Czechoslovak Mathematical Journal, 55 (130) (2005), 393–396

REMARKS ON RESTRAINED DOMINATION AND TOTAL RESTRAINED DOMINATION IN GRAPHS

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(Received July 22, 2002)

Abstract. The restrained domination number $\gamma^r(G)$ and the total restrained domination number $\gamma_t^r(G)$ of a graph G were introduced recently by various authors as certain variants of the domination number $\gamma(G)$ of (G). A well-known numerical invariant of a graph is the domatic number $d(G)$ which is in a certain way related (and may be called dual) to $\gamma(G)$. The paper tries to define analogous concepts also for the restrained domination and the total restrained domination and discusses the sense of such new definitions.

Keywords: domination number, domatic number, total domination number, total domatic number, restrained domination number, restrained domatic number, total restrained domination number, total restrained domatic number

MSC 2000: 05C35, 05C69

The research of the domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. A numerical invariant of a graph which is in a certain sense dual to it is the domatic number of a graph. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. Here we will study the restrained dominating set [4, 5] and the total restrained dominating set [1]. We consider finite undirected graphs without loops and multiple edges.

We start with definitions of various concepts concerning the domination in graphs. A subset $S \subseteq V(G)$ is called a dominating set (or a total dominating set) in G, if for each $x \in V(G) - S$ (or for each $x \in V(G)$, respectively) there exists a vertex $y \in S$ adjacent to x. A dominating set in G is called a restrained dominating set

Bohdan Zelinka passed away on February 2005.

This research was supported by Grant MSM 245100303 of the Ministry of Education, Youth and Sports of the Czech Republic.

in G, if each vertex $x \in V(G) - S$ is adjacent both to a vertex $y \in S$ and to a vertex $z \in V(G) - S$. A set S which is simultaneously total dominating and restrained dominating in G is called a total restrained dominating set in G. The minimum number of vertices of a dominating set in a graph G is the domination number $\gamma(G)$ of G. Analogously the total domination number $\gamma_t(G)$, the restrained domination number $\gamma^r(G)$ and the total restrained domination number $\gamma_t^r(G)$ are defined.

The domatic number of a graph was introduced in [2] and the total domatic number in [3]. In an analogous way we will define the restrained domatic number and the total restrained domatic number and then we will discuss the purpose of defining them. Let $\mathscr{D}$ be a partition of the vertex set $V(G)$ of G. If all classes of $\mathscr{D}$ are dominating sets (or total dominating sets) in G, then $\mathscr{D}$ is called a domatic (or total domatic, respectively) partition of G. Quite analogously we may go on. If all classes of $\mathscr{D}$ are restrained dominating sets (or total restrained dominating sets) in G then $\mathscr{D}$ is called a restrained domatic (or total restrained domatic, respectively) partition of G.

The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. Analogously the total domatic number $d_t(G)$, the restrained domatic number $d^r(G)$ and the total restrained domatic number $d_t^r(G)$ are defined. Note that $d^r(G)$ is well-defined for all graphs, so as $d(G)$ is, while $d_t^r(G)$ is well-defined for all graphs without isolated vertices, so as $d_t(G)$ is. The sense of introducing $d_t^r(G)$ is brought into doubt by the following theorem.

Theorem 1. *Let G be a graph without isolated vertices. Then $d_t^r(G) = d_t(G)$.*

P r o o f. Each total restrained dominating set in G is a total dominating set in G; therefore each total restrained domatic partition of G is a total domatic partition of G and $d_t^r(G) \leqslant d_t(G)$. Now denote $d(G)$ by d and let $\mathscr{D}$ be a total domatic partition of G with d classes $D_1, \ldots, D_d$. Choose a class of $\mathscr{D}$, without loss of generality let it be D_1. Let $x \in V(G)$. As D_1 is a total dominating set in G, there exists $y \in D_1$ which is adjacent to x. Now suppose $x \in V(G) - D_1$. Then $x \in D_i$ for some $i \in \{2, \ldots, d\}$. The set D_i is also a total dominating set in G, therefore there exists $z \in D_i$ adjacent to x and evidently $z \in V(G) - D_1$, because $D_1 \cap D_i = \emptyset$. We have proved that D_1 is a total restrained dominating set in G. The set D_1 was chosen arbitrarily, therefore $\mathscr{D}$ is a total restrained domatic partition of G and $d_t(G) \leqslant d_t^r(G)$, which together with the former inequality gives the required result. □

The following theorem is analogous, only a little more complicated.

Theorem 2. *Let G be a graph, let $d(G) \geqslant 3$. Then $d^r(G) = d(G)$.*

P r o o f. Each restrained dominating set in G is a dominating set in G; therefore each restrained domatic partition of G is a domatic partition of G and $d^r(G) \leqslant d(G)$. Now denote $d(G)$ by d and let $= \{D_1, \ldots, D_d\}$ be a domatic partition of G with d classes. Choose a class of $\mathscr{D}$; without loss of generality let it be D_1. Let $x \in V(G) - D_1 = \bigcup_{i=2}^{d} D_i$. Without loss of generality let $x \in D_2$. As D_1 is a dominating set in G, there exists $y \in D_1$ adjacent to x. Also D_3 is a dominating set in G and therefore there exists $z \in D_3$ adjacent to x. We have $z \in V(G) - D_1$, because $D_1 \cap D_3 = \emptyset$. We have proved that D_1 is a restrained dominating set in G. The set D_1 was chosen arbitrarily, therefore $\mathscr{D}$ is a restrained dominating set in G and $d^r(G) \geqslant d(G)_\gamma$, which together with the former inequality gives the required result. □

The case $d(G) \leqslant 2$ will be treated separately.

Theorem 3. *Let G be a graph, let $d(G) \leqslant 2$. If G has no isolated vertex, then $d^r(G) = d_t(G)$, otherwise $d^r(G) = 1$.*

P r o o f. If G has no isolated vertex, then $d_t^r(G)$ is well-defined and obviously $d^r(G) \leqslant d(G) \leqslant 2$. As any restrained dominating set in G is a dominating set in G, we have also $d^r(G) \leqslant d(G) \leqslant 2$. Suppose $d(G) = 2$ and let $\{D_1, D_2\}$ be a total domatic partition of G with two classes. Let $x \in D_1$. There exists $y \in V(G) - D_1 = D_2$ adjacent to x. As D_2 is a total dominating set in G, there exists $z \in D_2$ adjacent to y. Therefore D_1 is a restrained dominating set in G; analogously we prove that so is D_2 and thus $\{D_1, D_2\}$ is a restrained domatic partition of G and $d^r(G) = 2 = d_t(G)$. Now suppose $d^r(G) = 2$ and let $\{D'_1, D'_2\}$ be a restrained domatic partition of G with two classes. Each vertex of D is adjacent to a vertex of D'_1 and to a vertex of D'_2, because D'_2 is a restrained dominating set in G. Analogously also each vertex of D'_2 is adjacent to a vertex of $V(G) - D'_2 \equiv D'_1$ and to a vertex of D'_2. Both sets D'_1, D'_2 are total dominating sets in G and $\{D'_1, D'_2\}$ is a total domatic partition of G and $d_t(G) = 2 = d^r(G)$. We have proved that $d^r(G) = 2$ if and only if $d_t(G) = 2$. If $d(G) \leqslant 2$, then there is only one other possibility $d^r(G) = 1$ and $d_t(G) = 1$, therefore $d^r(G) = d_t(G)$ again. If G contains an isolated vertex r, then all dominating sets in G contain r and therefore no two of them are disjoint. We have $d(G) = 1$ and thus also $d^r(G) = 1$. □

The numbers $\gamma^r(G)$ and $\gamma_t^r(G)$ where studied in [1], [5], [6]. An interesting motivation for the research of $\gamma_t^r(G)$ is in [1] in applications in guarding prisons. But the concept of our paper shows that probably there is no reason to introduce $d^r(G)$ and $d_t^r(G)$ as new numerical invariants of graphs.

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