions of two-sided linear systems in max-min algebra have been given in [7]. A note on application of two-sided systems of $(\max,\min)-$linear equations and inequalities to some fuzzy set problems has been given in [9].

In this contribution, we will study systems of so called $(\max,\min)-$linear (or using an alternative notation $(\max,\wedge)-$linear) inequalities with variables on both sides. We consider optimization problems, the objective function of which is equal to the maximum of a finite number of continuous and unimodal functions of one variable. The set of feasible solutions is described by a system of $(\max,\wedge)-$linear inequalities with variables on both sides. Let us note that if we have variables x on the left hand sides and different variables y on the right hand sides, the system can be processed like the one-sided system considered e.g. in [6]. Including lower and upper bounds on $x,\ y$ is only a technical problem.

We can consider the practical problem, in which transportation means of different size are transporting goods from places $i \in I$ to one terminal T. The goods are unloaded in T and the transportation means (possibly with other goods are uploaded in T) have to return to i. We assume that the connection between i and T is only possible via one of the places (e.g. cities) $j \in J$ the roads between i and j are one-way roads, and the capacity of the road between $i \in I$ and $j \in J$ is equal to a_{ij}. We have to join places j with T by a two-way road with a capacity x_j in both directions. The total capacity of the connection between i and T is therefore equal to $\max_{j\in J}(a_{ij} \wedge x_j)$. The transport from T to i is carried out via other one-way roads between places $j \in J$ and $i \in I$ with (in general, different) capacities between j and i are equal to b_{ij}. Since the roads between T and j are two-way roads, the total capacity of the connection between T and i is equal to $\max_{j\in J}(b_{ij} \wedge x_j)$, for all $i \in I$. We assume that the transportation means can only pass through some roads with the capacity which is not smaller than the capacity of the transportation mean and our task is to choose appropriate capacities x_j, $j \in J$. In order that each of the transportation means may return to i, we may e.g. require for each i that the maximal attainable capacity of connections between i and T via j is greater than or equal to maximal attainable capacity of connections between T and i on the way back. In other words, we have to choose x_j, $j \in J$, which satisfy relation (1) below. In what follows, assume that we have the same variables on the left hand sides and right hand sides of the inequality system.

1 Systems of $(\max,\min)-$Linear Inequalities

Let us consider the following system of inequalities:

$$a_i(x) \geq b_i(x), i \in I, \tag{1}$$

where $a_i(x) = \max_{j\in J}(a_{ij} \wedge x_j)$, $b_i(x) = \max_{j\in J}(b_{ij} \wedge x_j)$, and a_{ij}, $b_{ij} \in R$, $i \in I$, $j \in J$ be given numbers. Let $M^{\geq}$ denote the set of all solutions of system (1). We will set for any $x, y \in R^n : x \leq y \Leftrightarrow x_j \leq y_j \ \ \forall j \in J$. Let us set $M^{\geq}(\underline{x}, \overline{x}) = \{x \ ; \ x \in M^{\geq} \ \& \ \underline{x} \leq x \leq \overline{x}\}$ for any finite $\underline{x} \leq \overline{x}$ and let $x^{\max}$ denote the maximum element of $M^{\geq}(\underline{x}, \overline{x})$. So that $M^{\geq}(\underline{x}, \overline{x}) \subset M^{\geq}$, and $M^{\geq}(\underline{x}, x^{\max}) \subset M^{\geq}$, also it is clear $M^{\geq}(\underline{x}, x^{\max}) \subseteq M^{\geq}(\underline{x}, \overline{x})$. To prove $M^{\geq}(\underline{x}, \overline{x}) \subseteq M^{\geq}(\underline{x}, x^{\max})$ there are two cases: the first one, if $\overline{x} \notin M^{\geq}$, then $x^{\max} < \overline{x}$. Therefore $\forall \ x \in M^{\geq}(\underline{x}, \overline{x})$, the inequality $x \leq x^{\max}$ verified, i.e. $x_j \leq x_j^{\max} \ \forall \ j \in J$ and if $x^* \in (x^{\max}, \overline{x}]$, (i.e. $x^{\max} < x^* \leq \overline{x}$, i.e. $x_{j_0}^{\max} < x_{j_0}^* \leq \overline{x}_{j_0}$ for at least one $j_0 \in J$ and $x_j^{\max} \leq x_j^* \leq \overline{x}_j$ for $j \in J \ \& \ j \neq j_0$) then $x^* \notin M^{\geq}$, otherwise x^* is the maximum element of $M^{\geq}(\underline{x}, \overline{x})$, but this contradicts the hypothesis $x^{\max}$ is the maximum element of $M^{\geq}(\underline{x}, \overline{x})$. So that for any $x \in M^{\geq}(\underline{x}, \overline{x})$, we have $x \leq x^{\max}$, and $x \in M^{\geq}(\underline{x}, x^{\max})$, then $M^{\geq}(\underline{x}, \overline{x}) \subseteq M^{\geq}(\underline{x}, x^{\max})$. The second case, if $\overline{x} \in M^{\geq}$, then $x^{\max} = \overline{x}$. Then we have $M^{\geq}(\underline{x}, x^{\max}) = M^{\geq}(\underline{x}, \overline{x}) \subset M^{\geq}$. In this section we will propose an algorithm, which find the maximum element of the set $M^{\geq}(\underline{x}, \overline{x})$, and calculates the maximum solution of system (1), take in account $\underline{x} \leq x \leq \overline{x}$. Note that, since any equation can be replaced by two inequalities, therefor we can use the next algorithm to find the maximum element of the set $M^{=}(\underline{x}, \overline{x})$, which is the set of all solutions of a system of equations, $(a_i(x) = b_i(x), i \in I)$.

Algorithm 1

$\boxed{0}$ Input I, J, $\overline{x}$, a_{ij} and b_{ij} for all $i \in I$ and $j \in J$.

$\boxed{1}$ Find $I^{<}(\overline{x}) \equiv \{i \in I \ ; \ a_i(\overline{x}) < b_i(\overline{x})\}$.

$\boxed{2}$ If $I^{<}(\overline{x}) = \emptyset$, then $x^{\max} := \overline{x}$, STOP.

$\boxed{3}$ Find $\alpha(\overline{x}) \equiv \min_{i\in I^{<}(\overline{x})} a_i(\overline{x})$.

$\boxed{4}$ Find $I^{<}(\alpha(\bar{x})) \equiv \{i \in I^{<}(\bar{x})\ ;\ a_i(\bar{x}) = \alpha(\bar{x})\}$.

$\boxed{5}$ Find $H_i^{<}(\bar{x}) \equiv \left\{j \in J\ ; b_{ij} \wedge \bar{x}_j > \alpha(\bar{x})\right\},\ \forall\ i \in I^{<}(\alpha(\bar{x}))$.

$\boxed{6}$ Set $H^{<}(\bar{x}) := \bigcup_{i \in I^{<}(\alpha(\bar{x}))} H_i^{<}(\bar{x})$.

$\boxed{7}$ Set $\bar{x}_j := \alpha(\bar{x})$ for all $j \in H^{<}(\bar{x})$ go to $\boxed{1}$.

We will illustrate the performance of this algorithm by the following small numerical example.

Example 1. : *Let $J = \{1,2,3,4\}$, $I = \{1,2,3\}$, $\bar{x} = (10,10,10,10)$, and consider system (1) of inequalities where a_{ij} & b_{ij} $\forall$ $i \in I$ and $j \in J$ are given by the matrices A and B as follows:*

$$A = \begin{pmatrix} 7 & 5 & 3 & 0 \\ 4 & 3 & 1 & 2 \\ 10 & 20 & 10 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 13 & 10 & -1 \\ 8 & 0 & 3 & 1 \\ 1 & 1 & 1 & -8 \end{pmatrix}$$

By substitution for these values in system (1) and using Algorithm 1:

***Iteration 1**:*

$\boxed{1}$ $I^{<}(\bar{x}) = \{1,2\}$.

$\boxed{2}$ $I^{<}(\bar{x}) \neq \emptyset$.

$\boxed{3}$ $\alpha(\bar{x}) = \min(7,4) = 4$.

$\boxed{4}$ $I^{<}(\alpha(\bar{x})) = \{2\}$.

$\boxed{5}$ $H_2^{<}(\bar{x}) = \{1\}$.

$\boxed{6}$ $H^{<}(\bar{x}) = \{1\}$.

$\boxed{7}$ $\bar{x}_1 = 4,\ \bar{x} = (4,10,10,10)$ *go to* $\boxed{1}$.

***Iteration 2**:*

$\boxed{1}$ $I^{<}(\bar{x}) = \{1\}$.

$\boxed{2}$ $I^{<}(\bar{x}) \neq \emptyset$.

$\boxed{3}$ $\alpha(\bar{x}) = 5$.

$\boxed{4}$ $I^{<}(\alpha(\bar{x})) = \{1\}$.

$\boxed{5}$ $H_i^{<}(\bar{x}) = \{2,3\}$.

$\boxed{6}$ $H^{<}(\bar{x}) = \{2,3\}$.

$\boxed{7}$ $\bar{x}_2 = 5, \bar{x}_3 = 5,\ \bar{x} = (4,5,5,10)$ *go to* $\boxed{1}$.

***Iteration 3**:*

$\boxed{1}$ $I^{<}(\bar{x}) = \emptyset$, *then* $x^{\max} = (4,5,5,10)$ *STOP.*

In the next part of this section we will introduce a method which finds the minimum upper bound $\tilde{x}$ for solution of system (1) such that $\tilde{x} \geq \underline{x}$. In other words $\tilde{x}$ has the properties $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$ and if $\underline{x} \leq \tilde{x}, \underline{x} \neq \tilde{x}$, then there exists $x^* \in M^{\geq}(\underline{x}, x^{\max})$ such that $x^* \nleq \tilde{x}$. It will be clear that $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$ and this element is suitable to find the optimal solution of the minimization problem as we will see in the next section. In what follows to simplify the notation we set for any $\alpha, \beta \in R : \alpha \vee \beta = \max(\alpha, \beta)$. Let us set

$$T_{ij} = \{x_j\, ; x_j \leq x_j^{\max} \& \ a_{ij} \wedge x_j \ \geq \ b_i(\underline{x}) \ \vee \ \underline{x}_j\}, \ \forall\, i \in I, \ j \in J.$$

Note that if $i_1, \ i_2$ are two different indices of I, $j \in J$, and $b_{i_2}(\underline{x}) \vee \underline{x}_j \leq b_{i_1}(\underline{x}) \vee \underline{x}_j$, then evidently $T_{i_1 j} \subseteq T_{i_2 j}$. It follows that for any subset of r indices of I, there exists such permutation $i_1, \ , \ldots, \ i_r$ of these indices that the inclusions $T_{i_1 j} \ \subseteq \ T_{i_2 j} \ \subseteq \ \ldots \ \subseteq T_{i_r j}$ hold so that $\bigcap_{h=1}^{r} T_{i_h j} = T_{i_1 j}$. Sets T_{ij} have the following properties:

$$T_{ij} \neq \emptyset \ \Leftrightarrow \ a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j,$$

$$T_{ij} \neq \emptyset \ \Rightarrow \ T_{ij} = [b_i(\underline{x}) \vee \underline{x}_j \ , \ x_j^{\max}].$$

Since we assumed that $\underline{x} \leq x^{\max}$, set $M^{\geq}(\underline{x}, x^{\max})$ is nonempty. Let us note that for any $x \in M^{\geq}(\underline{x}, x^{\max})$ and any $i \in I$, the inequalities $b_i(x) \geq b_i(\underline{x})$ & $x_j \geq \underline{x}_j \ \forall \ j \in J$ hold and further there exists for each $i \in I$ an index $j(i) \in J$ such that $T_{ij(i)} \neq \emptyset$ (otherwise set $M^{\geq}(\underline{x}, x^{\max})$ would be empty, because we would have $a_{ij} < b_i(\underline{x}) \vee \underline{x}_j \ \forall j \in J$ and therefore $a_i(x) < b_i(x)$ for any $x \in R^n$ and we have $\underline{x} \leq x^{\max}$ so that $M^{\geq}(\underline{x}, x^{\max}) \neq \emptyset$). Let us note further, that if $a_{ij} \wedge x_j < b_i(\underline{x}) \vee \underline{x}_j \ \forall j \in J$, then we have $a_i(x) < b_i(\underline{x})$ and thus $x \notin M^{\geq}(\underline{x}, x^{\max})$. If for some fixed $j \in J$ the inequalities $a_{ij} < b_i(\underline{x}) \vee \underline{x}_j$ hold , then $a_{ij} \wedge x_j < b_i(\underline{x}) \vee \underline{x}_j \ \forall \ x_j \in R$ so that $T_{ij} = \emptyset$ and x_j will never be "active" in $a_i(x)$ or $b_i(x)$ if $x \in M^{\geq}$ (i.e. it will never determine the values of $a_i(x)$ or $b_i(x)$). We will exclude such variables from our considerations and assume that for each $j \in J$ there exists at least one "row" index $i \in I$ such that $a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j$. We define sets $V_j, \ \ j \in J$

$$V_j = \{i \in I; a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j\},$$

and denote $\max_{k \in V_j}(b_k(\underline{x})) = b_{k(j)}(\underline{x})$. A vector $\tilde{x}$ will be defined as follows:

$$\tilde{x}_j = \max_{k \in V_j}(b_k(\underline{x})) \vee \underline{x}_j = b_{k(j)}(\underline{x}) \vee \underline{x}_j \ \ \forall j \in J. \tag{2}$$

The element $\tilde{x}$ defined by (2) has the following properties:

(1) $M^{\geq}(\underline{x}, \tilde{x}) \neq \emptyset$, & $\tilde{x} \in M^{\geq}(\underline{x}, \tilde{x})$.

(2) $\xi \in M^{\geq}(\underline{x}, \tilde{x}) \Rightarrow \underline{x} \leq \xi \leq \tilde{x}$.

(3) There may exist elements $\eta \in M^{\geq}(\underline{x}, \tilde{x})$ such that $\eta \neq \tilde{x}$.

If $\tilde{x}$ is the minimum element of $M^{\geq}(\underline{x}, x^{\max})$, then it would be $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$ and for any $x \in M^{\geq}(\underline{x}, x^{\max}) \Rightarrow x \geq \tilde{x}$. Therefore, because of the property (3) $\tilde{x}$ is not the minimum element of $M^{\geq}(\underline{x}, x^{\max})$, but we can say that $\tilde{x}$ is the minimum upper bound of $M^{\geq}(\underline{x}, x^{\max})$ such that $M^{\geq}(\underline{x}, \tilde{x}) \neq \emptyset$. Let us choose $\tau \leq x^{\max}$, & $\tau \neq x^{\max}$, and $\check{x} \in M^{\geq}(\underline{x}, \tau) \ \Rightarrow \ \check{x} \leq x^{\max}$ and $a_i(\check{x}) \geq b_i(\check{x}) \ \forall \ i \in I$ and $\underline{x} \leq \check{x} \leq \tau$. Let $H = \left\{x^{\max}(\tau) \mid x^{\max}(\tau) \ \textit{is the maximum element of} \ M^{\geq}(\underline{x}, \tau)\right\}$, then $\tilde{x}$ is the minimum element of H.

Theorem 1. : *Let $\tilde{x}$ be defined as in (2). Then $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$.*

Proof: Since evidently $\tilde{x} \geq \underline{x}$, we have to prove that only $a_i(\tilde{x}) \geq b_i(\tilde{x}),\ \forall i \in I$. Let $i \in I$ be arbitrarily chosen. We have

$$b_i(\tilde{x}) = \max_{j\in J}(b_{ij} \wedge \tilde{x}_j) = \max_{j\in J}(b_{ij} \wedge (\max_{k\in V_j}(b_k(\underline{x}) \vee \underline{x}_j))) = \max_{j\in J}(b_{ij} \wedge (b_{k(j)}(\underline{x}) \vee \underline{x}_j))$$

Let us assume that

$$b_i(\tilde{x}) = \max_{j\in J}(b_{ij} \wedge \tilde{x}_j) = b_{ij(i)} \wedge \tilde{x}_{j(i)}.$$

Since in this case $i \in V_{j(i)}$, we have $a_{ij(i)} \geq \tilde{x}_{j(i)}$ and we obtain $a_i(\tilde{x}) \geq a_{ij(i)} \wedge \tilde{x}_{j(i)} = \tilde{x}_{j(i)} \geq b_{ij(i)} \wedge \tilde{x}_{j(i)} = b_i(\tilde{x})$. Since $i \in I$ was arbitrarily chosen, the theorem is proved.

□

Element $\tilde{x}$ defined by (2) shows that the given lower bound $\underline{x}$ might not be an element of $M^{\geq}(\underline{x}, x^{\max})$. Moreover we obtained an explicit dependence of $\tilde{x}$ on the given lower bound $\underline{x}$ (compare (2)), which can be used for sensitivity analysis of the set $M^{\geq}(\underline{x}, \overline{x})$ or for a post optimal analysis of optimization problems, the set of feasible solutions equal to $M^{\geq}(\underline{x}, x^{\max})$. The properties of $\tilde{x}$ enable us to solve some of the optimization problems mentioned above explicitly.

2 Optimization Problems under Two-Sided $(\max,\min)-$Linear Inequalities Constraints

In this section we consider an optimization problem that is a combination of the problems solved in the above chapters but with a different feasible set. In other words, let us consider for instance the optimization problem:

$$f(x) \equiv \max_{j\in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{3}$$

subject to $x \in M^{\geq}(\underline{x}, x^{\max})$, where $f_j,\ j \in J$ are increasing functions. Let indices $j(i) \in J$ will be chosen for each $i \in I$ such that $\min_{j\in J} f_j(x_j^{(i)}) = f_{j(i)}(x_{j(i)})$, where $f_j(x_j^{(i)}) = \min_{x_j \in T_{ij}} f_j(x_j)$. Let $\tilde{x}$ be defined as in (2) and then we have to proceed as follows:

$$\tilde{T}_{ij} = \begin{cases} \emptyset & \text{if} \quad a_{ij} < b_i(\underline{x}), \\ b_i(\underline{x}) & \text{if} \quad a_{ij} > b_i(\underline{x}), \\ [\underline{x}_j, \tilde{x}] & \text{if} \quad a_{ij} = b_{ij}. \end{cases}$$

Set $f_j(\tilde{x}_j^{(i)}) = \min_{x_j \in \tilde{T}_{ij}} f_j(x_j)$, (if $\tilde{T}_{ij} = \emptyset$, we set minimum equal to $+\infty$). Let us set

$$\min_{j\in J} f_j(\tilde{x}_j^{(i)}) = f_{j(i)}(\tilde{x}_{j(i)}^{(i)}).$$

And $\tilde{R}_j = \{i \in I \mid j(i) = j\},\ \forall\ j \in J$, (it may be $\tilde{R}_j = \emptyset$ for some j). Then we have

$$f_k(x_k^{opt}) = \max_{i\in \tilde{R}_k} f_k(\tilde{x}_k^{(i)}),$$

if $\tilde{R}_k \neq \emptyset$, but when $\tilde{R}_k = \emptyset$, we set

$$f_k(x_k^{opt}) = f_k(\underline{x}_k).$$

The proof can be carried out in the same way as in the one sided case in [6]. We mentioned above that a system of inequalities can be transformed to a system of equations by making use of slack variables. Let us note that the other way round, systems of equations considered can be solved alternatively by the methods in this section, if we replace the equation system by the system of inequalities of the form

$$a_i(x) \geq b_i(x),\ i \in I$$
$$b_i(x) \geq a_i(x),\ i \in I$$
$$x_j \geq \underline{x}_j,\ j \in J.$$

We will describe now the corresponding algorithm explicitly step by step.

Algorithm 2

$\boxed{0}$ Input $m, n, \underline{x}, \overline{x}, A, B, f(x)$.

$\boxed{1}$ Find $x^{\max} \in M^{\geq}(\underline{x}, \overline{x})$.

$\boxed{2}$ If $\underline{x} \not\leq x^{\max}$, then $M^{\geq}(\underline{x}, \overline{x}) = \emptyset$, STOP.

$\boxed{3}$ $V_j := \{i \in I\ ;\ a_{ij} > b_i(\underline{x}) \vee \underline{x}_j\}\ \ \forall j \in J$.

$\boxed{4}$ $x_j^{(i)} := (b_i(\underline{x}) \vee \underline{x}_j)\ \ \forall i \in V_j$ for all $j \in J$ such that $V_j \neq \emptyset$.

$\boxed{5}$ Set $\tilde{x}_j := \max_{i \in V_j}(x_j^{(i)})$ if $V_j \neq \emptyset$, $\tilde{x}_j := \underline{x}_j$ if $V_j = \emptyset$.

$\boxed{6}$ $Q := \{k \in J\ ;\ f(\tilde{x}) = f_k(\tilde{x}_k)\},\ \ P := \{j \in J\ ;\ \tilde{x}_j = \underline{x}_j\}$.

$\boxed{7}$ If $Q \cap P \neq \emptyset$, then set $x^{opt} := \tilde{x}$, STOP.

$\boxed{8}$ $P_k := \{i \in I\ ;\ \tilde{x}_k = x_k^{(i)}\}\ \ \forall k \in Q$.

$\boxed{9}$ $V_k := V_k \setminus P_k\ \ \forall k \in Q$.

$\boxed{10}$ If $\bigcup_{j \in J} V_j = I$, go to $\boxed{4}$.

$\boxed{11}$ Set $x^{opt} := \tilde{x}$, STOP.

We will illustrate the performance of this algorithm by the following numerical examples.

Example 2. : *Let* $J = \{1, 2, \ldots, 5\}$, $I = \{1, 2, 3\}$, $\overline{x} = (10, 10, 10, 10, 10)$, $\underline{x} = (0, 3, 0, 0, 1)$ *and consider system (1) of inequalities where* $a_{ij}\ \&\ b_{ij}\ \forall\ i \in I$ *and* $j \in J$ *are given by the matrices A and B as follows:*

$$A = \begin{pmatrix} -10 & 10 & 15 & -9 & -8 \\ 5 & -8 & 10 & 20 & 7 \\ 3 & 4 & -18 & 19 & 11 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 2 & -10 & -20 & 6 \\ 8 & 9 & -15 & -25 & 5 \\ 13 & -17 & 12 & 10 & 9 \end{pmatrix}$$

and consider the objective function $f(x) = \max(x_1, x_2 - 3, x_3, x_4, x_5)$. *By substitution for these values in system (1) and using Algorithm 1 and Algorithm 2:*

$\boxed{1}$ $x^{\max} = \overline{x} = (10, 10, 10, 10, 10)$.

$\boxed{2}$ $\underline{x} \leq x^{\max}$.

$\boxed{3}$ $V_1 = \{2\},\ V_2 = \{1, 3\}, V_3 = \{1, 2\}, V_4 = \{2, 3\}, V_5 = \{2, 3\}$.

$\boxed{4}$ $x_1^{(1)}=2,\quad x_2^{(1)}=3,\quad x_3^{(1)}=2,\quad x_4^{(1)}=2,\quad x_5^{(1)}=2,\ x_1^{(2)}=3,\quad x_2^{(2)}=3,\quad x_3^{(2)}=3,$ $x_4^{(2)}=3,\quad x_5^{(2)}=3,\ x_1^{(3)}=1,\quad x_2^{(3)}=3,\quad x_3^{(3)}=1,\quad x_4^{(3)}=1,\quad x_5^{(3)}=1.$

$\boxed{5}$ $\tilde{x}=(3,3,2,3,3).$

$\boxed{6}$ $Q=\{1,4,5\},,\ f(\tilde{x})=3,\ P=\{2\}$ *then* $Q\cap P=\emptyset$.

$\boxed{8}$ $P_1=\{2\},\ P_2=\{1,2,3\}, P_3=\{1\}, P_4=\{2\}, P_5=\{2\}.$

$\boxed{9}$ $V_1=\emptyset,\ V_2=\emptyset, V_3=\{2\}, V_4=\{3\}, V_5=\{3\}.$

$\boxed{10}$ $\bigcup_{j\in J} V_j=\{2,3\}\neq I.$

$\boxed{11}$ $x^{opt}=\tilde{x}$, *STOP.*

Then $x^{opt}=(3,3,2,3,3)$ *is the optimal solution of the set* $M^{\geq}(\underline{x},\overline{x})$ *and* $f(x^{opt})=\max(3,0,2,3,3)$, *then the objective function is equal to* 3.

Example 3. *: Let* $J=\{1,2,\ldots,5\}$, $I=\{1,2,\ldots,6\}$, $\overline{x}=(20,20,20,20,20)$, $\underline{x}=(0,3,0,0,0)$ *and consider system (1) of inequalities where* $a_{ij}\ \&\ b_{ij}\ \forall\ i\in I$ *and* $j\in J$ *are given by the matrices A and B as follows:*

$$A=\begin{pmatrix} 2 & 2 & 6 & 0 & 13\\ 8 & 11 & 10 & 7 & 7\\ 4 & 3 & 0 & 13 & 8\\ 14 & 3 & 3 & 13 & 2\\ 1 & 3 & 13 & 4 & 2\\ 12 & 15 & 7 & 3 & 14 \end{pmatrix},\quad B=\begin{pmatrix} 0 & 10 & 9 & -1 & 5\\ 3 & -3 & 1 & -6 & -7\\ 4 & -8 & 2 & -14 & 11\\ 14 & -7 & 7 & -3 & 4\\ 6 & -8 & 12 & 2 & 0\\ 0 & -11 & 2 & -3 & 5 \end{pmatrix}$$

and consider the objective function $f(x)=\max_{j\in J}(f_j(x_j)$, *where* $f_j(x_j)=c_jx_j+d_j$, $c=(6,3,7,3,7)$ *and* $d=(10,0,5,1,7)$. *By substitution for these values in system (1) and using Algorithm 1 and Algorithm 2:*

$\boxed{1}$ $x^{\max}=\overline{x}=(20,20,20,20,20).$

$\boxed{2}$ $\underline{x}\leq x^{\max}.$

$\boxed{3}$ $V_1=\{2,3,4,5,6\},\ V_2=\{2,6\}, V_3=\{1,2,4,5,6\}, V_4=\{2,3,4,5,6\}, V_5=\{1,2,3,4,5,6\}.$

$\boxed{4}$ *find* $x_j^{(i)}$.

$\boxed{5}$ $\tilde{x}=(0,3,3,0,3).$

$\boxed{6}$ $Q=\{5\}, f(\tilde{x})=28,\ P=\{1,2,4\}$ *then* $Q\cap P=\emptyset$.

$\boxed{10}$ $\bigcup_{j\in J} V_j=\{1,2,3,4,5,6\}=I$ *go to* $\boxed{4}$.

$\boxed{4}$ *find* $x_j^{(i)}$.

$\boxed{5}$ $\tilde{x}=(0,3,3,0,0).$

$\boxed{6}$ $Q=\{3\}, f(\tilde{x})=26,\ P=\{1,2,4,5\}$ *then* $Q\cap P=\emptyset$.

$\boxed{10}$ $\bigcup_{j\in J}V_j=\{1,2,4,5,6\}\neq I$.

$\boxed{11}$ $x^{opt}=\tilde{x}$, *STOP.*

Then $x^{opt}=(0,3,3,0,0)$ *is the optimal solution of the set* $M^{\geq}(\underline{x},\overline{x})$ *and* $f(x^{opt})=\max(10,9,26,1,7)$, *then the objective function is equal to* 26.

Conclusion

We can summarize the properties of the systems of $(\max,\min)$-linear inequalities studied in this paper as follows:
(1) Any system of two-sided $(\max,\min)$-linear inequalities is solvable and has a unique maximum element $x^{\max}(A,B)$ depending on the matrices $A,\ B$ with finite elements $a_{ij},\ b_{ij}$ (note that including infinite elements can cause nonsolvability of the system).
(2) If we include an additional requirement $x\leq\overline{x}$, then the system is also solvable and has the maximum element $x^{\max}(A,\ B,\ \overline{x})\leq x^{\max}(A,B)$.
(3) The system with a finite lower bound on variables (i.e. with an additional constraint $x\geq\underline{x}$) is solvable if and only if $\underline{x}\leq x^{\max}(A,B)$, or in case of the additional upper bound $\overline{x}$ if and only if $\underline{x}\leq x^{\max}(A,\ B,\ \overline{x})$.

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OPTIMALIZAČNÍ PROBLÉMY PŘI OMEZENÍCH VE TVARU SOUSTAV DVOUSTRANNÝCH $(\max,\min)-$LINEÁRNÍCH NEROVNOSTÍ

Zkoumají se soustavy tzv. dvoustranných $(\max,\min)-$lineárních nerovností s proměnnými na obou stranách těchto nerovností. Zabýváme se optimalizačními úlohami, jejichž účelová funkce je rovna maximu konečného počtu spojitých funkcí jedné proměnné. Množina přípustných řešení těchto úloh je popsána soustavou dvoustranných $(\max,\min)-$lineárních nerovností. Je navržen konečný algoritmus pro nalezení optimálního řešení zkoumaného optimalizačního problému.

OPTIMALISIERUNGSPROBLEME BEI BEGRENZUNGEN IN DER FORM VON ZWEISEITIGEN $(\max,\min)-$LINEARER UNGLEICHHEITSSYSTEMEN

Es werden sog. zweiseitige $(\max,\min)-$lineare Ungleichheitssysteme mit Variablen auf beiden Seiten dieser Ungleichheiten untersucht. Wir befassen uns mit Optimalisierungsaufgaben, deren Zweckfunktion dem Maximum einer finiten Anzahl kontinuierlicher Funktionen einer Variablen gleich ist. Die Menge der zulässigen Lösungen dieser Aufgaben wird durch zweiseitige $(\max,\min)-$lineare Ungleichheitssysteme beschrieben. Es wird ein finiter Algorithmus zur Auffindung einer optimalen Lösung der untersuchten Optimalisierungsprobleme vorgeschlagen.

PROBLEMY OPTYMALIZACJI PRZY OGRANICZENIACH W POSTACI UKŁADÓW DWUSTRONNYCH $(\max,\min)-$LINIOWYCH NIERÓWNOŚCI

Badaniem objęto układy tzw. dwustronnych $(\max,\min)-$nierówności liniowych ze zmiennymi po obu stronach tych nierówności. W artykule przedstawiono zadania optymalizacyjne, których funkcja celowa jest równa maksymum skończonej liczby funkcji ciągłych jednej zmiennej. Zbiór możliwych rozwiązań tych zadań opisano przy pomocy układu dwustronnych $(\max,\min)-$nierówności liniowych. Zaproponowano ostateczny algorytm służący do znalezienia optymalnego rozwiązania badanego problemu optymalizacji.